DELTA SHOCK WAVES AS A LIMIT OF SHOCK WAVES

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Abstract. We discus the existence of delta shock waves obtained as a limit of two shock waves. For that purpose we perturb a prototype of weakly hyperbolic 2×2 system (sometimes called the "generalized pressureless gas dynamics model") by an additional term (which may be called the "generalized vanishing pressure"). The obtained perturbed system will be strongly hyperbolic and its Riemann problem is solvable by the means of usual elementary waves. As perturbation vanishes, the solution converges in the space of distribution. Specially, a pair of shock waves converge to a delta function.

After that we give a formal definition of approximate solution and prove a kind of entropy argument. The paper finishes by a discussion about delta shock wave interactions for the original system.

1. Introduction

Unbounded solutions to some conservation law systems called "delta shock waves" are described in a number of papers. Some of them are in listed as references, and one can look in [10] for additional ones. All of such solutions are not bounded variation functions but contain the Dirac delta function. They can be obtained by different procedures. All of them could be devided into two classes: measure theoretic (see [1], [7], [8], [12], [13] and [14]), and asymptotic solutions (see [2], [5], [6], [10] and [11]). Up to our knowledge, all of the systems in these papers are linear in one of dependant variables and a lot of them are weakly hyperbolic.

One can find different methods for delta shock wave verification in the papers cited above such as vanishing viscosity or some other physically motivated arguments. The present paper is an attempt to show that such kind of solution may exists as a "limit" of elementary waves for a given perturbation of the original system. For that purpose we shall take the Riemann problem for a 2×2 weakly hyperbolic system in a fairly general form and perturb it by an additional term in its flux. Unlike the standard procedure with vanishing viscosity mentioned above that produces hyperbolic-parabolic second order systems, we will get strictly hyperbolic first order system which can be solved by the means of elementary waves combination assuming the Riemann initial data. The solution to the perturbed system will represent a delta shock wave solution to the unperturbed system.

Let g be a non-decreasing function. The system we consider is

$$\begin{aligned} &\rho_t + (\rho g(u))_x = 0 \\ &(\rho u)_t + (\rho u g(u))_x = 0. \end{aligned} \tag{1}$$

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It serves as a prototype of a non-strictly hyperbolic 2×2 system sometimes called the "generalized pressureless gas dynamics" model (see [8], [13] and [14]). One can find definition of a weak solution belonging in a space of Borel measures in these papers and that is a desired limit for the perturbed solution.

Our method can be simply described as follows. First, we shall perturb the second equation in system (1) with "generalized vanishing pressure" term. Second, we shall discuss a distributional limit of a weak, bounded variation solution to the perturbed system. The limit is a Radon measure (sometimes called "signed Radon measure"), precisely the one given in the cited papers. The proposed perturbation generalizes the one given in [2] for the pressureless gas dynamics system with vanishing pressure:

$$\begin{aligned} &\rho_t + (\rho u)_x = 0 \\ &(\rho u)_t + (\rho u^2 + \varepsilon p(\rho))_x = 0, \end{aligned} \tag{2}$$

where $p(\rho) = \kappa \rho^{\gamma - 1}$, for $\gamma \in (1, 3)$. We tried to recover all results from that paper in the case when system (2) is substituted by the general one, (1). The result is satisfactory, as expected, since the weak solution for (1) is a generalization of the one for system (2).

After that, we shall propose a form of approximate solutions to the original system and prove some kind of entropy inequality. Approximated solution resembles nets of smooth function used in [10] as a solution description, but they have two crucial differences:

- they are piecewise constant functions
- instead of one shock, now we have two $x = c_{1,\varepsilon} t$ and $x = c_{2,\varepsilon} t$, where the speeds satisfy $\lim_{\varepsilon \to 0} c_{1,\varepsilon} - c_{2,\varepsilon} = 0$.

At the very end of the paper we try to give an answer on a problem of interaction of two delta shock waves in system (1). The result is only partial (also in the case of system (2)), so the interaction problem is still open. Let us mention that the delta shock wave interaction problem is solved by using the weak asymptotic method (see [5] and [6]) or splitted delta measures (see [12]) for some other, let say simpler systems. Unlike the systems above, there are some indications that the result of two delta shock waves interaction may not produce a delta shock wave with the vanishing pressure approach for system (1). One can find different clues in the literature: The solution concept from [13] does not allow two delta shock waves to interact, but with the solution concept in [8] one gets a single delta shock wave as a result of the interaction.

