DELTA SHOCK WAVE AND INTERACTIONS IN A SIMPLE MODEL CASE

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1. INTRODUCTION

It has been observed by various authors [3, 4, 5, 6, 8, 12] that the Riemann problem for certain equations from nonlinear elasticity and gasdynamics cannot be solved for all combinations of piecewise constant initial states with shock waves, rarefaction waves and contact discontinuities only. For that reason, the notion of a delta shock wave and a singular shock wave was introduced and employed by authors quoted above, and it was shown that a large class of Riemann problems can be solved globally with these additional building blocks. The aim of this paper is to study the interaction of one type of these new solutions, the delta shock waves, with the classical types of solutions.

We continue the investigation of the model equation

(1) {e1}
$$u_t + (u^2/2)_x = 0$$

(2) {e2}
$$v_t + ((u-1)v)_x = 0$$

initiated in [3]. This system is derived from a simplified model of magnetohydrodynamics. In [3], the authors found a solution for every Riemann problem with the initial data (u_0, v_0) on the left- and (u_1, v_1) on the right-hand side from zero in the following way.

The eigenvalues of the above system are $\lambda_1(u, v) = u - 1$, $\lambda_2(u, v) = u$, and the right-hand side eigenvectors are $r_1(u, v) = (0, 1)^T$, $r_2(u, v) = (1, v)^T$. The first characteristic field is linearly degenerate and the second is genuinely nonlinear. Thus, there are three types of solution.

(i) When $u_1 > u_0$ the solution is a contact discontinuity followed by a rarefaction wave,

$$u(x,t) = \begin{cases} u_0, & x \le u_0 t \\ \frac{x}{t}, & u_0 t < x < u_1 t \\ u_1, & x \ge u_1 t \end{cases}$$
$$v(x,t) = \begin{cases} v_0, & x \le (u_0 - 1)t \\ v_1 \exp(u_0 - u_1), & (u_0 - 1)t < x < u_0 t \\ v_1 \exp(\frac{x}{t} - u_1), & u_0 t \le x \le u_1 t \\ v_1, & x > u_1 t. \end{cases}$$

(ii) If $u_1 < u_0 < u_1 + 2$, the solution is given in the form of contact discontinuity followed by a shock wave,

$$u(x,t) = \begin{cases} u_0, & x \le ct \\ u_1, & x > ct \end{cases}$$
$$v(x,t) = \begin{cases} v_0, & x \le (u_0 - 1)t \\ v_*, & (u_0 - 1)t < x < ct \\ v_1, & x \ge ct, \end{cases}$$

where $v_* = v_1 \frac{2 - u_0 - u_1}{2 + u_1 - u_0}$. (iii) If $u_0 \ge u_1 + 2$ the solution is given in the form of delta shock wave,

$$u(x,t) = \begin{cases} u_0, & x \le ct \\ u_1, & x > ct \end{cases}$$
$$v(x,t) = \begin{cases} v_0, & x \le ct \\ v_1, & x > ct \end{cases} + \alpha_0(t)D^- + \alpha_1(t)D^+$$

where D^- and D^+ are the left- and right-hand side delta functions with the support on the line x = ct (see below), $c = (u_0 + u_1)/2$,

$$\alpha_0(t) = \frac{st(c - (u_1 - 1))}{u_0 - u_1}, \ \alpha_1(t) = \frac{st(c - (u_0 - 1))}{u_0 - u_1},$$

 $\alpha(t) := \alpha_0(t) + \alpha_1(t)$ is called the strength of the delta shock wave, and

$$s := c(v_1 - v_0) - ((u_1 - 1)v_1 - (u_0 - 1)v_0)$$

is called the Rankine-Hugoniot deficit (see [5]).

Our aim is to investigate various possible interactions of a solution in one of these forms with a delta shock wave. There are five possibilities for this to happen.

Case 1. delta shock wave interact with an another one

- Case 2. delta shock wave interact with a contact discontinuity followed by a shock wave from the left-hand side
- Case 3. delta shock wave interact with a contact discontinuity followed by a shock wave from the right-hand side
- Case 4. delta shock wave interact with a contact discontinuity followed by a rarefaction wave from the left-hand side
- Case 5. delta shock wave interact with a contact discontinuity followed by a rarefaction wave from the right-hand side

We shall always assume that the shock wave or the rarefaction wave starts from (0,0) and the delta shock wave from another point to left or right from zero. The initial data are determined by triplets (u_0, u_1, u_2) and (v_0, v_1, v_2) .

We shall now briefly describe what we mean by a solution in the form of a delta shock wave.

Suppose $\overline{R_{+}^{2}}$ is divided into finitely disjoint open sets $\Omega_{i} \neq \emptyset$, i = 1, ..., nwith piecewise smooth boundary curves Γ_i , i = 1, ..., m, that is $\Omega_i \cap \Omega_j = \emptyset$, $\bigcup_{i=1}^{n} \overline{\Omega}_{i} = \overline{R_{+}^{2}} \text{ where } \overline{\Omega}_{i} \text{ denotes the closure of } \Omega_{i}. \text{ Let } \mathcal{C}(\overline{\Omega}_{i}) \text{ be the space of bounded and continuous real-valued functions on } \overline{\Omega}_{i}, \text{ equipped with the } L^{\infty}-\text{ norm. Let } \mathcal{M}(\overline{\Omega}_{i}), \text{ be the space of measures on } \overline{\Omega}_{i}.$

We consider the spaces

$$\mathcal{C}_{\Gamma} = \prod_{i=1}^{n} \mathcal{C}(\overline{\Omega}_{i}), \quad \mathcal{M}_{\Gamma} = \prod_{i=1}^{n} \mathcal{M}(\overline{\Omega}_{i}).$$

The product of an element $G = (G_1, ..., G_n) \in C_{\Gamma}$ and $D = (D_1, ..., D_n) \in \mathcal{M}_{\Gamma}$ is defined as an element $D \cdot G = (D_1G_1, ..., D_nG_n) \in \mathcal{M}_{\Gamma}$, where each component is defined as the usual product of a continuous function and a measure.

