Generalized solution to multidimensional cubic Schrödinger equation with delta potential

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Abstract

This article addresses the Cauchy problem for the defocusing cubic Schrödinger equation in 2D and 3D and the equation with a delta well potential in 3D. Solutions belong to the Colombeau algebra of generalized functions $\mathcal{G}_{C^1,H^2}$ (see [15]). The physically significant homogeneous problem in 2D and 3D has not yet been treated in this framework, whereas no classical results exist on the equation with delta potential. The paper contains the construction of unique generalized solutions for both of these problems. One could also find two assertions about compatibility with classical solutions, again for both equations.
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1 Introduction

The subject of this paper are the following defocusing cubic Schrödinger equations

$$\begin{aligned} i\partial_t u + \triangle u &= u|u|^2 \quad \text{on} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) &= a(x), \end{aligned} \tag{1}$$

$n \in \{2,3\}$ and

$$\begin{aligned} i\partial_t u + \triangle u &= u|u|^2 + \delta u, \quad \text{on} \quad \mathbb{R}^3 \times (0,T) \\ u(0,x) &= a(x), \end{aligned} \tag{2}$$

where δ is the Dirac delta distribution and the initial data a are generalized functions.

The nonlinear Schrödinger equation (NLS) represents a universal model at the root of a wide range of physical and other natural phenomena and applications. For example, (1) is a model for the propagation of short temporal pulses in optical fibers. Because of its importance in quantum physics, it is natural to

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consider singular initial data, such as the delta function (representing an initially localized particle). It was shown in [14] that the one-dimensional cubic Schrdinger equation with the delta function as initial data is ill-posed in the class $L^\infty([0,\infty),\mathcal{S}'(\mathbb{R}))$. Contrary to this result, we obtain a unique solution to the corresponding problem in 2D and 3D in the setting of the Colombeau algebra. The critical regularity for global existence of solutions of (1) in 3D is in H^s for $s > \frac{4}{5}$, as is shown in [7]. Our result extends this to the space of distributions, embedded in the Colombeau algebra. Algebras of generalized functions allow performing multiplication and some other nonlinear operations within the space. This is another reason to use the setting, since the solution is expected to be as singular as the initial data and consequently, (1) an (2) involve a product of distributions.

Classical solutions of the equation (1) have been studied extensively in the framework of H^s spaces, where s is at least 0. For a summary of these results see [3]. On the other hand, there are no classical results for the equation (2), but its significance as a model for Bose-Einstein condensates with a well potential is reflected in the large amount of papers regarding solitons, bound states and exact solutions of (2), see for example [8], [9] and [11].

We will employ L^2-based Colombeau algebras $\mathcal{G}_{C^1,H^2}$, which we define below. A representative of a generalized function is then a net of smooth functions which permits the use of the classical estimates on each element of the net. This was the main idea used in the paper.

Up to our knowledge, there are three papers dealing with the Schrödinger equation in the setting of the Colombeau algebra of generalized functions. In [12], Hörmann solved the Cauchy problem in $\mathbb{R}^n$ for the linear Schrödinger equation with variable coefficients, provided the coefficients and initial data are generalized functions. In [13], the convergence properties of regularized solutions to the linear equation were studied. In [4], Bu showed that the cubic one-dimensional Schrdinger equation has a unique generalized solution. The next logical step is studying the limiting case. In this regard, we obtain a positive result for (1) and we leave open the question of convergence of regularized solutions of (2).

This paper is organized as follows.

In Section 2, the required results for (1) are recalled regarding the existence of strong local and global solutions and estimates of H^s norm of solutions. The main point needed for our setting is that for each $s > 0$, $\|u(t)\|_{H^s}$ is bounded uniformly in t both in 2D and 3D, but the bound in 2D is linear in $\|a\|_{H^s}$ and the bound in 3D is exponential in $\|a\|_{H^s}$. Further, the existence of a global solution of (1) in $\mathcal{G}C^1, H^2$ is proved. Uniqueness asks for new estimates of the solution of the regularized equation. Since we do this work in a more general setting for equation (2), we prove uniqueness using estimates from Section 3. Finally, we show that for any strong solution $u \in H^2$ of (1) with initial data in H^2 there is a net of solutions converging to u. Here, the initial data are regularized by $a_\varepsilon = a * \phi_\varepsilon$, where ϕ_ε is an appropriate mollifier. In Section 3, we study a regularized version of (2). Bounds for second order derivatives of the solution are obtained. We have used the operator $\mathcal{T}(t)$, which is a solution operator for the linear homogeneous equation. For NLS, Duhamel's formula holds and also a fundamental estimate for $\mathcal{T}(t)$. Once the estimates were derived, existence and uniqueness of a generalized solution followed.

The notation we use is as in [15]. Let $H^{m,p}(\mathbb{R}^n)$, $n \in \{2, 3\}$ denote the usual

Sobolev space, $H^{m,2}(\mathbb{R}^n) = H^m(\mathbb{R}^n)$. Then $\mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$ (respectively $\mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$), $T>0$ is the vector space of nets $(u_\varepsilon)_\varepsilon$ of functions

$$u_\varepsilon \in C([0,T),H^2(\mathbb{R}^n)) \cap C^1([0,T),L^2(\mathbb{R}^n)),\ \varepsilon\in(0,1),$$

with the property that there exists $N\in\mathbb{N}$ (respectively, for every $M\in\mathbb{N}$) such that

$$\max\{\sup_{t\in[0,T)}\|u_\varepsilon(t)\|_{H^2},\ \sup_{t\in[0,T)}\|\partial_t u_\varepsilon(t)\|_{L^2}\} = \mathcal{O}(\varepsilon^{-N}),$$
$$\max\{\sup_{t\in[0,T)}\|u_\varepsilon(t)\|_{H^2},\ \sup_{t\in[0,T)}\|\partial_t u_\varepsilon(t)\|_{L^2}\} = \mathcal{O}(\varepsilon^{M}),\quad \text{respectively.}$$

The quotient space

$$\mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^n) = \mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^n)/\mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$$

is a Colombeau type vector space. For $n\le 3$ this is a multiplicative algebra, since $H^2(\mathbb{R}^n)$ itself is an algebra for $n\le 3$.

Space $\mathcal{G}_{H^2}(\mathbb{R}^n)$ is defined in a similar way, but with representatives independent in the time variable t. This space is also an algebra in the case $n\le 3$. H^2 spaces are chosen for simplicity, especially in the case of delta potential.

Throughout the paper we use $\|\cdot\|_p$ to denote the $L^p(\mathbb{R}^n)$ norm. By D_x^k, we denote any partial derivative w.r.t. $x_1,...,x_n$ of the form $D_{x_1}^{\alpha_1}...D_{x_n}^{\alpha_n}$ with multindex $\alpha=(\alpha_1,...,\alpha_n)\in\mathbb{N}_0^n$ of order $|\alpha|=k$.

We will use the following lemmas in the sequel:

Lemma 1.1 (Gronwall's inequality)**.** *Let $A(t)$ be continuous and nonnegative on $[0,T]$ and satisfy*

$$A(t)\le E(t)+\int_0^t r(s)A(s)ds,\quad 0\le t\le T,$$

where $r(t)$ is a nonnegative integrable function on $[0,T]$ with $E(t)$ bounded on $[0,T]$. Then

$$A(t)\le |E(t)|\exp\left(\int_0^t r(s)ds\right),\quad 0\le t\le T.$$

Lemma 1.2 (Gagliardo-Nirenberg [5])**.** *let $1\le p,q,r\le\infty$ and let j,m be two integers such that $0\le j<m$. If*

$$\frac{1}{p}=\frac{j}{n}+b\left(\frac{1}{r}-\frac{m}{n}\right)+\frac{1-b}{q},$$

for some $b\in[j/m,1]$ ($b<1$ if $r>1$ and $m-j-\frac{n}{r}=0$) then there exists $C=C(n,m,j,q,r)$ so that

$$\sum_{|\alpha|=j}\|D^\alpha u(t)\|_p \le C\Big(\sum_{|\alpha|=m}\|D^\alpha u(t)\|_r\Big)^b\|u(t)\|_q^{1-b}\quad \forall u\in\mathcal{D}(\mathbb{R}^n) \tag{3}$$