2. BV solutions

Let $p(0) = 0$, g' and p' be non-negative functions, p' a smooth, increasing function on its domain $(0, \infty)$ satisfying $p((0, \infty)) = (0, \infty)$. Consider the following Riemann problem in the domain $\Omega = \{(\rho, u) \in \mathbb{R}_+ \cup \{0\} \times \mathbb{R}\}$

$$\begin{aligned} &\rho_t + (\rho g(u))_x = 0 \\ &(\rho u)_t + (\rho u g(u) + \varepsilon p(\rho))_x = 0 \\ &\rho|_{t=0} = \begin{cases} \rho_0, x < 0 \\ \rho_1, x > 0 \end{cases}, v|_{t=0} = \begin{cases} v_0, x < 0 \\ v_1, x > 0 \end{cases}, \end{aligned} \tag{3}$$

where $\varepsilon \ll 1$ is a perturbation parameter.

After a change of variables $\rho u \to v$, system (3) can be written in the evolution form

$$\begin{aligned} &\rho_t + (\rho g(v/\rho))_x = 0\\ &v_t + (v g(v/\rho) + \varepsilon p(\rho))_x = 0\\ &\rho|_{t=0} = \begin{cases} \rho_0, x<0\\ \rho_1, x>0\end{cases}, v|_{t=0} = \begin{cases} v_0, x<0\\ v_1, x>0\end{cases}, \end{aligned} \tag{4}$$

which is strictly hyperbolic, genuinely nonlinear system for $\rho > 0$.

Its eigenvalues are $\lambda_1 = g(v/\rho) - \sqrt{\varepsilon g'(v/\rho)p'(\rho)}$ and $\lambda_2 = g(v/\rho) + \sqrt{\varepsilon g'(v/\rho)p'(\rho)}$, while the characteristic vectors are chosen to be

$$\begin{aligned} r_1 &= \left(-1, -\frac{v\sqrt{g'(v/\rho)} - \rho\sqrt{\varepsilon p'(\rho)}}{\rho\sqrt{g'(v/\rho)}}\right) \text{ and}\\ r_2 &= \left(1, \frac{v\sqrt{g'(v/\rho)} + \rho\sqrt{\varepsilon p'(\rho)}}{\rho\sqrt{g'(v/\rho)}}\right), \end{aligned}$$

such that $\nabla\lambda_i \cdot r_i > 0$, $i = 1, 2$.

The unperturbed system (1) has only one eigenvalue $\lambda = g(v/\rho)$ with the eigenvector $r = (1, v/\rho)$, and $\nabla\lambda \cdot r \equiv 0$.

The rarefaction curves for system (3) are defined by the following systems of differential equations

$$\begin{aligned} &\frac{d\rho}{d\sigma} = -1\\ &\frac{dv}{d\sigma} = -\frac{v\sqrt{g'(v/\rho)} - \rho\sqrt{\varepsilon p'(\rho)})}{\rho\sqrt{g'(v/\rho)})}\\ &\rho|_{\sigma=0} = \rho_0, v|_{\sigma=0} = v_0 = \rho_0 u_0, \end{aligned} \tag{5}$$

for the first rarefaction curve, and

$$\begin{aligned} &\frac{d\rho}{d\sigma} = 1\\ &\frac{dv}{d\sigma} = \frac{v\sqrt{g'(v/\rho)} + \rho\sqrt{\varepsilon p'(\rho)})}{\rho\sqrt{g'(v/\rho)})}\\ &\rho|_{\sigma=0} = \rho_0, u|_{\sigma=0} = u_0, \end{aligned} \tag{6}$$

for the second one. In both cases σ is positive. The contact discontinuity curve for system (1) is defined by the solution to the system

$$\begin{aligned} &\frac{d\rho}{d\sigma} = 1\\ &\frac{dv}{d\sigma} = \frac{v}{\rho}\\ &\rho|_{t=0} = \rho_0, u|_{t=0} = u_0, \end{aligned} \tag{7}$$

