A CLASS OF UNBOUNDED SOLUTIONS TO CONSERVATION LAW SYSTEMS

MARKO NEDELJKOV
UNIVERSITY OF NOVI SAD

Part 1. Shadow wave type solutions

Since we were using the designation u for a fluid velocity, we will use U for arbitrary n-dimensional dependent variable in the sequel. Consider a following conservation law system

$$\partial_t f(U) + \partial_x g(U) = 0,\ U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n, \tag{0.1}$$

where $f = (f^1, \dots, f^n)$ and $g = (g^1, \dots, g^n)$ are continuous mapping from Ω in $\mathbb{R}^n$. A name of f is *density function*, while g is called *flux function*. The functions f and g are continuous mappings from a physical domain $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

1. Shadow waves

1.1. **Constant Shadow Waves.** The following notation will be used trough the paper. A parameter ε belongs to some interval $(0, \varepsilon_0)$ with ε_0 being as small as needed. Let a_ε be a net of reals and u_ε be a net of locally integrable functions over some domain $\omega \subset \mathbb{R}^m$. We say that

$$a_\varepsilon \sim \varepsilon \text{ if there exists } A \in (0,\infty) \text{ such that } \lim_{\varepsilon\to 0} \frac{a_\varepsilon}{\varepsilon} = A,$$

and

$$u_\varepsilon \approx g \in \mathcal{D}'(\omega) \text{ if } \int_\omega u_\varepsilon \phi \to \langle g, \phi\rangle \text{ as } \varepsilon \to 0 \text{ for every test function } \phi \in \mathcal{C}_0^\infty(\omega).$$

The relation $u_\varepsilon \approx v_\varepsilon$ means $u_\varepsilon - v_\varepsilon \approx 0$, and we called it *distributional equality* or just equality if there is no chance for misunderstanding.

In the sequel, relations $\sim$, $\approx$, a "growth order", Landau symbols $\mathcal{O}(\cdot)$ and $o(\cdot)$ will always be used assuming $\varepsilon \to 0$. The half-space $\{(x,t) \in \mathbb{R} \times \mathbb{R}_+\}$ is denoted by $\mathbb{R}^2_+$.

All calculations in the paper are based on exploitation of the Rankine-Hugoniot conditions. We will obtain all results by the following basic lemma and its minor revisions.

Lemma 1.1. *Let $f, g \in \mathcal{C}(\Omega : \mathbb{R}^n)$ and $U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n$ be a piecewise constant function given by*

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < c(t) - a_\varepsilon(t) - x_{1,\varepsilon} \\ U_{1,\varepsilon}, & c(t) - a_\varepsilon(t) - x_{1,\varepsilon} < x < c(t) \\ U_{2,\varepsilon}, & c(t) < x < c(t) + b_\varepsilon(t) + x_{2,\varepsilon} \\ U_1, & x > c(t) + b_\varepsilon(t) + x_{2,\varepsilon} \end{cases}. \tag{1.1}$$

Here $x_{1,\varepsilon}, x_{2,\varepsilon} \sim \varepsilon$, while a_ε, b_ε and their first order derivatives are smooth functions equal zero at $t=0$ with growth order less or equal to ε. Assume

$$\max_{i=1,2}\{\|f(U_{i,\varepsilon})\|_{L^\infty}, \|g(U_{i,\varepsilon})\|_{L^\infty}\} = \mathcal{O}(\varepsilon^{-1}). \tag{1.2}$$

Then

(1.3)

$$\begin{aligned}
\partial_t f(U_\varepsilon) \approx & -c'(t)\Big(f(U_1)-f(U_0)\Big)\delta + \Big(a'_\varepsilon(t)f(U_{1,\varepsilon}) + b'_\varepsilon(t)f(U_{2,\varepsilon})\Big)\delta \\
& - c'(t)\Big((a_\varepsilon(t)+x_{1,\varepsilon})f(U_{1,\varepsilon}) + (b_\varepsilon(t)+x_{2,\varepsilon})f(U_{2,\varepsilon})\Big)\delta' \\
\partial_x g(U_\varepsilon) \approx & \Big(g(U_1)-g(U_0)\Big)\delta + \Big((a_\varepsilon(t)+x_{1,\varepsilon})g(U_{1,\varepsilon}) + (b_\varepsilon(t)+x_{2,\varepsilon})g(U_{2,\varepsilon})\Big)\delta',
\end{aligned}$$

where δ and its derivative δ' are supported by the line $x=ct$.

Remark 1.1. The constants $x_{i,\varepsilon}$, $i=1,2$ are useful when initial data contains delta function: If $\sigma := \lim_{\varepsilon\to 0} x_{1,\varepsilon}U_{1,\varepsilon} + x_{2,\varepsilon}U_{2,\varepsilon} \in \mathbb{R}^n$ exists, then the function U from (1.1) satisfies

$$U|_{t=0} = \begin{cases} U_0, & x<0, \\ U_1, & x>0 \end{cases} + \sigma\delta_{(0,0)}.$$

Proof. We shall use the Taylor expansion formula for a test function $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2_+)$:

$$\phi((c(t)-a_\varepsilon(t)-x_{1,\varepsilon},t) = \phi(c(t),t) + \sum_{j=1}^m \partial_x^j\phi(c(t),t)\frac{(-a_\varepsilon(t)-x_{1,\varepsilon})^j}{j!} + \mathcal{O}(\varepsilon^{m+1})$$

$$\phi((c(t)+b_\varepsilon(t)+x_{2,\varepsilon},t) = \phi(c(t),t) + \sum_{j=1}^m \partial_x^j\phi(c(t),t)\frac{(b_\varepsilon(t)+x_{2,\varepsilon})^j}{j!} + \mathcal{O}(\varepsilon^{m+1}).$$

and the above growth assumptions on a_ε, b_ε, $f(U_{i,\varepsilon})$ and $g(U_{i,\varepsilon})$, $i=1,2$ to get the desired formulas. □

Remark 1.2. We used only constant mean-states $U_{1,\varepsilon}$, $U_{2,\varepsilon}$ and constant central SDW speed curve $(ct,t)_{t\geq 0}$ in (1.1). Such SDWs are not good enough for solving an SDW interaction problems for our model problems. The problem will be solved by introducing variable mean-states $U_{1,\varepsilon}(t)$ and $U_{2,\varepsilon}(t)$. Lemma 1.2 will be a natural modification of the above assertion.

Definition 1.1. Functions of the form (1.1) are called *constant shadow waves* or constant SDW for short. We shall drop the word "constant" in the sequel if there is no chance for confusion. The value

$$\sigma_\varepsilon(t) := (a_\varepsilon(t)+x_{1,\varepsilon})U_{1,\varepsilon} + (b_\varepsilon(t)+x_{2,\varepsilon})U_{2,\varepsilon}$$

is called the *strength* and $c'(t)$ is called the *speed* of the shadow wave. We assume that $\lim_{\varepsilon\to 0}\sigma_\varepsilon(t) = \sigma(t) \in \mathbb{R}^n$ exists for every $t \geq 0$ and

$$\lim_{\varepsilon\to 0} U_\varepsilon(x,t)\phi(x,t)\,dx\,dt = \sigma(t)\langle\delta(x-c(t)),\phi(x,t)\rangle, \text{ for } t\geq 0.$$

The SDW *central line* is given by $x=c(t)$, while $x=c(t)-a_\varepsilon(t)-x_{1,\varepsilon}$ and $x=c(t)+b_\varepsilon(t)+x_{2,\varepsilon}$ are called the *external SDW lines*. The values $x_{1,\varepsilon}$ and $x_{2,\varepsilon}$ are called the *shifts* while $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are called the *intermediate states* of a given SDW.

If $c(t)=t$, $a_\varepsilon(t)=a_\varepsilon t$, and $b_\varepsilon(t)=b_\varepsilon t$, an SDW is called *simple*.

Define

$$\kappa^i := c'(t)(f^i(U_1) - f^i(U_0)) - (g^i(U_1) - g^i(U_0))$$

to be so called *Rankine-Hugoniot deficit* (RH deficit for short) in the i-th equation.