2 The cubic Schrödinger equation

2.1 Estimates

Let $s_0 = \frac{n-2}{2}$ and $s \geq s_0$, where $s_0 \geq 0$ for $n \in \{2, 3\}$. For $a \in H^s$, the Cauchy problem (1) is well posed in $[0, T^*]$ for some T^* and

$$u \in C([0, T], H^s), \quad T < T^*$$

i.e. u is a local solution. It can be extended to a global solution if $a \in H^1$. Also, $|x|^m a \in L^2$ for every $m \in \mathbb{N}$ implies that the solution is smooth in x and t (see [10]). Standard energy equalities hold for (1):

$$\|u(t)\|_{L^2} = \|a\|_{L^2} \quad \text{and} \quad H(u(t)) = H(a)$$

where $H(u(t)) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^n} |u|^4 dx$ denotes the Hamiltonian. The energy equalities imply that the global solution satisfies

$$\sup_{t \in [0,\infty)} \|u(t)\|_{H^1} < \infty.$$

In one dimension, for every $s \geq 0$ the norm $\|u(t, \cdot)\|_{H^s}$ is uniformly bounded w.r.t. $t \in \mathbb{R}$. In two and three dimensions $u(t) \in H^s$ holds for every t and there exists $T = T(\|a\|_{H^s})$ such that

$$\|u(t)\|_{H^s} \leq C\|a\|_{H^s}, \quad t \in [0, T].$$

In [2], it was shown that in 3D there is scattering and a uniform bound

$$\|u(t)\|_{H^s} \leq C \exp(\|a\|_{H^s}), \quad \text{for all t} \geq 0, \quad \text{s} \geq 1. \tag{4}$$

In [6] (inequality (3.25)), it was shown by a similar argument that global in time solutions in 2D also satisfy a uniform bound

$$\|u(t)\|_{H^s} \leq c\|a\|_{H^s}, \quad \text{for all t} \geq 0, \quad \text{s} \geq 1. \tag{5}$$

2.2 Existence and uniqueness

We say that $u \in \mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^n)$ is a solution to (1) if there is its representative u_ε satisfying

$$\begin{aligned} i\partial_t u_\varepsilon + \triangle u_\varepsilon - |u_\varepsilon|^2 u_\varepsilon &= N_\varepsilon \\ u_\varepsilon(x, 0) &= a_\varepsilon(x) + n_\varepsilon(x), \end{aligned} \tag{6}$$

where $(N_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,H^2}([0, T) \times \mathbb{R}^n)$ and $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^n)$. The initial data $a \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ is represented by a_ε.

Theorem 2.1. *Let $n \in \{2, 3\}$, $T > 0$, $a \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ and*

$$\|a_\varepsilon\|_{H^2} \leq h_\varepsilon \tag{7}$$

where $h_\varepsilon \sim \varepsilon^{-N}$ for $n = 2$ and $h_\varepsilon \sim N \ln \varepsilon^{-1}$ for $n = 3$, for some $N \in \mathbb{N}$. Then there exists a solution $u \in \mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^n)$ of (1). If, additionally $h_\varepsilon \sim (\ln(\ln \varepsilon^{-1}))^{1/6}$ and $a \in \mathcal{G}_{H^3}(\mathbb{R}^n)$, $\|a_\varepsilon\|_{H^3} \sim (\ln \varepsilon^{-1})^{1/14}$, the solution is unique in $\mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^n)$.

Proof. Existence. Let us take the equation (1) written in the form of representatives

$$\begin{aligned} i\partial_t u_\varepsilon + \triangle u_\varepsilon - |u_\varepsilon|^2 u_\varepsilon &= 0 \\ u_\varepsilon(x,0) &= a_\varepsilon(x) \end{aligned} \tag{8}$$

There exists a unique solution $u_\varepsilon \in C^1([0,T], H^2(\mathbb{R}^n)$ for every $T > 0$ and ε (see [2] and [6]). Estimates (4) and (5) together with assumption (7) imply

$$\sup_{t\geq 0} \|D_x^\alpha u_\varepsilon(t)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}),\ \varepsilon \to 0,$$

for $|\alpha| \leq 2$. From the Gagliardo-Nirenberg inequality it follows that

$$\||u_\varepsilon(t)|^2 u_\varepsilon(t)\|_2 \leq \|\nabla u_\varepsilon(t)\|_2^3,$$

so boundedness of $\|\partial_t u_\varepsilon(t)\|_2$ follows easily from (8). We can conclude that u, represented by the net of functions $(u_\varepsilon)_\varepsilon$ belongs to the space $\mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$ that solves the problem (1) in the above sense.

Uniqueness. Let $u, v \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$, $n \in \{2,3\}$ be two solutions of (1) with representatives u_ε and v_ε satisfying (6). To obtain uniqueness we need to show that $(u_\varepsilon - v_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$. Let $w_\varepsilon = u_\varepsilon - v_\varepsilon$. Then w_ε solves:

$$\begin{aligned} i(w_\varepsilon)_t + \triangle w_\varepsilon - (|u_\varepsilon|^2 u_\varepsilon - |u_\varepsilon - w_\varepsilon|^2(u_\varepsilon - w_\varepsilon)) + N_\varepsilon &= 0, \\ w_\varepsilon(x,0) &= n_\varepsilon(x), \end{aligned} \tag{9}$$

where $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{H^3}(\mathbb{R}^n)$, $(N_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$. The first equation is simplified to

$$\begin{aligned} &i(w_\varepsilon)_t + \triangle w_\varepsilon - |u_\varepsilon|^2 u_\varepsilon + (u_\varepsilon - w_\varepsilon)(|u_\varepsilon|^2 - u_\varepsilon\overline{w_\varepsilon} - \overline{u_\varepsilon}w_\varepsilon + |w_\varepsilon|^2) + N_\varepsilon = 0, \\ &i(w_\varepsilon)_t + \triangle w_\varepsilon = u_\varepsilon^2\overline{w_\varepsilon} + 2w_\varepsilon|u_\varepsilon|^2 - 2u_\varepsilon|w_\varepsilon|^2 - w_\varepsilon^2\overline{u_\varepsilon} + w_\varepsilon|w_\varepsilon|^2 - N_\varepsilon \end{aligned}$$

If we multiply (9) by $\overline{w_\varepsilon}$, integrate over $\mathbb{R}^n$ and take the imaginary part (using the fact that $w_\varepsilon(x,t) \to 0$ fast enough as $\|x\| \to \infty$ for any t).

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} |w_\varepsilon|^2 dx = \ &\mathrm{Im}\int_{\mathbb{R}^n} \left(2\mathrm{Re}(u_\varepsilon\overline{w_\varepsilon})u_\varepsilon\overline{w_\varepsilon} - |w_\varepsilon|^2 u_\varepsilon\overline{w_\varepsilon} - N_\varepsilon\overline{w_\varepsilon}\right) dx \\ \leq &\int_{\mathbb{R}^n} \left(2|u_\varepsilon w_\varepsilon|^2 + |u_\varepsilon||w_\varepsilon|^3 + |N_\varepsilon w_\varepsilon|\right) dx. \end{aligned} \tag{10}$$

Integration with respect to t gives

$$\begin{aligned} \|w_\varepsilon(t)\|_2^2 \leq &\|n_\varepsilon\|_2^2 + \int_0^t \Big(2\|u_\varepsilon(t)\|_\infty^2\|w_\varepsilon(t)\|_2^2 + \|u_\varepsilon(t)\|_\infty\|w_\varepsilon(t)\|_\infty\|w_\varepsilon(t)\|_2^2 \\ &+ \|N_\varepsilon\|_2\|w_\varepsilon(t)\|_2\Big) d\tau \\ \sup_{[0,T)} \|w_\varepsilon(t)\|_2^2 \leq &\|n_\varepsilon\|_2^2 + 2\sup_{[0,T)}\left(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty\|w_\varepsilon(t)\|_\infty\right)\int_0^T \|w_\varepsilon(t)\|_2^2 d\tau \\ &+ \sup_{[0,T)}\|w_\varepsilon(t)\|_2\|N_\varepsilon\|_2, \\ \sup_{[0,T)} \|w_\varepsilon(t)\|_2^2 \leq &\varepsilon^M \exp(\sup_{[0,T)}\left(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty\|w_\varepsilon(t)\|_\infty\right)), \end{aligned} \tag{11}$$

for arbitrary $M \in \mathbb{N}$. Due to the construction above, the Sobolev inequality $\|u_\varepsilon(t)\|_\infty \leq \|u_\varepsilon(t)\|_{H^2}$ holds. We will use the following two facts that will be proved in Section 3.