where $\sigma \in \mathbb{R}$. Solution to system (7) is given by $v/\rho = v_0/\rho_0$ or $u = u_0$ if one uses the original variables. By the basic theorem in ODE's theory, one can see that the solutions to (5) and (6) tend to the solution to (7) as $\varepsilon \to 0$ (the right-hand sides of systems (5) and (6) tends to the right-hand side of system (7)). Since ρ is positive in the interior of the domain Ω, and p is an increasing function, one can see that v from (5) and (6) satisfies $v > \rho v_0/\rho_0$, i.e. the rarefaction curves for system (3) are above contact discontinuity curve for system (1). That means that rarefaction wave solution for (3) exists if $v_1/\rho_1 > v_0/\rho_0$, i.e. $u_1 > u_0$. For a given initial data (ρ_0, u_0) and (ρ_1, u_1) there exists a small enough ε such that one can always find a solution consisting of two rarefaction waves connected by the vacuum state ($\rho = 0$). In that vacuum state variable u takes values from u_m^1 to u_m^2, where $(0, u_m^1)$ is intersection point of the first rarefaction curve starting at (ρ_0, u_0) with the line $\rho = 0$ and $(0, u_m^2)$ is intersection point of the inverse second rarefaction curve starting at (ρ_1, u_1) with the line $\rho = 0$. One can find details of the above explanation in Section 4. Such a solution to (3) tends to the BV solution to (1) consisting of two contact discontinuities connected by vacuum state.

In the case when $u_0 = u_1$ (i.e. $v_0/\rho_0 = v_1/\rho_1$) a solution to (3) will be a rarefaction wave followed by a shock one or vice versa, depending on a sign of $\rho_0 - \rho_1$. In both cases the limit is a contact discontinuity. See Section 4 for a detailed explanation, too.

A situation is a bit more complex in the case $u_0 > u_1$ when we expect a delta shock wave as a limit (solution obtained previously in the literature, see [8] for example). Shock curves for (3) are below the contact discontinuity line $u_0 = u_1$ for system (1). One will see in next section that the solution is always a shock wave followed by another one. The crucial assumption is that $g'(u) > 0$. Unlike the pressureless gas dynamic model (where $g(u) \equiv u$), we do not have explicit values for an intermediate state $(\rho_\varepsilon, u_\varepsilon)$ between two shock waves. The mast we have is it exists. But that was enough to successfully finish the analysis of that case. The limit of such solution tends in the sense of distributions to the one obtained in [8].

3. Shock waves

The main result of the paper is given in the following theorem.

Theorem 1. *For each pair of initial states (ρ_0, v_0), (ρ_1, v_1) such that $u_1 = v_1/\rho_1 < u_0 = v_0/\rho_0$, there exists ε small enough such that the Riemann problem for (3) has a unique solution consisting of two shock waves. Let us denote the solution to that problem by $(\rho_\varepsilon, u_\varepsilon)$. The distributional limit of this solution is given by*

$$\begin{aligned} &\lim_{\varepsilon\to 0} \rho_\varepsilon = G(x-ct) + t\cdot \mathrm{const}\cdot\delta(x-ct),\\ &\lim_{\varepsilon\to 0} u_\varepsilon = H(x-ct), \end{aligned} \tag{8}$$

where G and H are step functions with values (ρ_0, ρ_1) and (u_0, u_1), resp.

Proof. We shall use variables ρ and u in the proof. Let $u_1 < u_0$. Assume that for $(\rho_\varepsilon, u_\varepsilon) \in \mathbb{R}_+ \times \mathbb{R}$ there exists a solution to (3) such that (ρ_0, u_0) is connected with $(\rho_\varepsilon, u_\varepsilon)$ by an one-shock wave and $(\rho_\varepsilon, u_\varepsilon)$ is connected with (ρ_0, u_0) by a two-shock wave. Denote by c_1 and c_2 the speeds of the first and second wave, respectively.

Rankine-Hugoniot conditions for the shocks are determined by the system

$$(u-u_0)(g(u)-g(u_0)) = \varepsilon\left(\tfrac{1}{\rho_0}-\tfrac{1}{\rho}\right)(p(\rho)-p(\rho_0)) \tag{9}$$

$$(u-u_1)(g(u)-g(u_1)) = \varepsilon\left(\tfrac{1}{\rho_1}-\tfrac{1}{\rho}\right)(p(\rho)-p(\rho_1)). \tag{10}$$

Our first task is to prove that there exists a solution $(\rho_\varepsilon, u_\varepsilon) \in (\max\{\rho_0, \rho_1\}, \infty) \times (u_1, u_0)$ to system of nonlinear equations (9-10), i.e. that the two shock wave combination is really a solution to (3).