Every measure on $\overline{\Omega}_i$ can be viewed as a measure on \mathbb{R}^2_+ with support in $\overline{\Omega}_i$. This way we obtain a mapping

$$m: \mathcal{M}_{\Gamma} \to \mathcal{M}(\mathbb{R}^2_+)$$
$$m(D) = D_1 + D_2 + \dots + D_n.$$

A typical example is obtained when $\overline{\mathbb{R}^2_+}$ is divided into two regions Ω_1 , Ω_2 by a piecewise smooth curve $x = \gamma(t)$. The delta function $\delta(x - \gamma(t)) \in \mathcal{M}(\overline{\mathbb{R}^2_+})$ along the line $x = \gamma(t)$ can be split in a non unique way into a left-hand side $D^- \in \mathcal{M}(\overline{\Omega}_1)$ and the right-hand component $D^+ \in \mathcal{M}(\overline{\Omega}_2)$ such that

$$\delta(x - \gamma(t)) = \alpha_0(t)D^- + \alpha_1(t)D^+$$
$$= m(\alpha_0(t)D^- + \alpha_1(t)D^+)$$

with $\alpha_0(t) + \alpha_1(t) = 1$. The solution concept which allows to incorporate such two sided delta functions as well as shock waves is modeled along the lines of the classical weak solution concept and proceeds as follows:

Step 1: Perform all nonlinear operations of functions in the space C_{Γ} .

Step 2: Perform multiplications with measures in the space \mathcal{M}_{Γ} .

Step 3: Map the space \mathcal{M}_{Γ} into $\mathcal{M}(\mathbb{R}^2_+)$ by means of the map m and embed it into the space of distributions.

Step 4: Perform the differentiation in the sense of distributions and require that the equation is satisfied in this sense.

Note that in the case of absence of a measure part (Step 2), this is the precisely the concept of a weak solution to equations in divergence form.

Following the reasoning in [5], delta shocks are required to satisfy the condition of overcompressibility, meaning that all characteristic curves run into the delta shock curve from both sides. It may happen that at a certain point on a delta shock curve, overcompressibility is lost. In this case we replace the delta shock by a new type of solution which we call a delta contact discontinuity. This new concept is introduced in Lemma 1 and Definition 1 below.

At interaction points our solutions are computed by continuation as continuous functions of time with values in the space of distributions (as solutions to a new initial value problem at the time of interaction).

The result of our investigation is that the interaction of a delta shock wave with any of three types of solutions to the Riemann problem, (i)–(iii) above, can be described by means of delta shocks and delta contact discontinuities. This is summarized in the following theorem.

{Theorem}

Theorem 1. The initial value problem for system (1, 2) with three constant states, one of which produces a delta shock, has a global weak solution consisting of a combination of rarefaction waves, shock waves, contact discontinuities, delta shock waves and delta contact discontinuities.

The remainder of the paper is devoted to proving this result by going throug all possible cases of interaction.

2. INTERACTIONS WITH SHOCK WAVES

Case 1. Here $u_0 \ge u_1 + 2$, $u_1 \ge u_2 + 2$. The speeds of the delta shock waves are $c_1 = (u_0 + u_1)/2$ and $c_1 = (u_1 + u_2)/2$. At the interaction point (x_0, t_0) , the new initial data are

$$u|_{t=t_0} = \begin{cases} u_0, & x < x_0 \\ u_2, & x > x_0 \end{cases}$$
$$v|_{t=t_0} = \begin{cases} v_0, & x < x_0 \\ v_2, & x > x_0 \end{cases} + \gamma \delta_{(x_0, t_0)}$$

where γ denotes a sum of the strengths of incoming delta shock waves.

Let u = G, $v = H + (\alpha_0(t)D^- + \alpha_1(t)D^+)$, where G and H are step functions

$$G = \begin{cases} u_0, & x - x_0 < (t - t_0)c \\ u_2, & x - x_0 > (t - t_0)c \end{cases} \quad H = \begin{cases} v_0, & x - x_0 < (t - t_0)c \\ v_2, & x - x_0 > (t - t_0)c, \end{cases}$$

and $D = \alpha_0(t)D^- + \alpha_1(t)D^+$ is a split delta function supported by the line $x = x_0 + (t - t_0)c$.

From (1) it follows

$$-c[G] + \frac{1}{2}[G^2] = 0,$$

i.e. $c = (u_0 + u_2)/2$. Since $c_1 > c > c_2$, the wave will be the overcompressive one, because of $u_0 - 1 > c > u_2$. Equation (2) gives

$$-c[H]\delta + [(G-1)H]\delta + (\alpha'_0(t) + \alpha'_1(t))\delta -c(\alpha_0(t) + \alpha_1(t))\delta' + ((u_0 - 1)\alpha_0(t) + (u_2 - 1)\alpha_1(t))\delta' = 0$$

This equation gives (with $\alpha(t) = \alpha_0(t) + \alpha_1(t)$) the following ODE

$$\alpha'(t) = c(v_2 - v_0) - ((u_2 - 1)v_2 - (u_0 - 1)v_0) =: s \in \mathbb{R}, \ \alpha(t_0) = \gamma.$$

The unique solution is given by $\alpha(t) = s(t - t_0) + \gamma$. Substitution of α in the equation gives

$$\alpha_0(t) + \alpha_1(t) = \alpha(t) = s(t - t_0) + \gamma$$

(u_0 - 1 - cs(t - t_0) - \gamma)\alpha_0(t) + (u_2 - 1 - cs(t - t_0) - \gamma)\alpha_1(t) = 0

which has a unique solution $\alpha_0(t)$, $\alpha_1(t)$, since $u_0 \neq u_2$.