When an SDW is simple one, then the formula (1.3) has a simpler form

$$\begin{aligned}\partial_t f(U_\varepsilon) \approx& - c(f(U_1) - f(U_0))\delta - c(a_\varepsilon f(U_{1,\varepsilon}) + b_\varepsilon f(U_{2,\varepsilon}))t\delta' \\ &+ (a_\varepsilon f(U_{1,\varepsilon}) + b_\varepsilon f(U_{2,\varepsilon}))\delta \\ \partial_t g(U_\varepsilon) \approx& (g(U_1) - g(U_0))\delta + (a_\varepsilon g(U_{1,\varepsilon}) + b_\varepsilon g(U_{2,\varepsilon}))t\delta'.\end{aligned} \tag{1.4}$$

The support of δ (and δ' consequently) is the line $x = ct$. Note that RH deficit for each equation of the system is now a real.

1.2. **Entropy conditions.** Let $\eta(U)$ be a (strictly) convex or semi-convex entropy function for (0.1), with entropy-flux function $q(U)$. We shall use entropy condition in the following form. A solution U_ε to the system (0.1) with initial data $U|_{t=0} = U_{0,\varepsilon}$ is *admissible* if for every $T > 0$ we have

$$\underline{\lim}_{\varepsilon\to 0} \int_{\mathbb{R}} \int_0^T \eta(U_\varepsilon)\partial_t\phi + q(U_\varepsilon)\partial_x\phi \, dt\, dx + \int_{\mathbb{R}} \eta(U_{0,\varepsilon}(x,0))\phi(x,0)\, dx \geq 0, \tag{1.5}$$

for all non-negative test functions $\phi \in C_0^\infty(\mathbb{R} \times (-\infty, T))$.

Take a simple SDW U_ε and use the equality (1.3) from Lemma 1.1 with f substituted by η and g by q. As the delta function is a non-negative distribution, the first condition becomes

$$\overline{\lim}_{\varepsilon\to 0} - c(\eta(U_1) - \eta(U_0)) + a_\varepsilon \eta(U_{1,\varepsilon}) + b_\varepsilon \eta(U_{2,\varepsilon}) + q(U_1) - q(U_0) \leq 0 \tag{1.6}$$

But a derivative of the delta function has no constant sign and the second condition becomes

$$\lim_{\varepsilon\to 0} -c(a_\varepsilon \eta(U_{1,\varepsilon}) + b_\varepsilon \eta(U_{2,\varepsilon})) + a_\varepsilon q(U_{1,\varepsilon}) + b_\varepsilon q(U_{2,\varepsilon}) = 0. \tag{1.7}$$

Here, U_0, U_1, $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ are constants.

In the most of papers with delta or singular shock solution, the authors use overcompressibility as the admissibility condition: A wave is called the *overcompressive* one if all characteristics from both sides of the SDW line run into a shock curve, i.e.

$$\lambda_i(U_0) \geq c'(t) \geq \lambda_i(U_1), \; i = 1, \dots, n,$$

where c is a shock speed and $x = \lambda_i(U)t$, $i = 1, \dots, n$ are the characteristics of the system.

The entropy condition is connected with a problem of uniqueness for a weak solution of a conservation law system. We give a definition of weak (distributional) uniqueness and some results about it afterward.

Definition 1.2. An SDW solution is called *weakly unique* if its distributional image is the unique. More precisely, a speed c of the wave has to be unique as well as the limit

$$\lim_{\varepsilon\to 0} a_\varepsilon U_{1,\varepsilon} + b_\varepsilon U_{2,\varepsilon}.$$

Let $i \in \{1, \dots, n\}$. If a limit $\lim\limits_{\varepsilon\to 0} a_\varepsilon U^i_{1,\varepsilon} + b_\varepsilon U^i_{2,\varepsilon}$ is unique, then we say that the i-th component is unique.

Note that all minor components of U_ε are unique by the above definition. The following proposition is a direct consequence of the SDW definition.

Proposition 1.1. *Suppose that (0.1) has an SDW solution.*

(a) *If there exists an equation of the system, say i-th one, such that a density function $f^i(U)$ is independent of major components of U, then a speed of the SDW is uniquely determined by the equation*

$$-c[f^i(U)] + [g^i(U)] = 0.$$

(b) *If there is an equation in the system, say i-th one, such that $f^i(U) = U^j$, where U^j is a major component, then it is uniquely determined by*

$$U^j_{1,\varepsilon} + U^j_{2,\varepsilon} = \kappa_i \in \mathbb{R}.$$

Consequently, if (a) holds and (b) holds for all major components, then a distributional limit of an SDW solution to (0.1) is unique. Specially, that is the case for a system given in evolutionary form.

Definition 1.3. We say that a solution to (0.1) is *weakly unique* if it consists from a unique combination of standard admissible elementary waves (shocks, rarefaction and contact discontinuities) and admissible SDW.

1.3. **Weighted Shadow Waves.** As in the case of constant SDWs we have the following basic lemma.

Lemma 1.2. *Let $f, g \in \mathcal{C}(\Omega : \mathbb{R}^n)$ and $U : \mathbb{R}^2_+ \to \Omega \subset \mathbb{R}^n$ be a piecewise constant function for every $t \geq 0$:*

$$U_\varepsilon(x,t) = \begin{cases} U_0, & x < c(t) - a_\varepsilon(t) \\ U_{1,\varepsilon}(t), & c(t) - a_\varepsilon(t) < x < c(t) \\ U_{2,\varepsilon}(t), & c(t) < x < c(t) + b_\varepsilon(t) \\ U_1, & x > c(t) + b_\varepsilon(t) \end{cases}. \tag{1.8}$$

The functions a_ε, b_ε are C^1-functions satisfying $a_\varepsilon(0) = x_{1,\varepsilon}$ and $b_\varepsilon(0) = x_{2,\varepsilon}$. Also, suppose that f and g satisfy (1.2). Then

$$\begin{aligned} \langle \partial_t f(U_\varepsilon), \phi \rangle \approx &\int_0^\infty \lim_{\varepsilon \to 0} \frac{d}{dt}\Big(a_\varepsilon(t) f(U_{1,\varepsilon}(t)) + b_\varepsilon(t) f(U_{2,\varepsilon}(t))\Big)\, \phi(c(t),t)\, dt \\ &- \int_0^\infty c'(t)\Big(f(U_1) - f(U_0)\Big)\, \phi(c(t),t)\, dt \\ &+ \int_0^\infty \lim_{\varepsilon \to 0} c'(t)\Big(a_\varepsilon(t) f(U_{1,\varepsilon}(t)) + b_\varepsilon(t) f(U_{2,\varepsilon}(t))\Big)\, \partial_x \phi(c(t),t)\, dt \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} \langle \partial_x g(U_\varepsilon), \phi \rangle \approx &\int_0^\infty \Big(g(U_1) - g(U_0)\Big)\, \phi(c(t),t)\, dt \\ &- \int_0^\infty \lim_{\varepsilon \to 0} \Big((a_\varepsilon(t) g(U_{1,\varepsilon}(t)) + (b_\varepsilon(t) g(U_{2,\varepsilon}(t))\Big)\, \partial_x \phi(c(t),t)\, dt. \end{aligned} \tag{1.10}$$

The proof of the first relation in (1.3) from Lemma 1.1 can be easily adopted for (1.9) and we omit it here (one just have to take care that $U_{i,\varepsilon}$ depends on t, $i = 1, 2$). The proof of (1.10) is the same as the one given for that lemma.