Immediately it follows that for every $u < u_0$ there exists a unique $\rho > \rho_0$ such that (9) is true. This is consequence of the fact that p and g are increasing functions. That means that there exists a function $\rho_{c1} : (u_1, u_0) \to (\rho_0, \infty)$, $\rho_{c1}(u_0) = \rho_0$, such that (9) holds true for $(\rho, u) = (\rho_{c1}(u), u)$. Additionally, one can check that ρ_{c1} is a decreasing function: Denote by

$$F(u) := \varepsilon \left(\frac{1}{\rho_0} - \frac{1}{\rho_{c1}(u)} \right) (p(\rho_{c1}(u)) - p(\rho_0)).$$

Then one can see that $F'(u) < 0$ using the left-hand side of (9):

$$F'(u) = \frac{\mathrm{d}}{\mathrm{d}u}((u - u_0)(g(u) - g(u_0))) = g(u) - g(u_0) + g'(u)(u - u_0) < 0,$$

since $g' > 0$ and $u < u_0$. On the other hand we have

$$\begin{aligned} F'(u) =& \frac{\varepsilon}{\rho_{c1}^2(u)} \rho_{c1}'(u)(p(\rho_{c1}(u)) - p(\rho_0)) \\ &+ \varepsilon \left(\frac{1}{\rho_0} - \frac{1}{\rho_{c1}(u)} \right) p'(\rho_{c1}(u))\rho_{c1}'(u) \\ =& \varepsilon \rho_{c1}'(u) \left(\frac{p(\rho_{c1}(u)) - p(\rho_0)}{\rho_{c1}^2(u)} + \left(\frac{1}{\rho_0} - \frac{1}{\rho_{c1}(u)} \right) p'(\rho_{c1}(u)) \right). \end{aligned}$$

Using the fact that $p' > 0$ and $\rho_{c1}(u) > \rho_0$ for $u \in (u_1, u_0)$, one can see that $\frac{p(\rho_{c1}(u)) - p(\rho_0)}{\rho_{c1}^2(u)} + \left(\frac{1}{\rho_0} - \frac{1}{\rho_{c1}(u)} \right) p'(\rho_{c1}(u)) > 0$. Thus, $\mathrm{sign}(F'(u)) = \mathrm{sign}(\rho_{c1}'(u))$, i.e. $\rho_{c1}' < 0$ in the interval $u \in (u_1, u_0)$.

In the same way as above one can see that there exists an increasing function $\rho_{c2} : (u_1, u_0) \to (\rho_1, \infty)$, $\rho_{c2}(u_1) = \rho_1$ defined by (10):

$$(u - u_1)(g(u) - g(u_1)) = \varepsilon \left(\frac{1}{\rho_1} - \frac{1}{\rho_{c2}(u)} \right) (p(\rho_{c2}(u)) - p(\rho_1)).$$

Additionally, we have that $\rho_{c1}(u_1) > \rho_1$ for ε small enough, since the term on the left-hand side of (9) is bounded with respect to ε, and $\rho_{c2}(u_0) > \rho_0$, because of the similar reasons. These properties of functions ρ_{c1} and ρ_{c2} ensures that there exist unique interaction point $(\rho_\varepsilon, u_\varepsilon) \in (\min\{\rho_0, \rho_1\}, \infty) \times (u_1, u_0)$ of curves defined by these functions. Therefore $\rho = \rho_\varepsilon$ and $u = u_\varepsilon$ define the intermediate point of two shock wave weak solution to (3). On the other hand, from (9) and (10) it follows that the value $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$, while u_ε stays bounded. More precisely $p(\rho_\varepsilon) \sim 1/\varepsilon$ as $\varepsilon \to 0$. Thus, we can assume that $(\rho_\varepsilon, u_\varepsilon) \in (\max\{\rho_0, \rho_1\}, \infty) \times (u_1, u_0)$.