- Condition (7) and $\|a_\varepsilon\|_{H^3} \sim (\ln \varepsilon^{-1})^{1/14}$ imply $\|u_\varepsilon(t)\|_{H^2} \sim \sqrt{\ln \varepsilon^{-1}}$ (the relation (36)).
- Estimates (4) and (5) can not be directly used bellow, since equation (6) is not homogeneous. These bounds are derived in the uniqueness proof of Theorem 3.4 (relation (35)).

Applying Gronwall's inequality, Lemma 1.1 to (11) we obtain

$$\sup_{0\leq t\leq T} \|w_\varepsilon(t)\|_2 = O(\varepsilon^M), \quad \varepsilon \to 0, \quad \text{for any } M \in \mathbb{N}. \tag{12}$$

To obtain a bound on the first derivative in x, differentiate (9) in x, multiply by $D_x \overline{w}_\varepsilon$, integrate over $\mathbb{R}^n$ and take the imaginary part. Similarly as before, this results in the following

$$\begin{aligned}
&\sup_{[0,T)} \|D_x w_\varepsilon(t)\|_2^2 \leq \sup_{[0,T)} (\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty + \|w_\varepsilon(t)\|_\infty^2) \\
&\cdot \int_0^T \|D_x w_\varepsilon(t)\|_2^2 d\tau + \|D_x n_\varepsilon\|_2^2 + T \sup_{[0,T)} (\|D_x N_\varepsilon(t)\|_2 \|D_x w_\varepsilon(t)\|_2) \\
&+ \sup_{[0,T)} (\|u_\varepsilon(t)\|_\infty + \|w_\varepsilon(t)\|_\infty) \int_0^T \|w_\varepsilon(t)\|_2 \|D_x w_\varepsilon(t) D_x u_\varepsilon(t)\|_2 d\tau \\
\leq &\ln \varepsilon^{-1} \int_0^T \|D_x w_\varepsilon(t)\|_2^2 d\tau + \varepsilon^M + T(\varepsilon^M \varepsilon^{-N} + \sqrt{\ln \varepsilon^{-1}} \varepsilon^{-N} \varepsilon^M).
\end{aligned}$$

The last inequality follows from (12) and

$$\begin{aligned}
&\|D_x w_\varepsilon(t) D_x u_\varepsilon(t)\|_2 \leq \|D_x w_\varepsilon(t)\|_4 \|D_x u_\varepsilon(t)\|_4 \\
\leq &\Big(\sum_{|\alpha|=2} \|D_x^2 w_\varepsilon(t)\|_2 \Big)^b \|w_\varepsilon(t)\|_2^{1-b} \Big(\sum_{|\alpha|=2} \|D_x^2 u_\varepsilon(t)\|_2 \Big)^b \|u_\varepsilon(t)\|_2^{1-b} \leq \varepsilon^{-N},
\end{aligned} \tag{13}$$

where $b = \frac{3}{4}$ for $n = 2$ and $b = \frac{7}{8}$ for $n = 3$, in both cases $p = 4,\ r = q = 2,\ j = 1, m = 2$. Here, the Hölder inequality and the Gagliardo- Nirenberg inequality were used. Applying Gronwall's lemma we obtain

$$\sup_{0\leq t\leq T} \|D_x w_\varepsilon(t)\|_2 = O(\varepsilon^M), \quad \varepsilon \to 0. \tag{14}$$

By differentiation of (9) it further follows that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^n} |D_x^2 w_\varepsilon|^2 dx \leq \int_{\mathbb{R}^n} |(u_\varepsilon + w_\varepsilon) D_x u_\varepsilon D_x w_\varepsilon D_x^2 w_\varepsilon + w_\varepsilon (D_x u_\varepsilon)^2 D_x^2 w_\varepsilon \\
&+ (u_\varepsilon w_\varepsilon + w_\varepsilon^2) D_x^2 u_\varepsilon D_x^2 w_\varepsilon + (u_\varepsilon + w_\varepsilon)(D_x w_\varepsilon)^2 D_x^2 w_\varepsilon| dx \\
&+ \int_{\mathbb{R}^n} |u_\varepsilon^2 + u_\varepsilon w_\varepsilon + w_\varepsilon^2| |D_x^2 w_\varepsilon|^2 dx + \int_{\mathbb{R}^n} |D_x^2 N_\varepsilon| |D_x^2 w_\varepsilon| dx.
\end{aligned}$$

For the first term on the right hand side we have

$$\|(u_\varepsilon(t)+w_\varepsilon(t))D_xu_\varepsilon(t)D_xw_\varepsilon(t)D_x^2w_\varepsilon(t)\|_1 \le (\|u_\varepsilon(t)\|_\infty+\|w_\varepsilon(t)\|_\infty)\|D_x^2w_\varepsilon(t)\|_2$$
$$\cdot\Big(\sum_{|\alpha|=2}\|D_x^\alpha u_\varepsilon(t)\|_2\Big)^{\frac{7}{8}}\Big(\sum_{|\alpha|=2}\|D_x^\alpha w_\varepsilon(t)\|_2\Big)^{\frac{7}{8}}(\|u_\varepsilon(t)\|_2\|w_\varepsilon(t)\|_2)^{\frac{1}{8}}. \tag{15}$$

Further on

$$\begin{aligned}&\|w_\varepsilon(t)(D_xu_\varepsilon(t))^2D_x^2w_\varepsilon(t)\|_1 \le \|D_x^2w_\varepsilon(t)\|_2\|w_\varepsilon(t)(D_xu_\varepsilon(t))^2\|_2\\ \le&\|D_x^2w_\varepsilon(t)\|_2\sum_{|\alpha|=1}\|D_xw_\varepsilon(t)\|_2\Big(\sum_{|\alpha|=2}\|D_x^\alpha u_\varepsilon(t)\|_2\Big)^2.\end{aligned} \tag{16}$$

The fourth term can be bounded in the following way

$$\begin{aligned}&\|(u_\varepsilon(t)+w_\varepsilon(t))(D_xw_\varepsilon(t))^2D_x^2w_\varepsilon(t)\|_1\\ \le&\|u_\varepsilon(t)+w_\varepsilon(t)\|_\infty\|D_x^2w_\varepsilon(t)\|_2\|D_xw_\varepsilon(t)\|_4^2\\ \le&\|u_\varepsilon(t)+w_\varepsilon(t)\|_\infty\|D_x^2w_\varepsilon(t)\|_2\Big(\sum_{|\alpha|=2}\|D_x^\alpha w_\varepsilon(t)\|_2\Big)^{\frac{7}{4}}\|w_\varepsilon(t)\|_2^{\frac{1}{4}}\end{aligned} \tag{17}$$

Also,

$$\begin{aligned}&\|(u_\varepsilon(t)w_\varepsilon(t)+w_\varepsilon^2(t))D_x^2u_\varepsilon(t)D_x^2w_\varepsilon(t)\|_1\\ \le&\|u_\varepsilon(t)+w_\varepsilon(t)\|_\infty\|w_\varepsilon(t)\|_\infty\|D_x^2u_\varepsilon(t)\|_2\|D_x^2w_\varepsilon(t)\|_2\\ \le&\|u_\varepsilon(t)+w_\varepsilon(t)\|_\infty(\|w_\varepsilon(t)\|_2+\|D_xw_\varepsilon(t)\|_2)\|D_x^2u_\varepsilon(t)\|_2\|D_x^2w_\varepsilon(t)\|_2\\ &+\|D_x^2w_\varepsilon(t)\|_2^2\|D_x^2u_\varepsilon(t)\|_2\|u_\varepsilon(t)+w_\varepsilon(t)\|_\infty.\end{aligned} \tag{18}$$