1.4. **The basic interaction theorem.** Suppose that two simple SDW solutions to (0.1)

$$\tilde{U}_\varepsilon(x,t)=\begin{cases}U_0, & x-a<(\tilde{c}-\tilde{a}_\varepsilon)t\\ \tilde{U}_{1,\varepsilon}, & (\tilde{c}-\tilde{a}_\varepsilon)t<x-a<\tilde{c}t\\ \tilde{U}_{2,\varepsilon}, & \tilde{c}t<x-a<(\tilde{c}+\tilde{b}_\varepsilon)t\\ U_1, & x-a>(\tilde{c}+\tilde{b}_\varepsilon)t\end{cases}$$

and

$$\hat{U}_\varepsilon(x,t)=\begin{cases}U_1, & x-b<(\hat{c}-\hat{a}_\varepsilon)t\\ \hat{U}_{1,\varepsilon}, & (\hat{c}-\hat{a}_\varepsilon)t<x-b<\hat{c}t\\ \hat{U}_{2,\varepsilon}, & \hat{c}t<x-b<(\hat{c}+\hat{b}_\varepsilon)t\\ U_2, & x-b>(\hat{c}+\hat{b}_\varepsilon)t\end{cases}$$

meat each other (when $\tilde{c}>\hat{c}$ and $a<b$). Denote by (X,T) the intersection point of the external SDW lines $x=a+(\tilde{c}+\tilde{b}_\varepsilon)t$ and $x=b+(\hat{c}-\hat{a}_\varepsilon)t$, i.e. $X=a+(\tilde{c}+\tilde{b}_\varepsilon)T=b+(\hat{c}-\hat{a}_\varepsilon)T$ and $T=(b-a)/(\tilde{c}-\hat{c}+\hat{a}_\varepsilon+\tilde{b}_\varepsilon)$. At the time $t=T$ a distributional limit of solution is a sum of a classical piecewise constant function and a delta function supported by the interaction point. So, it is natural to ask ourselves a question: When the interaction produces a shadow wave solution for $t>T$? Denote by $\hat{\kappa}\tilde{\kappa}\in\mathbb{R}^n$ the Rankine-Hugoniot deficits corresponding to $\hat{U}$, $\tilde{U}$, respectively. We have $\tilde{a}_\varepsilon f(\tilde{U}_{1,\varepsilon})T+\tilde{b}_\varepsilon f(\tilde{U}_{2,\varepsilon})T\approx T\tilde{\kappa}$ and $\hat{a}_\varepsilon f(\hat{U}_{1,\varepsilon})T+\hat{b}_\varepsilon f(\hat{U}_{2,\varepsilon})T\approx T\hat{\kappa}$.

So the interaction problem reduces to a problem of finding weighted shadow wave solution to (0.1) with the initial data

$$U(x,0)=\begin{cases}U_0, & x<0\\ U_2, & x>0.\end{cases}+T(\tilde{\kappa}+\hat{\kappa})\delta \tag{1.11}$$

Let $x_{1,\varepsilon},x_{2,\varepsilon}=\mathcal{O}(\varepsilon)$ be non-negative numbers for ε small enough described in (1.8). Then

$$f(U_\varepsilon(x,0))\approx\lim_{\varepsilon\to 0}x_{1,\varepsilon}f(U_{1,\varepsilon}(0))+x_{2,\varepsilon}f(U_{2,\varepsilon}(0))$$

and that should equal $T(\tilde{\kappa}+\hat{\kappa})$.

Suppose that such a solution exists and define

$$U_\varepsilon(x,t)=\begin{cases}U_0, & x-X<c(t-T)-a_\varepsilon(t),\ t>T\\ U_{1,\varepsilon}(t), & c(t-T)-a_\varepsilon(t)<x-X<c(t-T),\ t>T\\ U_{2,\varepsilon}(t), & c(t-T)<x-X<c(t-T)+b_\varepsilon(t),\ t>T\\ U_2, & x-X>c(t-T)+b_\varepsilon(t),\ t>T.\end{cases} \tag{1.12}$$

The solution before interaction is given by

$$(\tilde{U}_\varepsilon\wedge\hat{U}_\varepsilon)(x,t)=\begin{cases}U_0, & x<(\tilde{c}-\tilde{a}_\varepsilon)t+a,\ t<T\\ \tilde{U}_{1,\varepsilon}, & (\tilde{c}-\tilde{a}_\varepsilon)t+a<x<\tilde{c}t+a,\ t<T\\ \tilde{U}_{2,\varepsilon}, & \tilde{c}t+a<x<(\tilde{c}+\tilde{b}_\varepsilon)t+a,\ t<T\\ U_1, & (\tilde{c}+\tilde{b}_\varepsilon)t+a<x<(\hat{c}-\hat{a}_\varepsilon)t+b,\ t<T\\ \hat{U}_{1,\varepsilon}, & (\hat{c}-\hat{a}_\varepsilon)t+b<x<\hat{c}t+b,\ t<T\\ \hat{U}_{2,\varepsilon}, & \hat{c}t+b<x<(\hat{c}+\hat{b}_\varepsilon)t+b,\ t<T\\ U_2, & x>(\hat{c}+\hat{b}_\varepsilon)t+b,\ t<T.\end{cases}$$

The anticipated solution V_ε is obtained by gluing the solution for $t < T$ (denoted by $\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon$) with the one defined by (1.12) for $t > T$ (denoted by U_ε):

$$V_\varepsilon(x,t) = \begin{cases} (\tilde{U}_\varepsilon \wedge \hat{U}_\varepsilon)(x,t), & t < T \\ U_\varepsilon(x,t), & t > T, \end{cases} \tag{1.13}$$

Theorem 1.1. *With the above assumptions, V_ε, defined in (1.13), solves*

$$\partial_t f(V_\varepsilon) + \partial_x g(V_\varepsilon) \approx 0.$$

The above theorem applies also to the case when one of the incoming waves $\tilde{U}$ or $\hat{U}$ is a shock. The proof is the same with some obvious changes (for example, $\hat{a}_\varepsilon = \hat{b}_\varepsilon = 0$ if the second wave is a shock). Obviously, the assertion also holds if a contact discontinuity is in the place of the shock. Also, the above theorem is applicable in cases when one or two incoming SDWs are not simple. We have presented the proof for simple ones just to make the main ideas clear enough without going into detailed calculations.

Finally, the above theorem is useful for dealing with shadow and rarefaction wave interaction as already announced in the introduction of this part. When a rarefaction wave is substituted by a fan of non-entropy shocks of small strength (which solve the system in an approximated sense – see [5], for example), then the above theorem can be applied on interaction of an SDW and such non-entropy shock. After each such interaction we obtain a solution in the fan-form containing at least one SDW of the type (1.1) with $x_{1,\varepsilon}^2 + x_{2,\varepsilon}^2 > 0$ and the procedure can be continued. A trajectory of a resulting SDW is a broken line

$$\cup_{i=1}^m \{(c_i t + \alpha_i, t),\ t \in [T_{i-1}, T_i],\ \alpha_i \in \mathbb{R}\},$$

where T_i, $i = 1, \dots, m$ are time coordinates of interaction points. One can try to find an SDW central line $(c(t), t)$ as limit of the above trajectories by solving a governing ODE. Note that the resulting SDW can be of different nature, for example, with a constant or decreasing strength. That fact opens a door for solving the complete interaction problem.

Part 2. Model problems

The first system that we look at is the well known 3×3 pressureless gas dynamics model.

The other one is the Chaplygin gas model. Some of the cosmology theories uses it as a model of the so called dark energy of the Universe. It models a compressible fluid with the pressure is negative and inversely proportional to the gas energy density, $p = -A/\rho$, for some $A > 0$ (see [13] for physical explanations).

The third one will be more recent one, so called generalized Chaplygin gas, with the pressure defined by $p = -A/\rho^\alpha$, $0 < \alpha \leq 1$, introduced first (up to our knowledge) in [1].

In these models there is a significant mathematical difference between the cases $\alpha = 1$ and $\alpha \in (0.1)$. First, we will analyzed their classical solutions.

2. Pressureless gas dynamics

3. Chaplygin gas

The system modeling Chaplygin gas consists of mass and momentum conservation laws

$$\begin{gathered}\partial_t\rho+\partial_x(\rho u)=0\\ \partial_t(\rho u)+\partial_x\Big(\rho u^2-\frac{A}{\rho}\Big)=0,\end{gathered}$$

where u denotes the velocity of the gas. In this paper we shall fix $A=1$ and use the momentum variable $q=\rho u$,

$$\begin{gathered}\partial_t\rho+\partial_x q=0\\ \partial_t q+\partial_x\Big(\frac{q^2-1}{\rho}\Big)=0.\end{gathered}\tag{3.1}$$

The physical domain for the system is the hyperplane $\{(\rho,q)|\ \rho>0\}$ since a pressure in the vacuum state would be infinite otherwise. The sound speed of the system tends to zero as $\rho\to\infty$. That property allows the mass concentration in a finite time and one can expect delta shock or some other wave with similar properties to be a part of a solution at least for some initial data.