Thus, the result of the first type of interaction is a single delta shock wave.

Case 2. Suppose that the delta shock wave is given by

$$\begin{split} u(x,t) &= G(x+a^2-c_1t),\\ v(x,t) &= H(x+a^2-c_1t) + \beta_0(t)\delta^-(x+a^2-c_1t) + \beta_1(t)\delta^+(x+a^2-c_1t),\\ G &= \begin{cases} u_0, & x+a^2 < c_1t \\ u_1, & x+a^2 > c_1t \end{cases}, \quad H = \begin{cases} v_0, & x+a^2 < c_1t \\ v_1, & x+a^2 > c_1t, \end{cases} \end{split}$$

and that a contact discontinuity coupled with a shock wave is given by

$$u = \begin{cases} u_1, & x < c_2 t \\ u_1, & x > c_2 t \end{cases}, \quad v = \begin{cases} v_1, & x < (u_1 - 1)t \\ v_*, & (u_1 - 1)t \le x < c_2 t \\ v_2, & x \ge c_2 t, \end{cases}$$

where $c_2 = (u_1 + u_2)/2 < c_1$ (since $u_1 > u_2$, $u_0 \ge u_1 + 2$), and $v_* = v_2(2 + u_1 - u_2)/(2 + u_2 - u_1)$.

Let us denote by (t_0, x_0) the point where delta shock wave meets the contact discontinuity, i.e. this point is the intersection of the lines $x + a^2 = c_1 t$ and $x = (u_1 - 1)t$.

In the area bounded by the lines $x = (u_1 - 1)t$ and $x = u_1t$, the value of u is the constant u_1 . This implies that the delta shock wave runs through it with the same speed $c_1 = (u_0 + u_1)/2$ as before. Only the values of $\beta_0(t)$ and $\beta_1(t)$ are changed into, say, $\tilde{\beta}_0(t)$ and $\tilde{\beta}_1(t)$ due to the existing difference in v_0 and v_* . The new strength of the delta shock wave is now $s_1(t - t_0) + \gamma_0$, where

$$s_1 := c_1(v_* - v_0) - (u_0 - 1)(v_* - v_0),$$

and γ_0 is the strength of the previous delta shock wave in the point (t_0, x_0) . Obviously, the new delta shock wave is an overcompressive wave, since $u_0 - 1 \ge c_1 \ge u_1$.

Let us denote by (t_1, x_1) the point where the new delta shock meets the existing shock wave i.e. the point (t_1, x_1) is the intersection of the lines $x + c_1^2 = c_1 t$ and $x = c_2 t$. Let γ_1 be the strength of the delta shock wave at this point. Therefore, we obtain the new initial data

$$u = \begin{cases} u_0, & x < x_1 \\ u_2, & x > x_1 \end{cases} \quad v = \begin{cases} v_0, & x < x_1 \\ v_2, & x > x_1 \end{cases} + \gamma_1 \delta_{(t_1, x_1)}$$

A solution for the new initial data problem will be a delta shock wave with the speed $c = (u_0 + u_2)/2 < c_1$, and c is obtained directly from (1) in the usual way. Again this speed ensures that the obtained wave is an overcompressive one, since $u_0 - 1 \ge c \ge u_2$.

Substituting u and v into (2) gives the strength

$$\tilde{\alpha}_0(t) + \tilde{\alpha}_1(t) = s_1(t - t_1) + \gamma_1.$$

Using this equation and

$$(u_0 - 1 - s(t - t_1) - \gamma_1)\tilde{\alpha}_0(t) + (u_2 - 1 - s(t - t_1) - \gamma_1)\tilde{\alpha}_1(t) = 0,$$

where $s := c(v_2 - v_0) - ((u_2 - 1)v_2 - (u_0 - 1)v_0)$, one can find unique $\tilde{\alpha}_0(t)$ and $\tilde{\alpha}_1(t)$, and this proves the above statement.

Case 3. Now, $u_0 > u_1 > u_2 + 2$, and the speed of the shock wave $c_1 = (u_0 + u_1)/2$ is greater than the speed of the delta shock wave $c_2 = (u_1 + u_2)/2$. Let (t_0, x_0) be the interaction point of these two waves, and let γ_0 be the strength of the delta shock wave at this point. Initial data are now

$$u|_{t=0} = \begin{cases} u_0, & x < x_0 \\ u_2, & x > x_0 \end{cases} \quad v|_{t=0} = \begin{cases} v_*, & x < x_0 \\ v_2, & x > x_0 \end{cases} + \gamma_0 \delta_{(t_0, x_0)}$$

where the value of v_* is defined as before.

Similarly to the previous case, the result of the interaction is a single overcompressive delta shock wave with the speed $c = (u_0 + u_2)/2$ (since $u_0 - 1 \ge c \ge u_2$). As before, c is obtained from (1) and from (2) one can find $\tilde{\alpha}_0(t)$ and $\tilde{\alpha}_1(t)$ in the same way as above.