The last inequality was obtained using Sobolev inequality and (15) – (17) followed from the Gagliardo–Nirenberg inequality. All factors except the one for $\|D_x^2w_\varepsilon(t)\|_2^2\|D_x^2u_\varepsilon(t)\|_2\|u_\varepsilon(t)+w_\varepsilon(t)\|_\infty$ can be bounded by ε^M because of (12), (14) and again the fact that $\|u_\varepsilon(t)\|_\infty \le c\|u_\varepsilon(t)\|_{H^2} \sim \sqrt{\ln\varepsilon^{-1}}$. Integrating in t,

$$\begin{aligned}&\|D_x^2w_\varepsilon(t)\|_2^2\\ \le&(\|u_\varepsilon(t)\|_\infty^2+\|u_\varepsilon(t)\|_\infty\|w_\varepsilon(t)\|_\infty+\|w_\varepsilon(t)\|_\infty^2+\|D_x^2u_\varepsilon(t)\|_2\|u_\varepsilon+w_\varepsilon(t)\|_\infty)\\ &\cdot\int_0^t\|D_x^2w_\varepsilon(t)\|_2^2d\tau+\varepsilon^M,\end{aligned}$$

Gronwall's lemma again implies that $\sup_{0\le t\le T}\|D_x^2w_\varepsilon(t)\|_2 = O(\varepsilon^M),\ \varepsilon\to 0$, for any $M\in\mathbb{N}$.

Directly from equation (9), it follows that $\|\partial_tw_\varepsilon(t)\|_2 = O(\varepsilon^M),\ \varepsilon\to 0$, for any $M\in\mathbb{N}$. Thus, $(w_\varepsilon)_\varepsilon\in\mathcal{N}_{C^1,H^2}([0,T)\times\mathbb{R}^n)$ and the solution to (1) is unique. □

Remark 2.2. *Note that we have used the initial data $a\in\mathcal{G}_{H^3}(\mathbb{R}^n)$ for the uniqueness proof. The condition $\mathcal{G}_{H^2}(\mathbb{R}^n)$ suffices for the existence proof. That will be the case for the equation with a delta potential bellow, too.*

2.3 Compatibility with the classical solution

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \phi \, dx = 1$ and $\phi \geq 0$. Define $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$. The following holds

Theorem 2.3. *Let u be the classical H^2 solution of the cubic Schrdinger equation in $n \in \{2,3\}$ dimensions:*

$$\begin{aligned} i\partial_t u + \triangle u - |u|^2 u = 0 \quad &\text{on} \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) = a(x), & \end{aligned}$$

for $a \in H^3(\mathbb{R}^n)$. Let $T > 0$. The solution u_ε to the equation (8) with initial data $a_\varepsilon = a * \phi_\varepsilon$ converges to u in $H^2(\mathbb{R}^n)$–norm for every $t < T$.

Proof. Since

$$\|D_x^\alpha (a * \phi_\varepsilon)\|_2 = \|D_x^\alpha a * \phi_\varepsilon\|_2 \leq \|D_x^\alpha a\|_2 \|\phi_\varepsilon\|_1 = \|D_x^\alpha a\|_2$$

for $|\alpha| \leq 3$, uniformly with respect to ε, we obtain condition (7). It follows that the regularized initial data give rise to a unique solution in the space $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$.

We put $v_\varepsilon = u - u_\varepsilon$ now. Then $u \in H^2$ implies that $\|u(t)\|_\infty$ is finite, and $u_\varepsilon \in H^2$ for each $\varepsilon > 0$ gives, based on (4),

$$\begin{aligned} \|v_\varepsilon(t)\|_\infty &\leq \|u(t)\|_\infty + \|u_\varepsilon(t)\|_\infty \leq c_1 + \|u_\varepsilon(t)\|_{H^2} \\ &\leq c_1 + \exp(\|a * \phi_\varepsilon\|_{H^2}) \leq c_1 + c_2, \end{aligned}$$

Also,

$$\|D_x^\gamma v_\varepsilon(t)\|_2 \leq \|D_x^\gamma u(t)\|_2 + \|D_x^\gamma u_\varepsilon(t)\|_2 \leq c, \quad |\gamma| \leq 2$$

Further, v_ε satisfies

$$\begin{aligned} i\partial_t v_\varepsilon + \triangle v_\varepsilon - (|u|^2 u - |u - v_\varepsilon|^2 (u - v_\varepsilon)) = 0, & \\ v_\varepsilon(x,0) = a(x) - a * \phi_\varepsilon(x). & \end{aligned}$$

Like in the uniqueness proof, one can see that

$$\|v_\varepsilon(t)\|_2^2 \leq \|a - a * \phi_\varepsilon\|_2^2 \; \exp((\|u(t)\|_\infty^2 + \|u(t)\|_\infty \|v_\varepsilon(t)\|_\infty)T).$$

Therefore,

$$\|v_\varepsilon(t)\|_2^2 \leq C\|a - a * \phi_\varepsilon\|_2^2 \to 0, \quad \varepsilon \to 0.$$

In the same way,

$$\begin{aligned} \|D_x v_\varepsilon(t)\|_2^2 \leq T \; &\|D_x(a - a * \phi_\varepsilon)\|_2^2 \; (6\|u(t)\|_\infty \|v_\varepsilon(t)\|_\infty \\ &+ 3\|v_\varepsilon(t)\|_\infty^2) \; \|D_x u(t)\|_2 \|D_x v_\varepsilon(t)\|_2 \cdot \\ &\cdot \exp(3\|u(t)\|_\infty^2 + 6\|u(t)\|_\infty \|v_\varepsilon(t)\|_\infty + 2\|v_\varepsilon(t)\|_\infty)T \to 0, \quad \varepsilon \to 0, \end{aligned}$$

because all terms are bounded with respect to ε and $\|D_x(a - a * \phi_\varepsilon)\|_2 \to 0$, $\varepsilon \to 0$. The second order derivatives can be bounded in the following way

$$\begin{aligned} \frac{d}{dt}\|D_x^2 v_\varepsilon(t)\|_2^2 \leq \int_{\mathbb{R}^n} \Big(&|D_x^2 v_\varepsilon|^2 |u^2 + u v_\varepsilon + v_\varepsilon^2| \\ &+ |u D_x u D_x v_\varepsilon D_x^2 v_\varepsilon| + |v_\varepsilon (D_x u)^2 D_x^2 v_\varepsilon| + |u (D_x v_\varepsilon)^2 D_x^2 v_\varepsilon| \\ &+ |u D_x^2 u v_\varepsilon D_x^2 v_\varepsilon| + |v_\varepsilon^2 D_x^2 u D_x^2 v_\varepsilon| \Big) dx \end{aligned}$$

Integrating in t we obtain

$$\begin{aligned}
&\|D_x^2 v_\varepsilon(t)\|_2^2 \le \|D_x^2(a - a * \phi_\varepsilon)\|_2^2 \\
&+ (\|u(t)\|_\infty^2 + \|u(t)\|_\infty \|v_\varepsilon(t)\|_\infty + \|v_\varepsilon(t)\|_\infty^2) \int_0^t \|D_x^2 v_\varepsilon(t)\|_2^2 d\tau \\
&+ \|u(t)\|_\infty \|D_x^2 v_\varepsilon(t)\|_2 \|D_x u(t) D_x v_\varepsilon(t)\|_2 + \|v_\varepsilon(t)\|_\infty \|D_x^2 v_\varepsilon(t)\|_2 \|(D_x u(t))^2\|_2 \\
&+ \|u(t)\|_\infty \|D_x^2 v_\varepsilon(t)\|_2 \|(D_x v_\varepsilon(t))^2\|_2 \\
&+ (\|u(t)\|_\infty \|v_\varepsilon(t)\|_\infty + \|v_\varepsilon(t)\|_\infty^2) \|D_x^2 u(t)\|_2 \|D_x^2 v_\varepsilon(t)\|_2 \qquad (19)
\end{aligned}$$