The system is strictly hyperbolic with the characteristics $\lambda_1(\rho,q)=\frac{q-1}{\rho}<\lambda_2(\rho,q)=\frac{q+1}{\rho}$ and right eigenvectors $r_1=(1,\frac{q-1}{r})$, $r_2=(1,\frac{q+1}{r})$. One can see easily that both fields are linearly degenerate. So, there are only contact discontinuities as the elementary wave solutions for the Riemann data

$$(\rho,q)=\begin{cases}(\rho_0,q_0), & x<0\\ (\rho_1,q_1), & x>0.\end{cases}\tag{3.2}$$

The contact discontinuity curves are given by

$$\mathrm{CD}_1(\rho_0,q_0):\ q_1=1+\frac{q_0-1}{\rho_0}\rho_1,\ \mathrm{CD}_2(\rho_0,q_0):\ q_1=-1+\frac{q_0+1}{\rho_0}\rho_1.\tag{3.3}$$

Let us try to find all possible states (ρ_1,q_1) that can be connected to (ρ_0,q_0) by two contact discontinuities: CD_1+CD_2. That is, we need to find out when there is a solution (ρ_m,q_m), $\rho_m>0$ to the system

$$q_m=1+\frac{q_0-1}{\rho_0}\rho_m,\ q_1=-1+\frac{q_m+1}{\rho_m}\rho_1.$$

It is straightforward to see that

$$\rho_m=\frac{2\rho_0\rho_1}{(1+q_1)\rho_0+(1-q_0)\rho_1},\ q_m=\frac{(q_1+1)\rho_0+(q_0-1)\rho_1}{(1+q_1)\rho_0+(1-q_0)\rho_1}$$

is the algebraic solution to the above system.

Thus, (ρ_0,q_0) and (ρ_1,q_1) in (2.2) can be connected by a combination of classical elementary waves if and only if

$$\begin{aligned}&\lambda_2(\rho_1,q_1)=\frac{q_1+1}{\rho_1}>\frac{q_0-1}{\rho_0}=\lambda_1(\rho_0,q_0),\ \text{i.e. when}\\ &\rho_0(q_1+1)-\rho_1(q_0-1)>0.\end{aligned}\tag{3.4}$$

The solution then consists of two contact discontinuities connected by a constant state

$$(\rho_s, q_s) = \Big(\frac{2}{\lambda_1(\rho_1, q_1) - \lambda_2(\rho_0, q_0)}, \frac{\lambda_2(\rho_1, q_1) + \lambda_1(\rho_0, q_0)}{\lambda_2(\rho_1, q_1) - \lambda_1(\rho_0, q_0)}\Big).$$

We shall call such a solution the contact discontinuity combination (CDC) in the sequel.

Using the standard methods for finding entropies (**??**), one finds that the system posses an infinite number of convex entropies. Solving the system (here it reduces to a single hyperbolic PDE), one finds that a general form of an entropy function for (2.1) is given by

$$\eta = \frac{\rho}{2}\bigg(F\Big(\frac{q-1}{\rho}\Big) + G\Big(\frac{q+1}{\rho}\Big)\bigg) \tag{3.5}$$

with the entropy-flux function given by

$$Q = \frac{1}{2}\bigg((q+1)F\Big(\frac{q-1}{\rho}\Big) + (q-1)G\Big(\frac{q+1}{\rho}\Big)\bigg). \tag{3.6}$$

The entropy function η is convex if and only if both F and G are convex. The most important additional conservation law is the energy conservation (see [3])

$$\partial_t\Big(\frac{q^2+1}{\rho}\Big) + \partial_x\Big(\frac{q}{\rho}\frac{q^2-1}{\rho}\Big) = 0.$$

4. Generalized Chaplygin gas

A generalized Chaplygin gas appears in cosmology theories and it is a model for a compressible fluid with a pressure inversely proportional to a gas energy density, $p = -A/\rho^\alpha$, $A > 0$, $0 < \alpha < 1$, see [1]. We put $A = 1$ for simplicity bellow. It is used as a model for the dark energy in Universe. The system consists from the mass and momentum conservation laws

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= 0 \\ \partial_t(\rho u) + \partial_x\Big(\rho u^2 - \frac{1}{\rho^\alpha}\Big) &= 0, \end{aligned}$$

where u denotes a velocity of the gas. In this paper we use the momentum variable $q = \rho u$:

$$\begin{aligned} \partial_t\rho + \partial_x q &= 0 \\ \partial_t q + \partial_x\Big(\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big) &= 0. \end{aligned} \tag{4.1}$$

The physical region for both systems is $\rho > 0$ and the sound speed of the system tends to zero as $\rho \to \infty$. That property allows a mass concentration in a finite time and one could expect some kind of non-classical solutions. The choosing of proper solutions of a new type is a main subject of the paper.

Let us briefly give the properties of the system. It is strictly hyperbolic system with the eigenvalues $\lambda_1 = \frac{q}{\rho} - \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}$, $\lambda_1 = \frac{q}{\rho} + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}$ and appropriate eigenvectors $r_1 = \Big(-1, -\frac{q}{\rho} + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}\Big)$ and $r_2 = \Big(1, \frac{q}{\rho} + \sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}\Big)$. Both fields are genuinely nonlinear:

$$\begin{aligned}D\lambda_1\, r_1 =&\Big(-\frac{q}{\rho^2}+\frac{(1+\alpha)\sqrt{\alpha}}{2}\rho^{-\frac{3+\alpha}{2}},\frac{1}{\rho}\Big)\cdot\Big(-1,-\frac{q}{\rho}+\sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}\Big)\\ =&\sqrt{\alpha}\rho^{-\frac{3+\alpha}{2}}\Big(-\frac{1+\alpha}{2}+1\Big)>0,\end{aligned}$$

since$\alpha\in(0,1)$, and

$$\begin{aligned}D\lambda_2\, r_2 =&\Big(-\frac{q}{\rho^2}-\frac{(1+\alpha)\sqrt{\alpha}}{2}\rho^{-\frac{3+\alpha}{2}},\frac{1}{\rho}\Big)\cdot\Big(1,\frac{q}{\rho}+\sqrt{\alpha}\rho^{-\frac{1+\alpha}{2}}\Big)\\ =&\sqrt{\alpha}\rho^{-\frac{3+\alpha}{2}}\Big(\frac{1+\alpha}{2}+1\Big)>0.\end{aligned}$$

Using the standard procedures one can find rarefaction curves

$$\begin{aligned}R_1: q=\frac{\rho}{\rho_0}q_0+\frac{2\sqrt{\alpha}}{1+\alpha}\rho\big(\rho^{-\frac{1+\alpha}{2}}-\rho_0^{-\frac{1+\alpha}{2}}\big),\ \rho<\rho_0\\ R_2: q=\frac{\rho}{\rho_0}q_0-\frac{2\sqrt{\alpha}}{1+\alpha}\rho\big(\rho^{-\frac{1+\alpha}{2}}-\rho_0^{-\frac{1+\alpha}{2}}\big),\ \rho>\rho_0,\end{aligned}$$

as well as shock curves

$$\begin{aligned}S_1: q=\frac{\rho}{\rho_0}q_0-\sqrt{\frac{\rho}{\rho_0}(\rho-\rho_0)\Big(\frac{1}{\rho_0^\alpha}-\frac{1}{\rho^\alpha}\Big)},\ \rho>\rho_0,\\ S_2: q=\frac{\rho}{\rho_0}q_0-\sqrt{\frac{\rho}{\rho_0}(\rho-\rho_0)\Big(\frac{1}{\rho_0^\alpha}-\frac{1}{\rho^\alpha}\Big)},\ \rho<\rho_0.\end{aligned}$$

A speed of shock is given by the same relation in both cases

$$c=\frac{q_0}{\rho_0}+\sqrt{\frac{\rho}{\rho_0}\frac{\rho_0^\alpha-\rho^\alpha}{\rho_0-\rho}\frac{1}{\rho_0^\alpha\rho^\alpha}}.$$

Our aim is to solve Riemann problem: (3.1) with

$$(\rho,q)=\begin{cases}(\rho_0,q_0), & x<0\\ (\rho_1,q_1), & x>0\end{cases}. \tag{4.2}$$

It is not so hard to see that there is a classical solution consisting of shocks and rarefaction waves to that problem if (ρ_1,q_1) is above the curve

$$\Gamma_{ss}:\ q=\Big(\frac{q_0}{\rho_0}-\rho^{-\frac{1+\alpha}{2}}-\rho_0^{-\frac{1+\alpha}{2}}\Big)\rho. \tag{4.3}$$