All the way through the contact discontinuity, the delta shock wave has the same speed, only $\tilde{\alpha}_0(t)$ and $\tilde{\alpha}_1(t)$ are changing.

3. INTERACTIONS WITH RAREFACTION WAVES

One can easily see that interaction of a rarefaction and delta shock wave are much more complicated. Now we shall deal with this problem.

As one could see before, the new initial data include a delta function as a part. If the right-hand side of u is greater or equal to the left-hand one plus 2, the new initial value problem can be solved in a simple way as above and the result is a single overcompressive delta shock wave. But, when this is not a case, the types of admissible solution known so far are not enough to obtain a solution. The definition of a new type of admissible solution, called delta contact discontinuity, is given below. Its existence is justified by two facts. First, a contact discontinuity emerges in the case when one of the characteristic fields is linearly degenerate. Second, if a linear equation has a delta function as initial data, it propagates along the characteristic lines. These two facts inspired the following lemma and the definition of this new type of elementary waves.

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Lemma 1. Let the initial data for system (1-2) given by

$$u|_{t=0} = \begin{cases} u_0, & x < 0\\ u_1, & x > 0 \end{cases}, \quad v|_{t=0} = \begin{cases} v_0, & x < 0\\ v_1, & x > 0 \end{cases} + \gamma \delta_{(0,0)},$$

where $u_0 > u_1$, but $u_0 < u_1 + 2$. Then, the function

$$u = \begin{cases} u_0, & x < ct \\ u_1, & x > ct \end{cases} \quad v = \begin{cases} v_0, & x < (u_0 - 1)t \\ v_*, & (u_0 - 1)t < x < ct \\ v_1, x > ct \end{cases} + \gamma \delta_{x = (u_0 - 1)t},$$

where $c = (u_0 + u_1)/2$ weakly solves the Riemann problem for (1,2).

Proof. For every $\varphi \in C_0^{\infty}$, supp $\varphi \cap \{(x,t): x = (u_0 - 1)t, t > 0\} = \emptyset$, it holds that

$$\langle u_t, \varphi \rangle + \frac{1}{2} \langle (u^2)_x, \varphi \rangle = 0 \langle v_t, \varphi \rangle + \langle ((u-1)v)_x, \varphi \rangle = 0.$$

Our aim is to show that this still holds true when it is allowed that $\operatorname{supp} \varphi$ intersects the supports of D^- and D^+ , i.e. the line $x = (u_0 - 1)t$. Let us note that the condition $u_0 < u_1 + 2$ means that $(u_0 + u_1)/2 > u_0 - 1$ so the line $x = (u_0 - 1)t$ is on the left-hand side of the shock line $x = (u_0 + u_1)t/2$.

Equation (1) does not contain v, so it is still satisfied. From (2) we have that

$$v_t + ((u_0 - 1)v)_x = -(u_0 - 1)(v_* - v_0)\delta - \gamma\delta' + (u_0 - 1)(v_* - v_0)\delta + (u_0 - 1)\gamma\delta' = 0$$

near the line $x = (u_0 - 1)t$.

Usefulness of this lemma will be clear after the interaction of a delta shock and rarefaction wave is treated. Then one could roughly see how a solution looks like, since the rare faction wave could be approximated with a large number of small amplitude non-physical shock waves (see [1], for example).

Another possible use could be in a sort of a wave front tracking algorithm, where systems in question posses a solution containing a delta function.

Definition 1. Consider a region R where u is continuous function and a curve Γ in R of slope $\lambda_1(u, v)$. A distribution $(u, v) \in \mathcal{C}(R) \times \mathcal{D}'(R)$ is a delta contact discontinuity, if v is a sum of a locally integrable function on R and a delta function on Γ which weakly solves (1,2) on R.

Let us note that the overcompressiveness condition obviously need not hold in this case. But, as we already have mentioned before the lemma, a linearly degenerate field resembles a linear equation where this type of a solutions exists.

Now, we are returning to the last two cases which covers the rest of possible delta shock wave interactions.

Case 4. Suppose that a delta shock wave starts from the point $-a^2$, a > 0, with the speed $c_1 = (u_0 + u_1)/2$ and meets a contact discontinuity followed by the rarefaction wave centered at zero. Denote by $(\tilde{t}_0, \tilde{x}_0)$ the meeting point, i.e. it is the intersection of the lines $x + a^2 = c_1 t$ and $x = (u_1 - 1)t$. As we have already seen, the delta shock wave goes through the contact discontinuity without speed change (but its strength is changed) and meets the rarefaction wave at some point (x_0, t_0) ,

$$t_0 = \frac{2a^2}{u_0 - u_1}, \ x_0 = \frac{2u_1a}{u_0 - u_1}.$$

Let γ_0 be the strength of the delta shock wave at this point. In order to see what could happen, let us approximate the rarefaction wave with a set of nonphysical shock waves, supported by the lines $x = (u_1 + n\eta)t$, $\eta \ll 1$, $n \in \mathbb{N}$. (see Fig. 1.)