Using the Hölder inequality,

$$\begin{aligned}
&\|D_x u(t) D_x v_\varepsilon(t)\|_2 \le \|D_x u(t)\|_4 \|D_x v_\varepsilon(t)\|_4 \\
\le &\Big(\sum_{|\alpha|=2} \|D_x^2 u(t)\|_2\Big)^b \|u(t)\|_2^{1-b} \cdot \Big(\sum_{|\alpha|=2} \|D_x^2 v_\varepsilon(t)\|_2\Big)^b \|v_\varepsilon(t)\|_2^{1-b} \le C,
\end{aligned}$$

where the Gagliardo-Nirenberg inequality was also used with $b = \frac{3}{4}$ for $n = 2$ and $b = \frac{7}{8}$ for $n = 3$. Terms $\|(D_x u(t))^2\|_2$ and $\|(D_x v_\varepsilon(t))^2\|_2$ can be estimated in the same way. Convergence of $\|D_x^2 v_\varepsilon(t)\|_2$ to zero as $\varepsilon \to 0$ follows by applying Gronwall's inequality to (19). This completes the proof. $\square$

3 The delta potential

Consider now the equation with the delta potential

$$\begin{aligned}
i\partial_t u + \triangle u &= u|u|^2 + \delta u, \\
u(0,x) &= a(x),\ a \in \mathcal{G}_{H^2}(\mathbb{R}^3).
\end{aligned} \qquad (20)$$

A representative of δ is chosen such that the regularized version of (20) is

$$\begin{aligned}
i\partial_t u_\varepsilon + \triangle u_\varepsilon &= u_\varepsilon |u_\varepsilon|^2 + \phi_{h_\varepsilon} u_\varepsilon, \\
u_\varepsilon(0,x) &= a_\varepsilon(x),
\end{aligned} \qquad (21)$$

where $\phi_{h_\varepsilon}(x) = h_\varepsilon^3 \phi(h_\varepsilon x)$ and ϕ is a non-negative mollifier as before. Later on, one will see that we have to take $h_\varepsilon \sim (\ln \varepsilon^{-1})^{5/19}$. Let $\varepsilon > 0$. Multiplying by $\overline{u_\varepsilon}$, integrating over $\mathbb{R}^3$ and taking the imaginary part, we again get conservation of energy

$$\|u_\varepsilon(t)\|_{L^2(\mathbb{R}^3)} = \|a_\varepsilon\|_{L^2(\mathbb{R}^3)}.$$

Multiplying by $(\overline{u_\varepsilon})_t$, integrating over $\mathbb{R}^3$ and taking the real part , we have

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} |u_\varepsilon|^4 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{h_\varepsilon} |u_\varepsilon|^2 dx = H(a_\varepsilon),$$

where $H(a_\varepsilon) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla a_\varepsilon|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} |a_\varepsilon|^4 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{h_\varepsilon} |a_\varepsilon|^2 dx$. Then

$$H(u_\varepsilon(t)) = H(a_\varepsilon) \ge \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx$$

and

$$
\begin{aligned}
\|u_\varepsilon(t)\|_{H^1} \le& H(a_\varepsilon) + \|a_\varepsilon\|_2 \\
\le& \frac{1}{2}\int_{\mathbb{R}^3} |\nabla a_\varepsilon|^2 dx + \frac{1}{4}\cdot \|a_\varepsilon\|_2 \big(\sum_{|\alpha|=1} \|D_x^\alpha a_\varepsilon\|_2\big)^3 + \frac{1}{2}\|a_\varepsilon\|_\infty^2 \\
\le& \|a_\varepsilon\|_{H^2}^2 + \|a_\varepsilon\|_{H^2}^4. \qquad (22)
\end{aligned}
$$

We have used inequality (3) with $j = 0$, $m = 1$, $a = \frac{3}{4}$, $p = 4$, and $r = q = 2$.

3.1 Solution for a fixed $\varepsilon > 0$

The first step is to find a local solution in $H^1(\mathbb{R}^3)$ to

$$
\begin{aligned}
i\partial_t u_\varepsilon + \triangle u_\varepsilon + g_1(u_\varepsilon) + g_2(u_\varepsilon) = 0,& \\
u_\varepsilon(0,x) = a_\varepsilon(x),& \qquad (23)
\end{aligned}
$$

where $g_1(u_\varepsilon) = -\phi_\varepsilon u$, $g_2(u_\varepsilon) = -u_\varepsilon|u_\varepsilon|^2$, $\phi \in C_0^\infty(\mathbb{R}^3)$.

Theorem 3.1. (Theorem 4.3.1 in [5]) *Let g_1 and g_2 satisfy the following assumptions:*

$$
g_j = G_j', \quad \text{for some } G_j \in C^1(H^1(\mathbb{R}^3), \mathbb{R}),\ j \in \{1, 2\}.
$$

In particular, $g_j \in C(H^1(\mathbb{R}^3), H^{-1}(\mathbb{R}^3))$. We assume that there exist $r_j, \rho_j \in [2, 6)$ such that $g_j \in C(H^1(\mathbb{R}^3), L^{\rho_j'}(\mathbb{R}^3))$ and such that for every $M < \infty$ there exists $C(M) < \infty$ such that

$$
\|g_j(u) - g_j(v)\|_{L^{\rho_j'}} \le C(M)\|u - v\|_{L_j^r}
$$

for all $u, v \in H^1(\mathbb{R}^3)$ such that $\|u(t)\|_{H^1} + \|v\|_{H^1} \le M$. Finally, we assume that, for every $u \in H^1(\mathbb{R}^3)$,

$$
\operatorname{Im}(g_j(u)\overline{u}) = 0, \quad \text{a.e. in } \mathbb{R}^3.
$$

Let $G(u) = G_1(u) + G_2(u)$ and define the energy H by

$$
H(u) = \frac{1}{2}\int |\nabla u|^2 dx - G(u)
$$

for $u \in H^1(\mathbb{R}^3)$. Under these assumptions, the initial value problem is locally well posed in $H^1(\mathbb{R}^3)$. Furthermore, the energy is conserved,

$$
\|u(t)\|_{L^2} = \|a\|_{L^2}, \quad H(u(t)) = H(a)
$$

for all $t \in (-T_{min}, T_{max})$.

The functions $g_1(u) = -\phi_{h_\varepsilon} u$ and $g_2(u) = -u|u|^2$ satisfy the conditions of the theorem:

- Example 3.2.1 and Proposition 3.2.2 for g_1,
- Proposition 3.2.5 and Remark 3.2.6 for g_2,

- and Remark 3.3.4 for both functions, with
- $G_1(u) = -\frac{1}{2}\int \phi|u|^2$, $G_2(u) = -\frac{1}{4}\int |u|^4$.

Using (22), one can see that there exists a unique local solution satisfying

$$\|u_\varepsilon(t)\|_{L^2} = \|a_\varepsilon\|_{L^2}, \text{ and } H(u_\varepsilon(t)) = H(a_\varepsilon),\ t < T_\varepsilon. \tag{24}$$

The bounds (24) enable us to prolong a solution to any T using the same time steps T_ε. Thus, there exists a global solution in $H^1(\mathbb{R}^3)$ to (23).

Even more, there exists a solution in $C^1([0,T), H^2(\mathbb{R}^3))$ when the initial data belongs to $H^2(\mathbb{R}^3)$ The proof is based on the analysis from Section 5 in [5]: The right–hand side of (10) satisfies all assumptions in Theorem 5.3.1, so a H^2–solution solution exist in the same interval $[0,T)$ where we already have the H^1–solution determined above. The proofs of the theorems is based on the fact that for more regular initial data local solution in H^2 does not blow up before $t = T$, where T is a life–span of H^1 solution.