5. Generalized solution to Chaplygin model

Let us start with a piecewise constant function of the following form called the simple shadow wave (SDW for short)

$$(\rho,q)=\begin{cases}(\rho_0,q_0), & x<(c-\varepsilon)t\\ (\rho_{0,\varepsilon},q_{0,\varepsilon}), & (c-\varepsilon)t<x<ct\\ (\rho_{1,\varepsilon},q_{1,\varepsilon}), & ct<x<(c+\varepsilon)t\\ (\rho_1,q_1), & x>(c+\varepsilon)t.\end{cases}$$

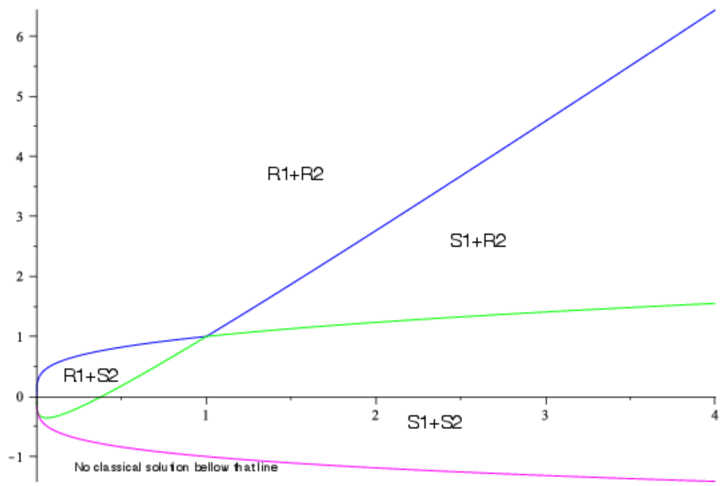

FIGURE 1. Classical waves

The SDW (p, q) solves (2.1,2.2) in the weak sense if

$$\lim_{\varepsilon\to 0}\langle \rho, \partial_t \phi\rangle + \langle q, \partial_x \phi\rangle = 0$$

$$\lim_{\varepsilon\to 0}\langle q, \partial_t \phi\rangle + \Big\langle \frac{q^2-1}{\rho}, \partial_x \phi\Big\rangle = 0$$

for every test function $\phi \in C_0^\infty(\mathbb{R}_+^2)$. Using Lemma 1.1 one gets the following formulas for the derivatives

$$\begin{aligned}
\partial_t\rho &\approx \big(-c[\rho] + (\varepsilon\rho_{0,\varepsilon} + \varepsilon\rho_{1,\varepsilon})\big)\delta - c(\varepsilon\rho_{0,\varepsilon} + \varepsilon\rho_{1,\varepsilon})t\delta' \\
\partial_x q &\approx [q]\delta + (\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})t\delta' \\
\partial_t q &\approx \big(-c[q] + (\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})\big)\delta - c(\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})t\delta' \\
\partial_x\Big(\frac{q^2-1}{\rho}\Big) &\approx \Big[\frac{q^2-1}{\rho}\Big]\delta + \Big(\varepsilon\Big(\frac{q_{0,\varepsilon}^2-1}{\rho_{0,\varepsilon}}\Big) + \varepsilon\Big(\frac{q_{1,\varepsilon}^2-1}{\rho_{1,\varepsilon}}\Big)\Big)t\delta'.
\end{aligned}$$

Here and bellow $a_\varepsilon \approx b_\varepsilon$ means $\lim_{\varepsilon\to 0} a_\varepsilon - b_\varepsilon = 0$ while $[y] := y_1 - y_0$ is the standard designation of a jump in the variable y across a shock front. The support of delta function and its derivative above is called a shock front. In the above formulas, the shock front is the line $x = ct$.

It is easy to see that the only possibility to avoid a trivial case when $\rho_{i,\varepsilon}$ and $q_{i,\varepsilon}$ disappear is that $\rho_{i,\varepsilon}, q_{i,\varepsilon} \sim \varepsilon^{-1}$, $i = 0, 1$. Denoting $\xi_i := \lim_{\varepsilon\to 0}\varepsilon\rho_i$, $\chi_i := \lim_{\varepsilon\to 0}\varepsilon q_i$, $i = 0, 1$, we have $\frac{q_{0,\varepsilon}^2-1}{\rho_{0,\varepsilon}} \approx \frac{\chi_i^2}{\xi_i}$, $i = 0, 1$. So the Riemann problem (2.1,2.2) reduces to the following system of algebraic equations

$$\begin{aligned}
-c[\rho] + (\xi_0 + \xi_1) + [q] &= 0 \\
c(\xi_0 + \xi_1) &= \chi_0 + \chi_1 \\
-c[q] + (\chi_0 + \chi_1) + \Big[\frac{q^2-1}{\rho}\Big] &= 0 \\
c(\chi_0 + \chi_1) &= \frac{\chi_0^2}{\xi_0} + \frac{\chi_1^2}{\xi_1}.
\end{aligned} \tag{5.1}$$

Denote by $\kappa_1 := c[\rho] - [q]$ and $\kappa_2 := c[q] - \left[\frac{q^2-1}{\rho}\right]$ so-called the Rankine-Hugoniot deficits. One immediately gets $\kappa_2 = c\kappa_1$ from the second equation. The third and fourth equation then imply

$$c = \frac{[q] \pm \sqrt{[q]^2 - [\rho]\left[\frac{q^2-1}{\rho}\right]}}{[\rho]} = \frac{[q] + \kappa_1}{[\rho]}. \tag{5.2}$$

From the fourth equation one can see that the only possible relations between the unknowns ξ_i and χ_i, $i = 0, 1$, are

$$\xi_0 = \frac{\chi_0}{c} \text{ and } \xi_1 = \frac{\chi_1}{c}.$$

The first and the third equation in (4.1) uniquely determine strength of the SDW (ξ, χ) defined by

$$\xi := \xi_0 + \xi_1 = \kappa_1,\ \chi := \chi_0 + \chi_1 = \kappa_2 = c\kappa_1.$$

The variable ρ denotes the density so $\kappa_1 > 0$ (the case $\kappa_1 = 0$ corresponds to the contact discontinuity solution). The positivity of κ_1 implies that one has to take the plus sign in expression (4.2) for the speed c. A simple computation gives

$$\begin{aligned}\kappa_1 =& \sqrt{\rho_0\rho_1\Big(\frac{q_0-1}{\rho_0} - \frac{q_1-1}{\rho_1}\Big)\Big(\frac{q_0+1}{\rho_0} - \frac{q_1+1}{\rho_1}\Big)} \\ =& \sqrt{\rho_0\rho_1(\lambda_1(\rho_0,q_0) - \lambda_1(\rho_1,q_1))(\lambda_2(\rho_0,q_0) - \lambda_2(\rho_1,q_1))}.\end{aligned}$$

It is obvious that the negation of condition (2.4) ensures the positivity of the term under the square root. Thus, we have well defined SDW solution whenever (2.4) is not satisfied i.e. when $\lambda_1(\rho_0, q_0) \geq \lambda_2(\rho_1, q_1)$. It remains to prove that the SDW is entropic. A SDW (ρ, q) is entropic (and thus admissible) if for every (semi)convex entropy function η and corresponding entropy flux function Q we have

$$\langle \partial_t \eta(\rho, q) + \partial_x Q(\rho, q), \phi\rangle \leq 0 \tag{5.3}$$

for every non-positive test function $\phi \in C_0^\infty$. According to (1.6) and (1.7), a SDW solution (ρ, q) to (2.1) is entropic if and only if

$$\begin{aligned}&\lim_{\varepsilon\to 0} -c(\varepsilon\eta(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon\eta(\rho_{1,\varepsilon}, q_{1,\varepsilon})) + \varepsilon Q(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon Q(\rho_{1,\varepsilon}, q_{1,\varepsilon}) = 0 \\ &\qquad -c(\eta(\rho_1, q_1) - \eta(\rho_0, q_0)) + Q(\rho_1, q_1) - Q(\rho_0, q_0) \\ &\qquad\qquad + \lim_{\varepsilon\to 0}(\varepsilon\eta(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon\eta(\rho_{1,\varepsilon}, q_{1,\varepsilon})) \leq 0.\end{aligned} \tag{5.4}$$