At least in the beginning, until $(u_1 + \eta n) + 2 \leq u_0$, the result of successive interactions of the delta shock wave with the non-physical shock waves are delta shock waves with increasing speeds, with values (u_0, v_0) on the left-hand side and the values on the right-hand side are the values of the rarefaction wave. This guide us to look for a curve $\Gamma_0 := (c(t), t)$, such that a delta function lives on it, $c(t_0) = x_0$. The value of u on the left-hand side of Γ is u_0 , and c(t)/t = x/t on the right-hand side. Inserting the above data for such a curve into (1), one gets the following ordinary differential equation

(3)
$$-c'(t)\left(\frac{c(t)}{t}-u_0\right) + \frac{1}{2}\left(\left(\frac{c(t)}{t}\right)^2 - u_0^2\right) = 0, \ c(t_0) = x_0,$$

which has the unique solution

(4)
$$c(t) = u_0 t - a\sqrt{2(u_0 - u_1)t}, \ t \ge t_0$$

Denote by v(t) the value of $v|_{\Gamma_0}$ in the rarefaction wave, $v(t) = v_2 \exp(c(t)/t - u_2)$. Substituting expected delta shock wave given by u = G, $v = H + \alpha_0(t)D_{\Gamma_0}^- + \alpha_1(t)D_{\Gamma_0}^+$, where G and H are the step functions with discontinuity line Γ_0 , gives

$$-c'(t)(v(t) - v_*)\delta + c(t)(\alpha_0(t) + \alpha_1(t))'\delta -c'(t)(\alpha_0(t) + \alpha_1(t))\delta' + \left(\left(\frac{c(t)}{t}\right)v(t) - (u_0 - 1)v_*\right)\delta + \left((u_0 - 1)\alpha_0(t) + \left(\frac{c(t)}{t} - 1\right)\alpha_1(t)\right)\delta' = 0.$$

Since the following ordinary differential equation

$$\alpha'(t) = \frac{c'(t)(v(t) - v_*) - (u_0 - 1)v_* + \left(\frac{c(t)}{t} - 1\right)v(t)}{c(t)}, \ \alpha(t_1) = \gamma_1$$

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has a unique solution (obtained in a simple manner by an integration), the strength of the delta shock wave, $\alpha(t)$, is determined.

Equating the coefficient of δ' with zero, we can compute the two summands α_0 and α_1 of α . Since

$$c'(t) = u_0 - \frac{a\sqrt{2(u_0 - u_1)}}{2\sqrt{t}} > \frac{c(t)}{t} = u_0 - \frac{a\sqrt{2(u_0 - u_1)}}{\sqrt{t}},$$

the obtained delta shock wave satisfies rhe right-hand overcompressibility condition. Overcompressibility condition for the left-hand side is

(5) {equ4}
$$u_0 - 1 \ge c'(t).$$

Now, we have the following two cases.

(i) If $u_2 \leq u_0 - 2$, relation (5) is satisfied trough all the rarefaction wave and the resulting solution is a single delta shock wave with the speed $c = (u_0 + u_2)/2$ starting from the point (\tilde{x}, \tilde{t}) which is the intersection of the curve Γ_0 and the line $x = u_2 t$.



After the time \tilde{t} , the solution in this case is given by

$$u|_{t>\tilde{t}} = \begin{cases} u_0, & x < \tilde{x}t \\ u_2, & x > \tilde{x}t \end{cases} \quad v|_{t>\tilde{t}} = \begin{cases} v_*, & x < \tilde{x}t \\ v_2, & x > \tilde{x}t \end{cases} + \gamma_{(\tilde{x}.\tilde{t})}\delta_{(\tilde{x},\tilde{t})}$$

(ii) Suppose that $u_2 > u_0 - 2$. Then the delta shock wave supported by Γ_0 is an overcompressive wave only until some point (x_s, t_s) lying inside the rarefaction wave. (See Fig. 3.)



So, the admissible solution cannot be prolonged along the same curve Γ_0 . Assuming that the rarefaction wave is approximated by a set of small non-physical shock waves, the present problem is described in Lemma 1: the right-hand side equals $u_0 - 2 + \eta$, $0 < \eta \ll 1$, while the left-hand one equals u_0 .

In this lemma, the problem is solved by using the new type of a solution – delta contact discontinuity. This is exactly what we shall try. That is, suppose that the solution consists of the delta function supported by a line Γ_1 : $(x - x_s) = (u_0 - 1)(t - t_s)$ going through an area where u has a constant value u_0 , and a shock wave supported by a curve Γ_2 : $x = c_2(t)$, where $c_2(t_s) = x_s$, with the left-hand side values u_0 of the function u and the right-hand side ones $c_2(t)/t$ (a part of the rarefaction wave). All that means that $c_2(t)$ should satisfy the same equation (3) as c(t) with the initial data $c_2(t_s) = x_s = c(t_s)$, i.e. the new shock wave is supported by the continuation of the curve Γ_0 .



Fig. 4.

Since $u_0 > c'_1(t)$ and $u_0 - 1 < c'_1(t)$ while $c'_1(t) > c(t)/t$, the obtained shock wave, supported by the curve Γ_2 is admissible.

The Rankine-Hugoniot conditions for Γ_2 after the time $t = t_s$ imply

(6) {e5}-
$$c'_2(t)(v(t) - w_*(t)) + \left(\frac{c_2(t)}{t} - 1\right)v(t) - (u_0 - 1)w_*(t) = 0,$$

where w_* denotes the left-hand side value of v along the curve Γ_2 . Equation (6) simply determines

$$w_*(t) = \frac{\sqrt{t} + B}{\sqrt{t} - B}v(t)$$
$$= \frac{\sqrt{t} + B}{\sqrt{t} - B}v_2 \exp(u_0 t - 2B\sqrt{t} - u_1),$$

where $B := a\sqrt{2(u_0 - u_1)}/2 = \sqrt{t_s}$.