3.2 Generalized solution

Let $\mathcal{T}(t) = e^{it\triangle}$ denote the Schrdinger evolution operator (propagator) defined in the following way (see [16]). Take the equation

$$i\partial_t u + \triangle u = 0, \quad u(0,x) = u_0(x), \tag{25}$$

for u_0 in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. Then,

$$\begin{aligned} &\widehat{u(t)}_t(\xi) = -i|\xi|^2\widehat{u(t)}(\xi) \quad \text{and} \\ &\widehat{u(t)}(\xi) = e^{-it|\xi|^2}\widehat{u_0}(\xi). \end{aligned}$$

It follows that

$$e^{it\triangle}u_0(x) = \int_{\mathbb{R}^n} e^{-it|\xi|^2 + ix\cdot\xi}\widehat{u_0}(\xi)d\xi$$

is a solution of (25). This operator is defined initially for Schwartz functions, but can be extended by standard density arguments to other spaces. $\mathcal{T}(t)$ is unitary on $L^2(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$. It is also a Fourier multiplier and as such, commutes with other Fourier multipliers, including constant coefficient differential operators. We use this fact in the proof of Theorem 3.4, along with the following theorems

Theorem 3.2. (Proposition 2.2.3 in [5]) *Let $p \in [2,\infty]$, $1/p + 1/p' = 1$ and $t > 0$. Then $\mathcal{T}(t)$ maps $L^{p'}(\mathbb{R}^n)$ continuously to $L^p(\mathbb{R}^n)$ and*

$$\|\mathcal{T}(t)\phi\|_p \le (4\pi|t|)^{-n(\frac{1}{2}-\frac{1}{p})}\|\phi\|_{p'}, \quad \text{for all } \phi \in L^{p'} \tag{26}$$

Theorem 3.3. (Duhamel's formula) *The function $u \in C(I, H^3(\mathbb{R}^3))$ is solution of (23) if and only if:*

$$u(t) = \mathcal{T}(t)a + i\int_0^t \mathcal{T}(t-s)(-u(s)|u(s)|^2 - \phi u(s))ds \text{ for all } t \in I, \tag{27}$$

where I is an interval in $\mathbb{R}$ such that $0 \in I$.

We can now state the theorem

Theorem 3.4. *Let*

$$\|a_\varepsilon\|_{H^3} = \mathcal{O}(\varepsilon^{-N}), \quad \text{and} \quad \|a_\varepsilon\|_{H^2} = \mathcal{O}(h_\varepsilon) \quad \textit{for some } N \in \mathbb{N}, \tag{28}$$

where $h_\varepsilon \sim (\ln \varepsilon^{-1})^{\frac{5}{19}}$*. Then for any* $T > 0$ *there exists a generalized solution* $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ *of* (20)*. Additionally, if* $h_\varepsilon \sim (\ln(\ln \varepsilon^{-1})^{\frac{1}{14}})^{\frac{1}{6}} \sim (\ln(\ln \varepsilon^{-1}))^{\frac{1}{6}}$ *and* $a \in \mathcal{G}_{H^3(\mathbb{R}^3)}$*,* $\|a_\varepsilon\|_{H^3} \sim (\ln \varepsilon^{-1})^{\frac{1}{14}}$*, the solution is unique in* $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$*.*

Proof. Existence. The following inequalities

$$\begin{aligned}
&\sum_{|\alpha|=1} \|D_x^\alpha u_\varepsilon(t)\|_2 \le c\sqrt{H(a_\varepsilon)},\\
&\qquad\qquad H(a_\varepsilon) \le \|a_\varepsilon\|_{H^1}^2 + \Big(\sum_{|\alpha|=1} \|D^\alpha a_\varepsilon\|_2\Big)^3 \cdot \|a_\varepsilon\|_2 + \|a_\varepsilon\|_\infty\\
&\qquad\qquad\qquad\quad \le c(\|a_\varepsilon\|_{H^2}^2 + \|a_\varepsilon\|_{H^2}^4)
\end{aligned}$$

hold as above. Then $H(a_\varepsilon) \le c(\ln \varepsilon^{-1})^{\frac{20}{19}}$ as $\varepsilon \to 0$, from (28). We have seen that for each $\varepsilon > 0$ there exists a unique solution $u_\varepsilon \in C^1([0,T), H^3(\mathbb{R}^3))$ to (21). Differentiating Duhamel's formula (27) twice in x and using (26) we obtain

$$\begin{aligned}
\|D_x^2 u_\varepsilon(t)\|_2 \le &\|D_x^2 a_\varepsilon\|_2 + c\int_0^t \|u_\varepsilon^2(t) D_x^2 u_\varepsilon(t) + u_\varepsilon(t)|D_x u_\varepsilon(t)|^2\|_2 ds\\
&+ \int_0^t \|u_\varepsilon(t) D_x^2 \phi_{h_\varepsilon} + D_x\phi_{h_\varepsilon} D_x u_\varepsilon(t) + \phi_{h_\varepsilon}(t) D_x^2 u_\varepsilon(t)\|_2 ds.
\end{aligned}\tag{29}$$

There holds

$$\begin{aligned}
\|u_\varepsilon^2 D_x^2 u_\varepsilon(t)\|_2 \le &\|D_x^2 u_\varepsilon(t)\|_{\frac{10}{3}} \|u_\varepsilon(t)\|_{10}^2\\
\le &\|D_x^2 u_\varepsilon(t)\|_{\frac{10}{3}} \Big(\sum_{|\alpha|=1} \|D_x^\alpha u_\varepsilon(t)\|_{\frac{10}{3}}\Big)^{\frac{3}{2}} \|u(t)\|_2^{\frac{1}{2}}.
\end{aligned}$$

Here we used the Hölder inequality and the Gagliardo–Nirenberg inequality for $j = 0$, $m = 1$, $p = 10$, $r = \frac{10}{3}$, $q = 2$, $b = \frac{3}{4}$. Similarly,

$$\begin{aligned}
\|u_\varepsilon(t)|D_x u(t)|^2\|_2 \le &\|u_\varepsilon(t)\|_6 \|D_x u_\varepsilon(t)\|_6^2\\
\le &\Big(\sum_{|\alpha|=1} \|D_x^\alpha u_\varepsilon(t)\|_2\Big)\Big(\sum_{|\alpha|=2} \|D_x^\alpha u_\varepsilon(t)\|_{\frac{10}{3}}\Big)^{\frac{20}{13}} \|u_\varepsilon(t)\|_2^{\frac{6}{13}},
\end{aligned}$$

where $j = 1$, $m = 2$, $p = 6$, $r = \frac{10}{3}$, $q = 2$, $b = \frac{10}{13}$ in the Gagliardo–Nirenberg inequality. Finally,

$$\begin{aligned}
&\|u_\varepsilon D_x^2 \phi_{h_\varepsilon}\|_2 \le \|D_x^2 \phi_{h_\varepsilon}\|_\infty \|u_\varepsilon(t)\|_2, \quad \|D_x u_\varepsilon D_x \phi_{h_\varepsilon}\|_2 \le \|D_x \phi_{h_\varepsilon}\|_\infty \|D_x u_\varepsilon(t)\|_2,\\
&\|\phi_{h_\varepsilon} D_x^2 u_\varepsilon(t)\|_2 \le \|D_x^2 u_\varepsilon(t)\|_{10/3} \|\phi_{h_\varepsilon}\|_5.
\end{aligned}$$

The norms $\|D_x^\alpha \phi_{h_\varepsilon}\|_p$ are controlled by h_ε^m for some m. It remains to obtain bounds for $\|D_x u_\varepsilon(t)\|_{\frac{10}{3}}$ and $\|D_x^2 u_\varepsilon(t)\|_{\frac{10}{3}}$. Again we use Duhamel's formula