In order to obtain the distributional limit of the above solution as $\varepsilon \to 0$ one has to prove that $\lim_{\varepsilon \to 0} u_\varepsilon$ and $\lim_{\varepsilon \to 0} \varepsilon p(\rho_\varepsilon)$ do exist. Let us define

$$\begin{aligned} F(\varepsilon, \rho, u) &:= (u_0 - u)(g(u_0) - g(u)) - \varepsilon \left(\frac{1}{\rho_0} - \frac{1}{\rho} \right) (p(\rho) - p(\rho_0)) = 0 \\ G(\varepsilon, \rho, u) &:= (u_1 - u)(g(u_1) - g(u)) - \varepsilon \left(\frac{1}{\rho_1} - \frac{1}{\rho} \right) (p(\rho) - p(\rho_1)) = 0. \end{aligned} \tag{11}$$

Since

$$
\begin{aligned}
D_s =& \begin{vmatrix} \frac{\partial F}{\partial \rho} & \frac{\partial F}{\partial u} \\ \frac{\partial G}{\partial \rho} & \frac{\partial G}{\partial u} \end{vmatrix} \\
=& \begin{vmatrix} -\varepsilon\left(\frac{1}{\rho^2}(p(\rho)-p(\rho_0)) + \left(\frac{1}{\rho_0}-\frac{1}{\rho}\right)p'(\rho)\right) & g(u)-g(u_0)+(u-u_0)g'(u) \\ -\varepsilon\left(\frac{1}{\rho^2}(p(\rho)-p(\rho_1)) + \left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)p'(\rho)\right) & g(u)-g(u_1)+(u-u_1)g'(u) \end{vmatrix} \\
=& -\varepsilon\left(\frac{1}{\rho^2}(p(\rho)-p(\rho_0)) + \left(\frac{1}{\rho_0}-\frac{1}{\rho}\right)p'(\rho)\right)(g(u)-g(u_1)+(u-u_1)g'(u)) \\
&+\varepsilon\left(\frac{1}{\rho^2}(p(\rho)-p(\rho_1)) + \left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)p'(\rho)\right)(g(u)-g(u_0)+(u-u_0)g'(u)) < 0,
\end{aligned}
$$

on the domain $(0,1)\times(\max\{\rho_0,\rho_1\},\infty)\times(u_1,u_0)$, the Implicit Function Theorem implies that it is possible to define $\rho_\varepsilon := \rho(\varepsilon)$ and $u_\varepsilon := u(\varepsilon)$ from (11). Also, from the same theorem one can see that

$$
\frac{\partial \rho}{\partial \varepsilon} = \begin{vmatrix} -\frac{\partial F}{\partial \varepsilon} & \frac{\partial F}{\partial u} \\ -\frac{\partial G}{\partial \varepsilon} & \frac{\partial G}{\partial u} \end{vmatrix} \cdot D_s^{-1}, \quad \frac{\partial u}{\partial \varepsilon} = \begin{vmatrix} \frac{\partial F}{\partial \rho} & -\frac{\partial F}{\partial \varepsilon} \\ \frac{\partial G}{\partial \rho} & -\frac{\partial G}{\partial \varepsilon} \end{vmatrix} \cdot D_s^{-1}.
$$

By simple calculations one finds that

$$
\begin{aligned}
\begin{vmatrix} -\frac{\partial F}{\partial \varepsilon} & \frac{\partial F}{\partial u} \\ -\frac{\partial G}{\partial \varepsilon} & \frac{\partial G}{\partial u} \end{vmatrix} =& \begin{vmatrix} \left(\frac{1}{\rho_0}-\frac{1}{\rho}\right)(p(\rho)-p(\rho_0)) & g(u)-g(u_0)+(u-u_0)g'(u) \\ \left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)(p(\rho)-p(\rho_1)) & g(u)-g(u_1)+(u-u_1)g'(u) \end{vmatrix} \\
=& \left(\frac{1}{\rho_0}-\frac{1}{\rho}\right)(p(\rho)-p(\rho_0))(g(u)-g(u_1)+(u-u_1)g'(u)) \\
&-\left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)(p(\rho)-p(\rho_1))(g(u)-g(u_0)+(u-u_0)g'(u)) > 0.
\end{aligned}
$$

Since D_s¡0, the above means that $\rho(\varepsilon)$ is a decreasing function, i.e. $\rho_\varepsilon \to \infty$ as $\varepsilon \to 0$.