Substituting (2.5) and (2.6) into the first relation above we obtain

$$-\frac{c}{2}\Big(\xi_0 F(c) + \xi_1 G(c)\Big) + \frac{1}{2}\Big(\chi_0 F(c) + \chi_1 G(c)\Big) = 0$$

since $\chi_i = c\xi_i$, $i = 0, 1$. The second relation from (4.4) reduces to

$$\begin{aligned}-c\Bigg(\frac{\rho_1}{2}\Big(F\Big(\frac{q_1-1}{\rho_1}\Big) + G\Big(\frac{q_1+1}{\rho_1}\Big)\Big) - \frac{\rho_0}{2}\Big(F\Big(\frac{q_0-1}{\rho_0}\Big) + G\Big(\frac{q_0+1}{\rho_0}\Big)\Big)\Bigg) \\ + \frac{1}{2}\Bigg((q_1+1)F\Big(\frac{q_1-1}{\rho_1}\Big) + (q_1-1)G\Big(\frac{q_1+1}{\rho_1}\Big) \\ -(q_0+1)F\Big(\frac{q_0-1}{\rho_0}\Big) - (q_0-1)G\Big(\frac{q_0+1}{\rho_0}\Big)\Bigg) + \frac{\xi_0+\xi_1}{2}\Big(F(c) + G(c)\Big) \leq 0.\end{aligned}$$

Here we give the proof for $G \equiv 0$ only. The proof for $F \equiv 0$ (and thus for a general case since one can prove the inequality for each addend separately) goes along the same lines so we omit it.

One has to prove

$$-\Big(\big(q_0+1-c\rho_0\big)F\Big(\frac{q_0-1}{\rho_0}\Big)+\big(c\rho_1-(q_1+1)\big)F\Big(\frac{q_1-1}{\rho_0}\Big)\Big)+\kappa_1 F(c)\le 0.$$

In order to do it, let us put

$$I:=-\Big(\frac{q_0+1-c\rho_0}{\kappa_1}F\Big(\frac{q_0-1}{\rho_0}\Big)+\frac{c\rho_1-(q_1+1)}{\kappa_1}F\Big(\frac{q_1-1}{\rho_0}\Big)\Big)+F(c).$$

Using

$$\frac{q_0+1-c\rho_0}{\kappa_1}+\frac{c\rho_1-(q_1+1)}{\kappa_1}=\frac{c(\rho_1-\rho_0)-(q_1-q_0)}{\kappa_1}$$

and the convexity of F one has

$$\begin{aligned} I\le&-F\Big(\frac{q_0+1-c\rho_0}{\kappa_1}\frac{q_0-1}{\rho_0}+\frac{c\rho_1-(q_1+1)}{\kappa_1}\frac{q_1-1}{\rho_1}\Big)+F(c)\\ =&-F\Bigg(\frac{1}{\kappa_1}\Bigg(\underbrace{c(\rho_1-\rho_0)-\Big(\frac{q_1^2-1}{\rho_1}-\frac{q_0^2-1}{\rho_0}\Big)}_{=\kappa_2=c\kappa_1}\Bigg)\Bigg)+F(c)=-F(c)+F(c)=0.\end{aligned}$$

Note that it is necessary that both of $\dfrac{q_0+1-c\rho_0}{\kappa_1}$ and $\dfrac{c\rho_1-(q_1+1)}{\kappa_1}$ are non-negative. Thus the SDW has to satisfy

$$\lambda_2(\rho_0,q_0)\ge c\ge\lambda_2(\rho_1,q_1). \tag{5.5}$$

If one of the terms $q_0+1-c\rho_0$ or $c\rho_1-(q_1+1)$ is negative then it is easy to find an F such that $I>0$. Therefore $I\le 0$ for every convex F if and only if (4.5) holds.

Using the same procedure for G one can get that the wave is entropic if and only if

$$\lambda_1(\rho_0,q_0)\ge c\ge\lambda_1(\rho_1,q_1). \tag{5.6}$$

A wave satisfying (4.5) and (4.6) is said to be overcompressive.

So, we have proved the following theorem.

Theorem 5.1. *The Riemann problem (2.1,2.2) has a unique entropic solution which consists of two contact discontinuities if (2.4) holds. If (2.4) does not hold, the solution is a single SDW represented by*

$$(\rho,q)=\begin{cases}(\rho_0,q_0), & x<(c-\varepsilon)t\\ (\xi_0/\varepsilon,\chi_0/\varepsilon), & (c-\varepsilon)t<x<ct\\ (\xi_1/\varepsilon,\chi_1/\varepsilon), & ct<x<(c+\varepsilon)t\\ (\rho_1,q_1), & x>(c+\varepsilon)t,\end{cases}$$

with $c=\frac{q_1-q_0+\kappa_1}{\rho_1-\rho_0}$, $\xi_0+\xi_1=\kappa_1$, $\chi_0+\chi_1=c\kappa_1$, *where the Rankine-Hugoniot deficit is given by* $\kappa_1=\sqrt{\rho_0\rho_1(\lambda_1(\rho_0,q_0)-\lambda_1(\rho_1,q_1))(\lambda_2(\rho_0,q_0)-\lambda_2(\rho_1,q_1))}$.

Remark 5.1. As we have already said, the term "unique solution" in Theorem 4.1 should be understood in a weakly unique sense defined above. Next, all assertions are valid for any combination of ξ_0 and ξ_1 or χ_0 and χ_1 as long as ξ_0 and ξ_1 stay non-negative and have sums determined in the theorem. Particularly, it is safe to take $\rho_{0,\varepsilon} = \rho_{1,\varepsilon} =: \rho_\varepsilon$ (i.e. $\xi_0 = \xi_1$) and $q_{0,\varepsilon} = q_{1,\varepsilon} = q_\varepsilon$ (i.e. $\chi_0 = \chi_1$).

6. Solution to generalized Chaplygin model

Lemma 6.1. *There exists a simple shadow wave (SDW for short) written in the form*

$$(\rho, q) = \begin{cases} (\rho_0, q_0), & x < (c-\varepsilon)t \\ (\rho_{0,\varepsilon}, q_{0,\varepsilon}), & (c-\varepsilon)t < x < ct \\ (\rho_{1,\varepsilon}, q_{1,\varepsilon}), & ct < x < (c+\varepsilon)t \\ (\rho_1, q_1), & x > (c+\varepsilon)t, \end{cases} \tag{6.1}$$

that solves (3.1,3.2) if and only if

$$(q_0\rho_1 - q_1\rho_0)^2 > (\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)\rho_0\rho_1.$$

Proof. Using Lemma 1.1 one get the following formulas for its derivatives

$$\begin{aligned}
\partial_t \rho &\approx \big(-c[\rho] + (\varepsilon\rho_{0,\varepsilon} + \varepsilon\rho_{1,\varepsilon})\big)\delta - c(\varepsilon\rho_{0,\varepsilon} + \varepsilon\rho_{1,\varepsilon})t\delta' \\
\partial_x q &\approx [q]\delta + (\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})t\delta' \\
\partial_t q &\approx \big(-c[q] + (\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})\big)\delta - c(\varepsilon q_{0,\varepsilon} + \varepsilon q_{1,\varepsilon})t\delta' \\
\partial_x\Big(\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big) &\approx \Big[\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big]\delta + \bigg(\varepsilon\Big(\frac{q_{0,\varepsilon}^2}{\rho_{0,\varepsilon}} - \frac{1}{\rho_{0,\varepsilon}^\alpha}\Big) + \varepsilon\Big(\frac{q_{1,\varepsilon}^2}{\rho_{1,\varepsilon}} - \frac{1}{\rho_{1,\varepsilon}^\alpha}\Big)\bigg)t\delta'.
\end{aligned}$$

The supports of delta function and its derivative is again the line $x = ct$. One immediately sees that the only possibility to avoid a trivial case (when both $\rho_{i,\varepsilon}$ and $q_{i,\varepsilon}$, $i = 0,1$, are zero) is $\rho_{i,\varepsilon}, q_{i,\varepsilon} \sim \varepsilon^{-1}$, $i = 0,1$. So, let us denote

$$\xi_i := \lim_{\varepsilon\to 0} \varepsilon\rho_i,\ \chi_i := \lim_{\varepsilon\to 0} \varepsilon q_i,\ i = 0,1.$$