The value of v between Γ_1 and Γ_2 , denoted by w(x,t) has to satisfy the equation

(7) {e6}
$$w_t + (u_0 - 1)w_x = 0, \ w_{\Gamma_1} = w_*(t).$$

The solution to (7) is of the form w(x,t) = V(y), $y = x - (u_0 - 1)t$. More precisely, using the initial data one gets (8) {e7}

(b) (e)

$$V(y) = v_2 \left(1 + \frac{2B}{\sqrt{B^2 + y}} \right) \exp((u_0 B + u_0 \sqrt{B^2 + y} - 2B)(B + \sqrt{B^2 + y}) - u_1).$$

The curve Γ_1 is given by

(9) {e8}
$$x - (u_0 - 1)t = x_s - (u_0 - 1)t_s = (u_s - u_0 - 1)t_s = -t_s.$$

Substitution of (9) into (8) yields $V|_{\Gamma_2} = \infty$ and $V(y) \in \mathbb{R}$ for (t, x) lying between Γ_1 and Γ_2 , since $B^2 = t_s$. But $w \in L^1_{\text{loc}} \subset \mathcal{D}'$.

In order to verify that it is a solution we note that

$$v(t,x) = \left\{ \begin{array}{l} v_*, \ x < (u_0 - 1)t \\ w(x,t), \ x > (u_0 - 1)t, \ x < c_2(t) \end{array} \right\} + \gamma_s \delta_{\Gamma_2},$$

when $\gamma_s \delta_{\Gamma_2}$ is the delta function with the strength γ_s obtaining from the initial data at (t_s, x_s) . Since v(x, t) is constant along the lines parallel to $x = (u_0 - 1)t$ in a region where $u \equiv u_0$ it is clear that it is a solution of (2).

In order to see what is going on after the interaction of the delta shock and rarefaction wave, one has to consider three different possibilities.

(a) $u_0 \leq u_2$.

Then the delta contact discontinuity and shock wave supported by Γ_1 lies inside the rarefaction wave since $\Gamma_1 \cap \{(t,x) : x = u_2t\} = \emptyset$ and $\Gamma_2 \cap \{(t,x) : x = u_2t\} = \emptyset$ (actually, $c_2(t)$ has the line $x = u_0t$ as an asymptote, as $t \to \infty$ (see 4)).

(b) $u_0 > u_2 \ge u_0 - 1$.

Then the delta contact discontinuity stays inside the rarefaction wave and the shock wave supported by Γ_2 intersects the line $x = u_2 t$ at some point (\tilde{t}, \tilde{x}) .

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Now, at the point (\tilde{t}, \tilde{x}) we have the new Cauchy problem for (1,2) with the initial data $(u_0, w(x, t))$, (u_2, v_2) . Since $u_0 > u_2$, but $u_0 < u_2 + 2$, the solution is given by

$$u = \begin{cases} u_0, & x - \tilde{x} < (u_0 + u_2)(t - \tilde{t})/2 \\ u_2, & x - \tilde{x} > (u_0 + u_2)(t - \tilde{t})/2 \end{cases}$$
$$v = \begin{cases} w(t, x), & x - \tilde{x} < (u_0 - 1)(t - \tilde{t}) \\ \tilde{v}_*, & (u_0 - 1)(t - \tilde{t}) < x - \tilde{x} < (u_0 + u_2)(t - \tilde{t})/2 \\ v_2, & x - \tilde{x} > (u_0 + u_2)(t - \tilde{t})/2, \end{cases}$$

where $\tilde{v}_* = v_2(2 + u_0 - u_2)/(2 + u_2 - u_0)$. Let us remark that the function w equals a constant value along lines with the slope $(u_0 - 1)$. Denote by Γ_4 the shock line $x - \tilde{x} = (u_0 + u_2)(t - \tilde{t})/2$. This line is a tangent to the curve Γ_2 at (\tilde{x}, \tilde{t}) .

Let Γ_3 be the line of slope $u_0 - 1$ starting at (\tilde{t}, \tilde{x}) . Since \tilde{t} is a solution to

$$u_0 t - 2B\sqrt{t} = u_2 t,$$

we have

$$\tilde{t} = \frac{4B^2}{(u_0 - u_2)^2}, \quad \tilde{x} = \frac{4u_2B^2}{(u_0 - u_2)^2}$$

and

$$w|_{\Gamma_3} = \frac{\sqrt{\tilde{t}} + B}{\sqrt{\tilde{t}} - B} v(\tilde{t}) = \frac{2 + u_0 - u_2}{2 + u_2 - u_0} v_2 = \tilde{v}_*,$$

after the substitution of \tilde{t} and the ending value of the rarefaction wave, $v(\tilde{t}) = v_2$. So, the function w(t, x) is continuously prolonged by \tilde{v}_* into the area between the lines $x - \tilde{x} = (u_0 - 1)(t - \tilde{t})$ (the contact discontinuity line) and $x - \tilde{x} = (u_0 - u_2)(t - \tilde{t})/2$ (the shock curve).

The slope of Γ_3 is the same as the one of Γ_1 . That is, there are no interactions, and this case is finished.

(c) $u_2 < u_0 - 1$. In this case both of Γ_1 and Γ_2 intersects the line $x = c_2 t$. But as Γ_2 reaches this line at the time $\tilde{t} = 4B^2/(u_0 - u_2)^2$, before the time when Γ_1 would intersect it, analysis is the same as in the case (b) (see Fig. 5) below.



Case 5. Suppose that a delta shock wave starts from the point $(0, a^2)$, a > 0 and meets a coupled pair of contact discontinuity and rarefaction wave at some point (x_0, t_0) . This is possible if $u_0 < u_1$, $u_1 > u_2 + 2$. Suppose that the rarefaction wave is centered (starts from (0, 0)).