(27), estimate (26) for $p = \frac{10}{3}$, $p' = \frac{10}{7}$ and the fact that $\mathcal{T}(t)$ commutes with D_x

$$\begin{aligned}
&\|D_x u_\varepsilon(t)\|_{\frac{10}{3}} \le \|D_x(\mathcal{T}(t)a_\varepsilon)\|_{\frac{10}{3}} \\
&+ c\int_0^t \frac{1}{(t-s)^{\frac{3}{5}}}\|D_x(u_\varepsilon(s)|u_\varepsilon(s)|^2 + \phi_{h_\varepsilon}u_\varepsilon)(s)\|_{\frac{10}{7}}\,ds, \\
&\|D_x(\mathcal{T}(t)a_\varepsilon)\|_{\frac{10}{3}} \le \big(\sum_{|\alpha|=2}\|D_x^\alpha(\mathcal{T}(t)a_\varepsilon)\|_2\big)^{\frac{4}{5}}\|\mathcal{T}(t)a_\varepsilon\|_2^{\frac{1}{5}} \le \|a_\varepsilon\|_{H^2}
\end{aligned}$$

where the Gagliardo-Nirenberg inequality (3) was used, $j = 1$, $m = 2$, $p = 10/3$, $q = r = 2$, $b = 4/5$. Applying the Hölder inequality and (3) again with $j = 0$, $m = 1$, $p = 5$, $r = q = 2$, $b = \frac{9}{10}$, we derive the following inequalities

$$\begin{aligned}
\|D_x\phi_{h_\varepsilon}u_\varepsilon(t)\|_{\frac{10}{7}} \le& \|a_\varepsilon\|_2\|D_x\phi_{h_\varepsilon}\|_5, \\
\|\phi_{h_\varepsilon}D_x u_\varepsilon(t)\|_{\frac{10}{7}} \le& \|D_x u_\varepsilon(t)\|_2\|\phi_{h_\varepsilon}\|_5 \le H(a_\varepsilon)^{\frac{1}{2}}\|\phi_{h_\varepsilon}\|_5, \text{ and} \\
\|D_x u_\varepsilon(t)|u_\varepsilon(t)|^2\|_{\frac{10}{7}} \le& \|D_x u_\varepsilon(t)\|_{\frac{10}{3}}\|u_\varepsilon(t)\|_5^2 \\
\le& \|D_x u_\varepsilon(t)\|_{\frac{10}{3}}\Big(\sum_{|\alpha|=1}\|D_x^\alpha u_\varepsilon(t)\|_2\Big)^{\frac{9}{5}}\|u_\varepsilon(t)\|_2^{\frac{1}{5}} \\
\le& \|D_x u_\varepsilon(t)\|_{\frac{10}{3}}H(a_\varepsilon)^{\frac{9}{10}}\|a_\varepsilon\|_2^{\frac{1}{5}}.
\end{aligned}$$

Gronwall's inequality implies

$$\begin{aligned}
\sup_{[0,T)}\|D_x u_\varepsilon(t)\|_{\frac{10}{3}} \le& \Big(\|a_\varepsilon\|_{H^2} + T^{\frac{2}{5}}(\|a_\varepsilon\|_2\|D_x\phi_{h_\varepsilon}\|_5 + H(a_\varepsilon)^{\frac{1}{2}}\|\phi_{h_\varepsilon}\|_5)\Big) \\
&\cdot\exp\Big(T^{\frac{2}{5}}H(a_\varepsilon)^{\frac{9}{10}}\|a_\varepsilon\|_2^{\frac{1}{5}}\Big) \qquad (30)
\end{aligned}$$

Denote by f_ε the expression on the right hand side. It follows

$$f_\varepsilon \le (\ln\varepsilon^{-1})^m\exp(T^{\frac{2}{5}}(\ln\varepsilon^{-1})^p),$$

for some $m > 0$ and $p = \frac{20}{19}\cdot\frac{9}{10} + \frac{5}{19}\cdot\frac{1}{5} = 1$. Finally,

$$\sup_{[0,T)}\|D_x u_\varepsilon(t)\|_{\frac{10}{3}} \le c\varepsilon^{-N}, \quad \varepsilon \to 0, \text{ for some } N.$$

Similarly

$$\begin{aligned}
&\|D_x^2 u_\varepsilon(t)\|_{\frac{10}{3}} \le \|D_x^2(\mathcal{T}(t)a_\varepsilon)\|_{\frac{10}{3}} \\
&+ c\int_0^t \frac{1}{(t-s)^{\frac{3}{5}}}\|D_x^2(u_\varepsilon(s)|u_\varepsilon(s)|^2 + \phi_{h_\varepsilon}u_\varepsilon(s))\|_{\frac{10}{7}}\,ds, \\
&\|D_x^2(\mathcal{T}(t)a_\varepsilon)\|_{\frac{10}{3}} \le \Big(\sum_{|\alpha|=3}\|D^\alpha(\mathcal{T}(t)a_\varepsilon)\|_2\Big)^{\frac{13}{15}}\|\mathcal{T}(t)a_\varepsilon\|_2^{\frac{2}{15}} \le \|a_\varepsilon\|_{H^3}, \\
&\|D_x^2 u_\varepsilon(t)\|_{\frac{10}{3}} \le \Big(\|a_\varepsilon\|_{H^3} + T^{\frac{2}{5}}\Big(\|a_\varepsilon\|_2\|D_x^2\phi_{h_\varepsilon}\|_5 + H(a_\varepsilon)^{\frac{1}{2}}\|D_x\phi_{h_\varepsilon}\|_5 \\
&+ \|a_\varepsilon\|_2^{2/3}\Big(\sum_{|\alpha|=1}\|D_x u_\varepsilon(t)\|_{\frac{10}{3}}\Big)^{\frac{7}{3}}\Big)\Big)\exp\Big(T^{\frac{2}{5}}(\|\phi_{h_\varepsilon}\|_{\frac{5}{2}} + H(a_\varepsilon)^{\frac{9}{10}}\|a_\varepsilon\|_2^{\frac{1}{5}})\Big) \quad (31)
\end{aligned}$$

Denote by g_ε the expression on the right hand side. It follows that $g_\varepsilon \leq c\varepsilon^{-N}$ for $\varepsilon \to 0$ and for some N, since

$$\|\phi_{h_\varepsilon}\|_{\frac{5}{2}} \leq c h_\varepsilon^{\frac{9}{5}} \sim ((\ln \varepsilon^{-1})^{\frac{5}{19}})^{\frac{9}{5}} \leq ((\ln \varepsilon^{-1})^{\frac{5}{9}})^{\frac{9}{5}} = \ln \varepsilon^{-1},\ \varepsilon \to 0.$$

Note that $\|D_x u_\varepsilon(t)\|_{\frac{10}{3}}$ and $\|D_x^2 u_\varepsilon(t)\|_{\frac{10}{3}}$ are bounded on $[0,T)$ (an assumption needed for Gronwall's inequality), since the Gagliardo–Nirenberg inequality implies

$$\|D_x^2 u_\varepsilon(t)\|_{\frac{10}{3}} \leq \Big(\sum_{|\alpha|=3} \|D_x^\alpha u_\varepsilon(t)\|_2\Big)^{\frac{13}{15}} \|u_\varepsilon(t)\|_2^{\frac{2}{15}} < \infty \text{ for each } t \in [0,T).$$

One can bound $\|D_x u_\varepsilon(t)\|_{\frac{10}{3}}$ similarly. Returning to (29) we see that

$$\begin{aligned}
\sup_{[0,T)} \|D_x^2 u_\varepsilon(t)\|_2 \leq & \|a_\varepsilon\|_{H^2} + g_\varepsilon f_\varepsilon^{\frac{3}{2}} \|a_\varepsilon\|_2^{\frac{1}{2}} + H(a_\varepsilon)^{\frac{1}{2}} g_\varepsilon^{\frac{20}{13}} \|a_\varepsilon\|_2^{\frac{6}{13}} \\
& + \|D_x^2 \phi_{h_\varepsilon}\|_\infty \|a_\varepsilon\|_2 + H(a_\varepsilon)^{\frac{1}{2}} \|D_x \phi_{h_\varepsilon}\|_\infty + g_\varepsilon \|\phi_{h_\varepsilon}\|_5, \\
\sup_{[0,T)} \|D_x^2 u_\varepsilon(t)\|_2 = & \mathcal{O}(\varepsilon^{-N}),\ \text{for some } N.
\end{aligned}$$

Moderateness of $\sup_{[0,T)} \|\partial_t u_\varepsilon(t)\|_2$ follows easily from (21), moreover

$$u_\varepsilon \in C([0,T), H^2(\mathbb{R}^n)) \cap C^1([0,T), L^2(\mathbb{R}^n)),\ \varepsilon \in (0,1).$$