Then

$$\frac{q_{0,\varepsilon}^2}{\rho_{0,\varepsilon}} - \frac{1}{\rho_{0,\varepsilon}^\alpha} \approx \frac{\chi_i^2}{\xi_i},\ i = 0,1.$$

and Riemann problem (3.1,3.2) reduces to the system of the following equations

$$\begin{aligned}
-c[\rho] + (\xi_0 + \xi_1) + [q] &= 0 \\
c(\xi_0 + \xi_1) &= \chi_0 + \chi_1 \\
-c[q] + (\chi_0 + \chi_1) + \Big[\frac{q^2}{\rho} - \frac{1}{\rho^\alpha}\Big] &= 0 \\
c(\chi_0 + \chi_1) &= \frac{\chi_0^2}{\xi_0} + \frac{\chi_1^2}{\xi_1}.
\end{aligned} \tag{6.2}$$

Denote by $\kappa_1 := c[\rho] - [q]$ and $\kappa_2 = c[q] - \Big[\dfrac{q^2}{\rho} - \dfrac{1}{\rho^\alpha}\Big]$ so called Rankine-Hugoniot deficits for the first and second equation of the system, resp. One immediately gets $\kappa_2 = c\kappa_1$ from the second equation. The third and fourth equation determines c

$$c = \frac{[q] \pm \sqrt{[q]^2 - [\rho]\Big[\frac{q^2 - \rho^{1-\alpha}}{\rho}\Big]}}{[\rho]}. \tag{6.3}$$

The only possible relation between unknowns ξ_i, χ_i, $i=0,1$, is

$$\xi_0 = \frac{\chi_0}{c} \text{ and } \xi_1 = \frac{\chi_1}{c},$$

and it fixes the fourth equation. The first and the third equation in (5.2) uniquely determines a strength of SDW

$$\xi := \xi_0 + \xi_1 = \kappa_1,\ \chi := \chi_0 + \chi_1 = \kappa_2 = c\kappa_1.$$

The variable ρ denotes the density so $\kappa_1 > 0$ (the case $\kappa_1 = 0$ corresponds to a shock). From the first equation in (5.2) we have

$$c = \frac{q_1 - q_0 + \kappa_1}{\rho_1 - \rho_0},$$

and the positivity of κ_1 implies that one has to take plus sign in (5.3). a simple computation gives

$$\kappa_1 = \sqrt{\frac{(q_0\rho_1 - q_1\rho_0)^2}{\rho_0\rho_1} - (\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)}.$$

Thus, an SDW solution to (3.1), (3.2) exists if and only if

$$(q_0\rho_1 - q_1\rho_0)^2 > (\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)\rho_0\rho_1$$

i.e. a point (ρ_1, q_1) has to be bellow the curve

$$q = \frac{\rho}{\rho_0}q_0 - \sqrt{\frac{\rho_1}{\rho_0}(\rho_0 - \rho_1)\Big(\frac{1}{\rho_1^\alpha} - \frac{1}{\rho_0^\alpha}\Big)}. \tag{6.4}$$

□

That curve given by (5.4) is above Γ_{ss} meaning that a solution to Riemann problem is not unique: For (ρ_1, q_1) between these curves both S1+S2 and SDW solution exists. One has to exclude SDW or S1+S2 solution. The overcompressibility condition is often used in order to gain a uniqueness of delta shock – type solutions. It means that $\lambda_i(\rho_0, q_0) \geq c \geq \lambda_i(\rho_1, q_1)$ should be true for $i = 1, 2$. That relation for system (3.1) is satisfied if

$$\frac{q_0}{\rho_0} - \sqrt{\alpha}\rho_0^{-\frac{1+\alpha}{2}} \geq \frac{q - q_0 + \kappa_1}{\rho - \rho_0} \geq \frac{q_1}{\rho_1} + \sqrt{\alpha}\rho_1^{-\frac{1+\alpha}{2}}. \tag{6.5}$$

Let us denote by $x := q_0\rho_1 - q_1\rho_0$. Relatioon (5.5) imply $\dfrac{q_0}{\rho_0} - \dfrac{q_1}{\rho_1} \geq \sqrt{\alpha}\rho_0^{-\frac{1+\alpha}{2}} + \sqrt{\alpha}\rho_1^{-\frac{1+\alpha}{2}} > 0$, so $x > 0$.

One can find that the first inequality in (5.5) is always satisfied if $\rho_1^{-\alpha} \leq (1 - \alpha\rho_0^2)\rho_0^{-\alpha}$. Otherwise, it is satisfied if

$$x \leq \sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}\rho_1 - \rho_1^{\frac{1}{2}}\sqrt{\alpha\rho_0^{2-\alpha} + \rho_1^{-\alpha} - \rho_0^{-\alpha}} \text{ or}$$
$$x \geq \sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}\rho_1 + \rho_1^{\frac{1}{2}}\sqrt{\alpha\rho_0^{2-\alpha} + \rho_1^{-\alpha} - \rho_0^{-\alpha}}.$$

The second one is satisfied if $\rho_0^{-\alpha} \leq (1 - \alpha\rho_1^2)\rho_1^{-\alpha}$, or

$$x \leq \sqrt{\alpha}\rho_0\rho_1^{\frac{1-\alpha}{2}} - \rho_0^{\frac{1}{2}}\sqrt{\alpha\rho_1^{2-\alpha} + \rho_0^{-\alpha} - \rho_1^{-\alpha}} \text{ or}$$
$$x \geq \sqrt{\alpha}\rho_0\rho_1^{\frac{1-\alpha}{2}} + \rho_0^{\frac{1}{2}}\sqrt{\alpha\rho_1^{2-\alpha} + \rho_0^{-\alpha} - \rho_1^{-\alpha}}$$

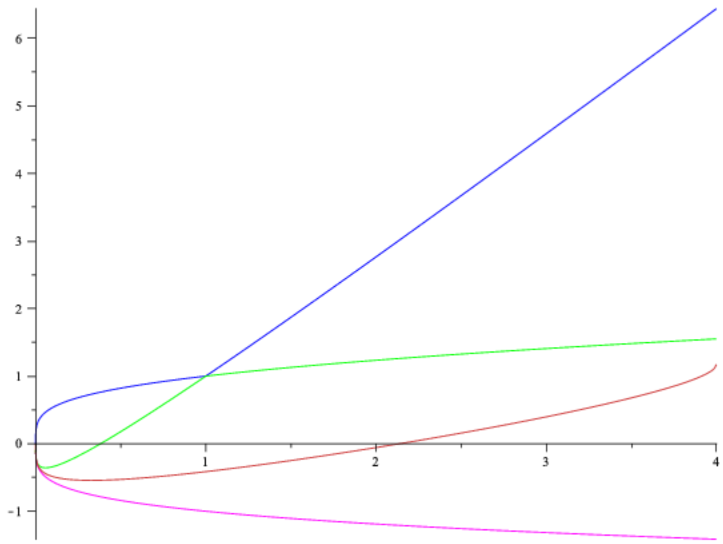

FIGURE 2. Overcompressive SDW vs. S1+S2

otherwise.

Let us analyse a bit these solutions to (5.5). In the case $\rho_1 > \rho_0$ the term $\sqrt{\alpha}\rho_0\rho_1^{\frac{1-\alpha}{2}} - \rho_0^{\frac{1}{2}}\sqrt{\alpha\rho_1^{2-\alpha}+\rho_0^{-\alpha}-\rho_1^{-\alpha}}$ is negative. Similarly, for $\rho_1 < \rho_0$, $\sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}\rho_1 - \rho_1^{\frac{1}{2}}\sqrt{\alpha\rho_0^{2-\alpha}+\rho_1^{-\alpha}-\rho_0^{-\alpha}} < 0$. These contradicts the relation $x > 0$, so upper bounds for x in both of the above solution of the entropy inequalities are irrelevant.

Next, at least one of the terms $\rho_1^{-\alpha} \le (1-\alpha\rho_0^2)\rho_0^{-\alpha}$ and $\rho_0^{-\alpha} \le (1-\alpha\rho_1^2)\rho_1^{-\alpha}$ has to be positive. Otherwise $\rho_0^{-\alpha} \le \rho_0^{-\alpha}(1-\alpha\rho_0^2)(1-\alpha\rho_1^2) < \rho_0^{-\alpha}$ leads to a contradiction. Thus at least one lower part of the above solutions is present.