The point (x_0, t_0) can be easily found by solving the equations

$$x - a^2 = \frac{u_1 + u_2}{2}t, \quad x = u_1t, \text{ i.e.}$$

 $t_0 = \frac{2a^2}{u_1 - u_2}, \quad x_0 = \frac{2a^2u_1}{u_1 - u_2}.$

In the beginning of the interaction of the rarefaction and the delta shock wave the situation is quite similar to the one in the previous case. The solution is given by a delta shock wave supported by $\Gamma_0 = \{(t, c(t)) : t > t_0\}$, where c(t) is a solution to

(10) {e9}
$$-c'(t)\left(\frac{c(t)}{t}-u_2\right)+\frac{1}{2}\left(\left(\frac{c(t)}{t}\right)^2-u_2^2\right)=0, \ c(t_0)=x_0,$$

i.e.

$$c(t) = u_2 t + a \sqrt{2(u_1 - u_2)t}, \ t > t_0.$$

Equation (10) is in fact Rankine-Hugoniot condition for (1).

The left- and right-hand side coefficients of the new delta shock wave, $\alpha_0(t)$ and $\alpha_1(t)$, can be found in the same way as in the previous case. If (u(t), v(t))is the value of the rarefaction wave, then on the left-hand side of Γ_0 the new delta shock wave takes value $(u(t), v(t))|_{\Gamma_0} = (c(t)/t, v_1 \exp(c(t)/t - u_1))$ and on the right-hand side it equals (u_2, v_2) . Only the overcompressibility condition is still in question. The first condition for overcompressibility on the right-hand side is always satisfied, since $c'(t) = u_2 + a\sqrt{2(u_1 - u_2)}/\sqrt{4t} > u_2$. For the overcompressibility it is necessary that also characteristic lines run into the shock from the left-hand side.

$$c'(t) = u_2 + \frac{a\sqrt{2(u_1 - u_2)}}{2\sqrt{t}} \le u(c(t), t) - 1 = u_2 + \frac{a\sqrt{2(u_1 - u_2)}}{\sqrt{t}} - 1,$$

i.e.

$$\frac{a\sqrt{2(u_1-u_2)}}{2\sqrt{t}} \ge 1$$

This is true until the time $t = t_s$, where

$$t_s = \frac{a^2(u_1 - u_2)}{2}, \ x_s = c(t_s) = a^2(u_1 - u_2)\left(\frac{u_2}{2} + 1\right).$$

Thus, $u_s = x_s/t_s = u_2 + 2$.

The first case: $u_2 + 2 > u_0$.

Then the termination of overcompressibility takes place within the rarefaction fan and again we are in a position to use the intuition behind Lemma 1, i.e. to look for a solution consisting of a delta contact discontinuity supported by a curve Γ_1 and a shock wave supported by some other curve Γ_2 . Γ_1 should be below Γ_2 .

 Γ_1 is the characteristic line of the equation

$$v_t + (u-1)v_x = 0$$

passing trough (x_s, t_s) . Using the fact that $u = c_1(t)/t$ on $\Gamma_1 = \{(t, c_1(t)), t > t_s\}$, one can find such a function c_1 by solving the initial value problem

$$c_1'(t) = \frac{c_1(t)}{t} - 1, \ c_1(t_s) = x_s.$$

The unique solution to the above problem can be easily found

$$c_1(t) = t\left(-\log t + \log\left(\frac{a^2(u_1 - u_2)}{2}\right) + u_2 + 2\right).$$

Using (1) and the Rankine-Hugoniot condition, the curve $\Gamma_2 = \{(c_2(t), t) | t > t_s\}$ is uniquely determined by a solution to

$$-c_2'(t)\left(u_2 - \frac{c_2(t)}{t}\right) + \frac{1}{2}\left(u_2^2 - \left(\frac{c_2(t)}{t}\right)^2\right) = 0, \ c_2(t_s) = x_s,$$

i.e.

$$c_2(t) = u_2 t + a\sqrt{2(u_1 - u_2)t}.$$

One can see that it equals to the function c(t) from the previous case.

One has to prove that Γ_1 is actually strictly below the curve Γ_2 .

Since $c_1(t_s) = c_2(t_s)$ and $c'_1(t_s) = c'_2(t_s) = u_s - 1$, it is enough to compare $c'_1(t)$ and $c'_2(t)$, for $t > t_s$.

$$c_1'(t) = u_2 + 1 - \log t + \log \left(\frac{a^2(u_1 - u_2)}{2}\right)$$
$$= u_2 + 1 + \log \left(\frac{a^2(u_1 - u_2)}{2t}\right)$$
$$c_2'(t) = u_2 + \frac{a\sqrt{u_1 - u_2}}{\sqrt{2t}}.$$

 $c_{2}'(t) > c_{1}'(t)$ if

(11)
$$\frac{a\sqrt{u_1 - u_2}}{\sqrt{2t}} - 1 > \log\left(\frac{a^2(u_1 - u_2)}{2t}\right), \ t > t_s.$$

But the last relation is true; one can check it by changing the variables, and noticing that $\sqrt{y} - 1 > \log(y)$, for $y \in (0, 1)$.

Denote by A the region between Γ_1 and Γ_2 for $t > t_s$.

The value of u is u(x,t) = x/t inside A. Therefore, (1) is satisfied. Now, Γ_1 is the support of the delta contact discontinuity, and we are trying to find the value of v in this area.