Also,

$$
\begin{aligned}
& \begin{vmatrix} \frac{\partial F}{\partial \rho} & -\frac{\partial F}{\partial \varepsilon} \\ \frac{\partial G}{\partial \rho} & -\frac{\partial G}{\partial \varepsilon} \end{vmatrix} \\
=& \begin{vmatrix} -\varepsilon\left(\frac{1}{\rho^2}(p(\rho)-p(\rho_0)) + \left(\frac{1}{\rho_0}-\frac{1}{\rho}\right)p'(\rho)\right) & \left(\frac{1}{\rho_0}-\frac{1}{\rho}\right)(p(\rho)-p(\rho_0)) \\ -\varepsilon\left(\frac{1}{\rho^2}(p(\rho)-p(\rho_1)) + \left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)p'(\rho)\right) & \left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)(p(\rho)-p(\rho_1)) \end{vmatrix} \\
=& \varepsilon\frac{1}{\rho^2}(p(\rho)-p(\rho_0))(p(\rho)-p(\rho_1))\left(\frac{1}{\rho_0}-\frac{1}{\rho_1}\right) \\
&+\varepsilon\left(\frac{1}{\rho_0}-\frac{1}{\rho}\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho}\right)(p(\rho_1)-p(\rho_0)).
\end{aligned}
$$

If $\rho_0 \le \rho_1$, then $\frac{1}{\rho_0}-\frac{1}{\rho_1} \le 0$ and $p(\rho_1)-p(\rho_0) \le 0$. Now, the above determinant is positive for ε small enough and $u(\varepsilon)$ is an increasing function, i.e. u_ε decreases as $\varepsilon \to 0$. On the other hand, if $\rho_0 > \rho_1$, then $\frac{1}{\rho_0}-\frac{1}{\rho_1} > 0$, $p(\rho_1)-p(\rho_0) > 0$, and $u(\varepsilon)$ is a decreasing function, i.e. u_ε increases as $\varepsilon \to 0$.

In both of the cases, the sequence u_ε is a monotone one, and since it is bounded, there exists $\lim_{\varepsilon\to 0} u_\varepsilon =: u_s$. Now, it easy to see that (9) implies

$$\lim_{\varepsilon\to 0} \varepsilon p(\rho_\varepsilon) = \rho_0(u_0-u_s)(g(u_0)-g(u_s)),$$

or that (10) implies

$$\lim_{\varepsilon\to 0} \varepsilon p(\rho_\varepsilon) = \rho_1(u_1-u_s)(g(u_1)-g(u_s)).$$

Thus, u_s is a unique solution to the equation

$$f(u_s) := \rho_0(u_0-u_s)(g(u_0)-g(u_s)) - \rho_1(u_1-u_s)(g(u_1)-g(u_s)) = 0. \tag{12}$$

Uniqueness follows from the fact that

$$\begin{aligned} f'(u_s) =& \rho_0(g(u_s)-g(u_0)+(u_s-u_0)g'(u_s)) \\ &+ \rho_1(g(u_1)-g(u_s)+(u_1-u_s)g'(u_s)) < 0. \end{aligned}$$

Now we have to show that these shock waves are admissible (in Lax sense).The speeds of the shock waves are given by

$$c_{1,\varepsilon} = \frac{\rho_\varepsilon g(u_\varepsilon)-\rho_0 g(u_0)}{\rho_\varepsilon-\rho_0} \text{ and } c_{2,\varepsilon} = \frac{\rho_1 g(u_1)-\rho_\varepsilon g(u_\varepsilon)}{\rho_1-\rho_\varepsilon}. \tag{13}$$

The first shock is admissible if

$$g(u_0) - \sqrt{\varepsilon g'(u_0)p'(\rho_0)} > \frac{\rho_\varepsilon g(u_\varepsilon)-\rho_0 g(u_0)}{\rho_\varepsilon-\rho_0} > g(u_\varepsilon) - \sqrt{\varepsilon g'(u_\varepsilon)p'(\rho_\varepsilon)}.$$

These inequalities imply

$$\frac{\rho_\varepsilon(g(u_\varepsilon)-g(u_0))}{\rho_\varepsilon-\rho_0} < -\sqrt{\varepsilon g'(u_0)p'(u_0)} \tag{14}$$

and

$$\frac{\rho_0(g(u_\varepsilon)-g(u_0))}{\rho_\varepsilon-\rho_0} > -\sqrt{\varepsilon g'(u_\varepsilon)p'(\rho_\varepsilon)}. \tag{15}$$

The left-hand side of (14) tends to $g(u_s)-g(u_0)<0$, while the term on the right-hand side tends to 0 as $\varepsilon\to 0$. Thus, (14) is satisfied for ε small enough. To prove inequality (15) we substitute ε by $1/p(\rho_\varepsilon)$, and using the fact that $\rho_\varepsilon\to\infty$ we have

$$\frac{p(\rho)}{\rho^2} \sim \frac{p'(\rho)}{\rho} \ll p'(\rho) \text{ as } \rho\to\infty.$$

This proves (15).