Therefore, one sees that (ρ_1, q_1) can be connected by an overcompressive SDW with (ρ_0, q_0) if and only if it lies bellow the curve

$$\begin{aligned}\Gamma_{oc}:\ q = \frac{\rho}{\rho_0}q_0 - \frac{1}{\rho_0}\max\{&\sqrt{\alpha}\rho_0^{\frac{1-\alpha}{2}}\rho_1 + \rho_1^{\frac{1}{2}}\sqrt{\alpha\rho_0^{2-\alpha}+\rho_1^{-\alpha}-\rho_0^{-\alpha}},\\ &\sqrt{\alpha}\rho_0\rho_1^{\frac{1-\alpha}{2}} + \rho_0^{\frac{1}{2}}\sqrt{\alpha\rho_1^{2-\alpha}+\rho_0^{-\alpha}-\rho_1^{-\alpha}}\}.\end{aligned} \tag{6.6}$$

The operation max above is understood in the following sense: If a is a real and b is not, then $\max\{a,b\} = \max\{b,a\} = a$.

As one could see, if (ρ_0, q_0) lies bellow Γ_{oc} and above Γ_{ss} a solution to (3.1), (3.2) is not unique: One can construct both S1+S2 and SDW solution to that problem. Our aim is to use a possibility of using convex entropy – entropy flux pair for SDWs. That possibility was one of the major reasons of use SDWs to reconstruct non-classical solution to conservation law systems.

6.1. **Convex entropies.** Suppose that a conservation laws system posses convex entropy – entropy flux pair (called convex entropy pair bellow) (η, A). According

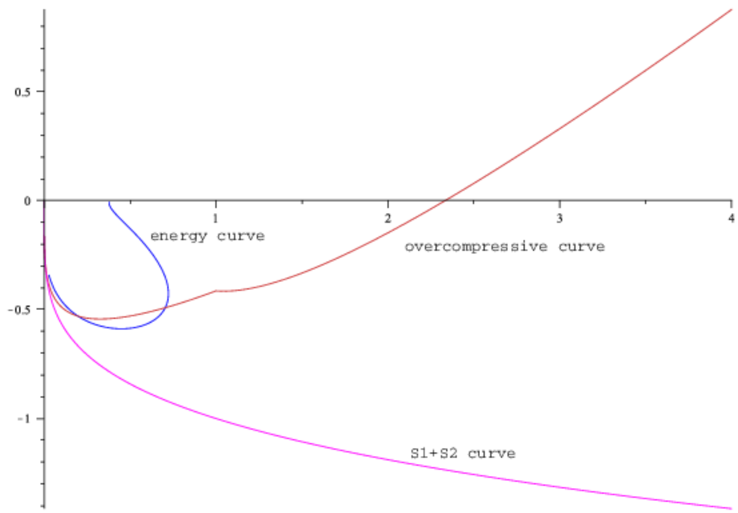

FIGURE 3. Energy entropy condition

to the entropy conditions from [17], a SDW solution (ρ, q) to (3.1) is admissible if

$$\begin{aligned}\lim_{\varepsilon\to 0} -c(\varepsilon\eta(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon\eta(\rho_{1,\varepsilon}, q_{1,\varepsilon})) + \varepsilon Q(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon Q(\rho_{1,\varepsilon}, q_{1,\varepsilon}) = 0 \\ -c(\eta(\rho_1, q_1) - \eta(\rho_0, q_0)) + Q(\rho_1, q_1) - Q(\rho_0, q_0) \\ + \lim_{\varepsilon\to 0}(\varepsilon\eta(\rho_{0,\varepsilon}, q_{0,\varepsilon}) + \varepsilon\eta(\rho_{1,\varepsilon}, q_{1,\varepsilon})) \leq 0.\end{aligned} \tag{6.7}$$

It is not so hard to find one convex entropy pair. One just have to imitate a known energy function for different gas dynamic models:

$$\eta = \frac{1}{2}\frac{q^2}{\rho} - \frac{1}{1+\alpha}\rho^{-\alpha},\ Q = \frac{1}{2}\frac{q^3}{\rho^2} - \frac{\alpha}{1+\alpha}q\rho^{-(1+\alpha)}.$$

Substitution of these functions in (5.7) gives a different set of admissible points (ρ_1, q_1) than the overcompressibility condition. But there is still a non-empty intersection of that set with a set of (ρ_1, q_1) for which there is a S1+S2 solution. Even more, the overcompressive and entropic sets of admissible states $(\rho_1, q_1 0$ are not comparable as one could see on the Figure 5.1. One can see that a situation is different in the case of Chaplygin gas when $\alpha = 1$ (see [19]). Use of the energy $\eta = \dfrac{q^2+1}{\rho}$ as a convex entropy singles out a unique solution to Riemann problem in that case.

Using the standard procedure (see [7] for example) one can find that entropy function satisfies

$$\partial_{\rho\rho}\eta + \frac{2q}{\rho}\partial_{\rho q} + \Big(\frac{q^2}{\rho^2} - \frac{\alpha}{\rho^{1+\alpha}}\Big)\partial_{qq}\eta = 0,$$

or, after a change of variables

$$v = \frac{q}{\rho} + \frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}} \text{ and } w = \frac{q}{\rho} - \frac{2\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}},$$

$$(v-w)\partial_{vw} = \frac{3+\alpha}{2(1+\alpha)}(\partial_v\eta - \partial_w\eta).$$

If we separate variables by $\eta(v,w) = f(v-w)g(v+w)$, the equation reduces to

$$\frac{g''(v+w)}{g(v+w)} = \frac{2A}{v-w}\frac{f(v-w)}{f'(v-w)} + \frac{f''(v-w)}{f(v-w)} = \lambda \in \mathbb{R}.$$

For $\lambda \le 0$ a function g can not be convex and so is η. Let $\lambda > 0$. Then $g(v+w) = C_1 e^{\sqrt{\lambda}(v+w)}$ f solves $f''(v-w) + \frac{2A}{v-w}f'(v-w) - \lambda f(v-w) = 0$. It is known that f is a linear combination of Bessel function of the second kind (denoted by $K_a(x)$). Using the original variables (ρ, q) one can see that all convex η obtained by the separation of variables are linear combination of the functions

$$\eta_1(\rho,q) := e^{\frac{2q}{\rho}\lambda}\rho^{\frac{1}{2}}K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big),\ \eta_2(\rho,q) := e^{-\frac{2q}{\rho}\lambda}\rho^{\frac{1}{2}}K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big),$$

for every $\lambda > 0$. Appropriate entropy flux functions are given by

$$\begin{aligned} Q_1(\rho,q) :=& \frac{1}{2\lambda}\rho^{\frac{1}{2}}e^{\frac{2q}{\rho}\lambda}\Bigg((2\lambda q - \rho)K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big) \\ &\cdot + 2\lambda\sqrt{\alpha}\rho^{\frac{1-\alpha}{2}}K_{\frac{2+\alpha}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\Bigg) \\ Q_2(\rho,q) :=& \frac{1}{2\lambda}\rho^{\frac{1}{2}}e^{\frac{2q}{\rho}\lambda}\Bigg((2\lambda q + \rho)K_{\frac{1}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big) \\ &\cdot - 2\lambda\sqrt{\alpha}\rho^{\frac{1-\alpha}{2}}K_{\frac{2+\alpha}{1+\alpha}}\Big(\frac{4\sqrt{\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}\lambda\Big)\Bigg). \end{aligned}$$

Definition 6.1. An SDW solution to 3.1 is said to be admissible (or entropic) one if (5.7) holds true for all entropy pairs (η_1, Q_1), (η_2, Q_2), $l > 0$.

Remarks about the literature

Good books for Conservation Law System theory: [5], [7], [21].

Other approaches to unbounded shocks:

- Early attempts [11], [22]
- Variational method [9]
- Sticky particles method [4]
- Singular shocks and special measure spaces [10]
- Weak asymptotic methods [8] and more formal version in [12]
- Split delta function [16] (for experimental verification of delta splitting phenomenon [14])
- Vanishing pressure method (two shocks converge to a delta shock) [6] (with a generalization in [15])
- Shadow Waves [17]

Chaplygin gas [3], [18]

Generalized Chaplygin gas [1], [13], [19]

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