First, let $v_*(t)$ denote the value of v on the left-hand side of Γ_2 . The value of u there is given by

$$u|_{\Gamma_2} = \frac{x}{t}|_{\Gamma_2} = \frac{c_2(t)}{t} = u_2 + \frac{a\sqrt{2(u_1 - u_2)}}{\sqrt{t}}.$$

The values of u and v on the right-hand side of Γ_2 are u_2 and v_2 , respectively. The Rankine-Hugoniot condition for (2) gives

$$-c_2'(t)(v_2 - v_*(t)) + \left((u_2 - 1)v_2 - \left(\frac{c_2(t)}{t} - 1\right)v_*(t)\right) = 0.$$

Solving the above equation, one gets

$$v_*(t) = \frac{\sqrt{t+B}}{\sqrt{t-B}}v_2, \ B = \frac{a\sqrt{2(u_1-u_2)}}{2} = \sqrt{t_s}.$$

Denote by w(x,t) the value of v inside D. Then w is the solution to the linear partial differential equation

$$w_t + \left(\frac{x}{t} - 1\right)w_x = 0, \ w|_{\Gamma_2} = v_*(t).$$

The solution of the above equation is a constant along the characteristic curves

$$\gamma := \frac{dx}{dt} = \frac{x}{t} - 1$$
, where $\gamma|_{\Gamma_2}$ is known.

In particular, w tends to infinity near Γ_1 , but in locally integrable fashion because $v_*(t) = \mathcal{O}(1/(\sqrt{t} - \sqrt{t_s}))$ as $t \to t_s$.

Now, we shall look for an exit of the delta contact discontinuity and the shock wave supported by Γ_2 trough the rarefaction wave.

The line $x = u_0 t$ and the curve $x = c_1(t)$ always has an interaction point, say (\tilde{t}, \tilde{x}) , for $u_0 < u_2 + 2$. The equation

$$u_0 t = (u_2 + 2)t + t \log(B^2/t) = \log(t_s/t).$$

This equation has a unique solution $\tilde{t} > t_s$.

The next question is whether the curve $x = c_2(t)$ intersects the line $x = u_0 t$ or not. An intersection takes place, if the equation

$$u_0 t = u_2 t + 2B\sqrt{t}$$
, i.e. $u_0 - u_2 = 2B/\sqrt{t}$

has a solution $t > t_s$. If $u_0 < u_2$, there is no solution. If $u_0 > u_2$, then the solution $t = 4B^2/(u_0 - u_2)^2 = 4t_s/(u_0 - u_2)^2$ is bigger than t_s , because $2 > u_0 - u_2$.

In both cases we have to solve initial data problem for (1,2), given by

$$u = \begin{cases} u_0, & x < \tilde{x} \\ u_0, & x > \tilde{x}, \end{cases} \quad v = \begin{cases} v_*, & x < \tilde{x} \\ w(x, x/u_0), & x > \tilde{x} \end{cases} + \gamma_s \delta_{(\tilde{t}, \tilde{x})},$$

where w is the right-hand side of v in the region A, constant along the characteristics of

$$w_t + \left(\frac{x}{t} - 1\right)w_x = 0, \quad \gamma : \frac{dx}{dt} = \frac{x}{t} - 1.$$

long the line $x = u_0 t$ the slope of these characteristic curves is $u_0 - 1$. Thus we may continue the solution to the left of $x = u_0 t$ as a delta contact discontinuity

$$u(t,x) = u_0, \quad v(t,x) = \left\{ \begin{array}{cc} v_*, \ x - \tilde{x} < (u_0 - 1)(t - \tilde{t}) \\ w(t,x), \ x - \tilde{x} > (u_0 - 1)(t - \tilde{t}) \end{array} \right\} + \gamma_s \delta_{x - \tilde{x} = (u_0 - 1)(t - \tilde{t})}$$

where w(t, x) is again constant along the lines with slope $u_0 - 1$. Denote by Γ_3 the line $x - \tilde{x} = (u_0 - 1)(t - \tilde{t})$. There is no further intersection with the original contact discontinuity along the (parallel) line $x = (u_0 - 1)t$. In the case $u_0 < u_2$, the solution is complete (See Fig. 6).



In the case $u_0 > u_2$ we still have to consider the region above the intersection point $(\tilde{\tilde{x}}, \tilde{\tilde{t}})$ of $x = c_2(t)$ with $x = u_0 t$. In this case

$$u_2 < u_0 < u_2 + 2$$

and we can connect a constant left-hand state u_0 to the constant right-hand state u_2 by by a shock wave in u.

This shock wave supported by Γ_5 has speed $(u_0+u_2)/2$ and actually is tangent to the line $x = c_2(t)$ at the intersection point $(\tilde{\tilde{x}}, \tilde{\tilde{t}})$. It follow a classical contact discontinuity starting from the point $(\tilde{\tilde{x}}, \tilde{\tilde{t}})$ with speed $u_0 - 1$, supported by the line Γ_3 and this connects the region when $v = w(\xi, t)$ has been determined by the initial data along the line $x = u_0 t$. The value of v between Γ_4 and Γ_5 is $\tilde{v}_* = (2 + u_0 - u_2)/(2 + u_2 - u_0)$. (See Fig. 7).



The second case is $u_0 > u_s = u_2 + 2$. Then there is no bifurcation of the delta contact discontinuity supported by Γ_0 . After Γ_0 intersects the line $x = u_0 t$ at (t_1, x_1) , say, the solution can be continued into the region $(u_0 - 1)t < x < u_0 t$ by a simple delta contact discontinuity on the line $\Gamma_3 : x - x_1 = (u_0 - 1)(t - t_1)$, where u has the constant value u_0 and v has the value v_* and v_2 on the left and right-hand side, respectively, and a constant strength delta function is placed on the line Γ_3 . This concludes investigation of all possible cases.

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