The second shock wave is admissible if

$$g(u_\varepsilon) + \sqrt{\varepsilon g'(u_\varepsilon)p'(\rho_\varepsilon)} > \frac{\rho_1 g(u_1)-\rho_\varepsilon g(u_\varepsilon)}{\rho_1-\rho_\varepsilon} > g(u_\varepsilon) + \sqrt{\varepsilon g'(u_1)p'(\rho_1)}.$$

These inequalities imply

$$\frac{\rho_1(g(u_1)-g(u_\varepsilon))}{\rho_1-\rho_\varepsilon} < \sqrt{\varepsilon g'(u_\varepsilon)p'(u_\varepsilon)} \tag{16}$$

and

$$\frac{\rho_\varepsilon(g(u_1)-g(u_\varepsilon))}{\rho_1-\rho_\varepsilon} > \sqrt{\varepsilon g'(u_1)p'(\rho_1)}. \tag{17}$$

The left-hand side of (17) tends to $-(g(u_1)-g(u_s))>0$, while the term on the right-hand side tends to 0 as $\varepsilon\to 0$. Thus, (17) is satisfied for ε small enough. Proof of inequality (16) can be done in the same way as the one for (15).

The direct calculation gives

$$c_{2,\varepsilon} - c_{1,\varepsilon} = (\rho_0 g(u_0) - \rho_1 g(u_1) + g(u_\varepsilon)(\rho_1 - \rho_0))/\rho_\varepsilon \to 0,$$

and since $(c_{2,\varepsilon} - c_{1,\varepsilon})t\rho_\varepsilon \to \rho_0 g(u_0) - \rho_1 g(u_1) + g(u_s)(\rho_1 - \rho_0)$, the solution component $\rho(x,t) \to G(x-ct) + t(\rho_0 g(u_0) - \rho_1 g(u_1) + g(u_s)(\rho_1 - \rho_0))\delta(x-ct)$, where $c = \lim_{\varepsilon\to 0} c_{1,\varepsilon} = \lim_{\varepsilon \to 0} c_{2,\varepsilon}$. This completes the proof. □

Remark 1. Lax entropy conditions can be true only if $\rho_\varepsilon \geq \max\{\rho_0, \rho_1\}$ because S_1 is given for ρ increasing while S_2 is given for ρ decreasing. Therefore the choice of the domain where (ρ, u) lie for system (9-10) is appropriate.

4. Other elementary wave combinations

In the previous section we were dealing with the initial data satisfying $u_0 > u_1$. We are now looking at other two possibilities.

4.1. $u_0 = u_1$**.** Let the initial conditions for system (3) be

$$\begin{aligned} \rho|_{t=0} &= \begin{cases} \rho_0, & x<0 \\ \rho_1, & x>0 \end{cases} \\ u|_{t=0} &= u_0. \end{aligned} \tag{18}$$

There are two cases (the case $\rho_0 = \rho_1$ is trivial: $u \equiv u_0$, $\rho \equiv \rho_0$ is the solution).

In the first case we assume that $\rho_1 < \rho_0$. We will show that the states (u_0, ρ_1) and (u_0, ρ_0) can be connected by a rarefaction wave (R_1) followed by shock one (S_2). An S_2 connection of the state (ρ, u) with the end state (ρ_1, u_0) satisfies

$$(u-u_0)(g(u) - g(u_0)) = \varepsilon\left(\frac{1}{\rho_1} - \frac{1}{\rho}\right)(p(\rho) - p(\rho_1)), \tag{19}$$

where $u > u_0$ and $\rho > \rho_1$ (see (10)). Let us prove that for every fixed $\rho > \rho_1$ there exists a unique $u > u_0$ such that (19) holds. Define

$$G(u) := (u-u_0)(g(u) - g(u_0)).$$

Since $F(u_0) = 0$ and g is increasing we have $G(u) \to \infty$ as $u \to \infty$, and $G([u_0, \infty)) = [0, \infty)$. The right-hand side of (19) is positive for every $\rho > \rho_1$, so there exists a $u > u_0$ such that

$$G(u) = \varepsilon\left(\frac{1}{\rho_1} - \frac{1}{\rho}\right)(p(\rho) - p(\rho_1)).$$

Also,

$$G'(u) = g(u) - g(u_0) + (u-u_0)g'(u) > 0,$$

since g is increasing and $u > u_0$. That means that such u is unique.