Uniqueness. Let $u, v \in \mathcal{G}_{H^2}$ be two solutions with representatives $u_\varepsilon, v_\varepsilon$ and let $w_\varepsilon = u_\varepsilon - v_\varepsilon$. Also, let now $h_\varepsilon = (\ln(\ln \varepsilon^{-1})^{1/14})^{1/6} \sim (\ln(\ln \varepsilon^{-1}))^{1/6}$. Then w_ε solves

$$\begin{aligned}
i(w_\varepsilon)_t + \triangle w_\varepsilon = & u_\varepsilon |u_\varepsilon|^2 - (u_\varepsilon - w_\varepsilon)(|u_\varepsilon|^2 - u_\varepsilon \overline{w}_\varepsilon - w_\varepsilon \overline{u}_\varepsilon + |w_\varepsilon|^2) + \phi_{h_\varepsilon} w_\varepsilon + N_\varepsilon \\
w_\varepsilon(0,x) = & n_\varepsilon(x),
\end{aligned} \tag{32}$$

where $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{H^3}(\mathbb{R}^n)$, $(N_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$. Multiplying by $\overline{w}_\varepsilon$, integrating on $\mathbb{R}^3$ and taking the imaginary part we obtain again equation (10), since $\phi_{h_\varepsilon}|w_\varepsilon|^2$ is real. Furthermore, for arbitrary $M \in \mathbb{N}$

$$\begin{aligned}
\sup_{[0,T)} \|w_\varepsilon(t)\|_2^2 \leq & \|n_\varepsilon(x)\|_2^2 + \sup_{[0,T)} \big(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty\big) \int_0^T \|w_\varepsilon(t)\|_2^2 d\tau \\
& + \sup_{[0,T)} \|w_\varepsilon(t)\|_2 \|N_\varepsilon(t)\|_2, \\
\sup_{[0,T)} \|w_\varepsilon(t)\|_2^2 \leq & \varepsilon^M \exp\big(\sup_{[0,T)} \big(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty\big)\big).
\end{aligned} \tag{33}$$

The following estimates are needed for completing the proof of Theorem 2.1 above. We know that u_ε and v_ε satisfy

$$\begin{aligned}
i\partial_t u_\varepsilon + \triangle u_\varepsilon = & u_\varepsilon |u_\varepsilon|^2 + \phi_{h_\varepsilon} u_\varepsilon + N_\varepsilon, \\
u_\varepsilon(0,x) = & a_\varepsilon(x) + n_\varepsilon(x)
\end{aligned} \tag{34}$$

where $N_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^n)$ and $n_\varepsilon \in \mathcal{N}_{H^3}(\mathbb{R}^n)$.

Using the same procedure as in the existence proof we obtain

$$\sup_{[0,T)} \|D_x^2 u_\varepsilon(t)\|_2 \leq \|a_\varepsilon\|_{H^2} + g_\varepsilon f_\varepsilon^{\frac{3}{2}} \|a_\varepsilon\|_2^{\frac{1}{2}} + H(a_\varepsilon)^{\frac{1}{2}} g_\varepsilon^{\frac{20}{13}} \|a_\varepsilon\|_2^{\frac{6}{13}} + c(\|a_\varepsilon\|_{H^2})\|n_\varepsilon\|_{H^2}$$
$$+ \|D_x^2 \phi_{h_\varepsilon}\|_\infty \|a_\varepsilon\|_2 + H(a_\varepsilon)^{\frac{1}{2}} \|D_x \phi_{h_\varepsilon}\|_\infty + g_\varepsilon \|\phi_{h_\varepsilon}\|_5 + T \sup_{[0,T)} \|D_x^2 N_\varepsilon(t)\|_2, \tag{35}$$

We can now use the fact that $\|u_\varepsilon(t)\|_\infty^2 \leq \|u_\varepsilon(t)\|_{H^2}^2$. Recall that

$$\|a_\varepsilon\|_{H^2} \sim h_\varepsilon,\ H(a_\varepsilon) \sim h_\varepsilon^4,\ \|\phi_\varepsilon\|_{\frac{5}{2}} \sim h_\varepsilon^{\frac{9}{5}},\ \|\phi_\varepsilon\|_5 \sim h_\varepsilon^{\frac{12}{5}},$$
$$\|D_x \phi_\varepsilon\|_\infty \sim h_\varepsilon^4,\ \|D_x^2 \phi_\varepsilon\|_\infty \sim h_\varepsilon^5,\ \|D_x^2 \phi_\varepsilon\|_5 \sim h_\varepsilon^{\frac{22}{5}},\ \|D_x \phi_\varepsilon\|_5 \sim h_\varepsilon^{\frac{17}{5}}.$$

From (30) and (31) we derive

$$f_\varepsilon \sim (\ln \varepsilon^{-1})^{\frac{2}{14}},\ g_\varepsilon \sim (\ln \varepsilon^{-1})^{\frac{3}{14}}.$$

Using the fact that $\ln^s \ln^q \varepsilon^{-1} \leq \ln^q \varepsilon^{-1},\ \varepsilon \to 0$ for $s \leq 1$ we see that each factor in (35) can be estimated by $\sqrt{\ln \varepsilon^{-1}}$. Thus,

$$\|u_\varepsilon(t)\|_{H^2} \sim \sqrt{\ln \varepsilon^{-1}}. \tag{36}$$

Returning to (33), it follows that for any $M \in \mathbb{N}$ $\|w_\varepsilon(t)\|_2^2 \leq \varepsilon^M$.

A similar procedure leads to

$$\begin{aligned}
&\sup_{[0,T)} \|D_x w_\varepsilon(t)\|_2^2 \\
\leq &\|D_x n_\varepsilon(x)\|_2^2 + \sup_{(0,T)} \Big(\|u_\varepsilon(t)\|_\infty^2 + \|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty + \|w_\varepsilon(t)\|_\infty^2 \Big) \\
&\cdot \int_0^T \|D_x w_\varepsilon(t)\|_2^2 ds + T \sup_{(0,T)} \Big((\|u_\varepsilon + w_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_2 \|D_x u_\varepsilon D_x w_\varepsilon(t)\|_2 \\
&+ \|w_\varepsilon(t)\|_2 \|D_x \phi_{h_\varepsilon}\|_\infty \|D_x w_\varepsilon(t)\|_2 + \|D_x w_\varepsilon(t)\|_2 \|D_x N_\varepsilon\|_2 \Big)
\end{aligned}$$

and as in the homogeneous case (equation (13)), $\sup_{[0,T)} \|D_x u_\varepsilon(t) D_x w_\varepsilon(t)\|_2$ can be bounded by ε^{-N}. It follows that $\sup\limits_{[0,T)} \|D_x w_\varepsilon(t)\|_2^2 \leq \varepsilon^M$ for any $M \in \mathbb{N}$. Also,

$$\begin{aligned}
\|D_x^2 w_\varepsilon(t)\|_2^2 \leq &\|D_x^2 n_\varepsilon(x)\|_2^2 + T \sup_{(0,T)} \Big(\|D_x \phi_{h_\varepsilon}\|_\infty \|D_x w_\varepsilon(t)\|_2 \|D_x^2 w_\varepsilon(t)\|_2 \\
&+ \|D_x^2 \phi_{h_\varepsilon}\|_\infty \|w_\varepsilon(t)\|_2 \|D_x^2 w_\varepsilon(t)\|_2 \Big) + \sup_{[0,T)} \Big(\|u_\varepsilon(t)\|_\infty \|w_\varepsilon(t)\|_\infty + \|u_\varepsilon(t)\|_\infty^2 \\
&+ \|w_\varepsilon(t)\|_\infty^2 + \|D_x^2 u_\varepsilon(t)\|_2 \Big) \int_0^T \|D_x^2 w_\varepsilon(t)\|_2^2 ds + \varepsilon^M,
\end{aligned}$$

where ε^M was obtained in the same way as in the homogeneous case (equations (15) $-$ (18)). The remaining factors can also be bounded by ε^M. Applying Gronwall's lemma $\|D_x^2 w_\varepsilon(t)\|_2^2 \leq \varepsilon^{-N} \varepsilon^M$, for some N and any $M \in \mathbb{N}$.

Once again, $\|(w_\varepsilon)_t\|_2 = \mathcal{O}(\varepsilon^M),\ \varepsilon \to 0,\ \forall M \in \mathbb{N}$ follows directly from (32) and this completes the proof. □

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