Denote by $u_{S_2} = u_{S_2}(\rho)$ the function defined by

$$(u_{S_2}(\rho) - u_0)(g(u_{S_2}(\rho)) - g(u_0)) = \varepsilon\left(\frac{1}{\rho_1} - \frac{1}{\rho}\right)(p(\rho) - p(\rho_1)). \tag{20}$$

We have seen above that $u_{S_2}(\rho_1) = u_0$, $u_{S_2} : [\rho_1, \rho_0) \to [u_0, +\infty)$ and the function u_{S_2} is well defined. It is increasing in the interval (ρ_1, ρ_0): Differentiation of (20) gives

$$\begin{aligned} &u'_{S_2}(\rho)\left(g(u_{S_2}(\rho)) - g(u_0) + (u_{S_2}(\rho) - u_0)g'(u_{S_2}(\rho))\right) \\ =&\frac{\varepsilon}{\rho^2}(p(\rho) - p(\rho_0)) + \varepsilon\left(\frac{1}{\rho_1} - \frac{1}{\rho}\right)p'(\rho). \end{aligned}$$

Since p and g are increasing functions and $u > u_0$ as well as $\rho > \rho_1$ we conclude that $u'_{S_2}(\rho) > 0$, i.e. $u_{S_2}(\rho_0) > u_0$.

Similarly, we define the function $u_{R_1} : (\rho_1, \rho_0) \to (u_0, +\infty)$ to be the solution to the differential equation for 1-rarefaction wave:

$$\frac{\mathrm{d}u_{R_1}}{\mathrm{d}\rho} = -\frac{\sqrt{\varepsilon p'(\rho)}}{\rho\sqrt{g'(u_{R_1}(\rho))}}, \quad u_{R_1}(\rho_0) = u_0.$$

Since p and g are increasing function one can see that u_{R_1} is decreasing. Therefore, $u_{R_1}(\rho_1) > u_0$.

From the above we conclude that there exists $\rho_m \in (\rho_1, \rho_0)$ such that $u_m = u_{S_2}(\rho_m) = u_{R_1}(\rho_m)$, i.e. the solution of problem (3), (18) is given by:

$$\begin{aligned}
\rho_\varepsilon(x,t) &= \begin{cases} \rho_0, & x < \lambda_1(u_0,\rho_1)t \\ R_{1,\rho}(x/t), & \lambda_1(u_0,\rho_1)t < x < \lambda_1(u_m,\rho_m)t \\ \rho_m, & \lambda_1(u_m,\rho_m)t < x < ct \\ \rho_1, & x > ct \end{cases} \\
u_\varepsilon(x,t) &= \begin{cases} u_0, & x < \lambda_1(u_0,\rho_1)t \\ R_{1,u}(x/t), & \lambda_1(u_0,\rho_1)t < x < \lambda_1(u_m,\rho_m)t \\ u_m, & \lambda_1(u_m,\rho_m)t < x < ct \\ u_1, & x > ct. \end{cases}
\end{aligned} \tag{21}$$

Here, $c = \dfrac{g(u_1)\rho_1 - g(u_m)\rho_m}{\rho_1 - \rho_m}$ is the speed of S_2 given by Rankine-Hugoniot condition, and $(R_{1,\rho}, R_{1,u})$ is a non-constant part of R_1.

The case when $\rho_0 < \rho_1$ can be done in a similar way. In that case solution consists from S_1 followed by R_2 and is given by

$$\begin{aligned}
\rho_\varepsilon(x,t) &= \begin{cases} \rho_0, & x < ct \\ \rho_m, & ct < x < \lambda_2(u_m,\rho_m)t \\ R_{2,\rho}(x/t), & \lambda_2(u_m,\rho_m)t < x < \lambda_2(u_0,\rho_1)t \\ \rho_1, & x > \lambda_2(u_0,\rho_1)t \end{cases} \\
u_\varepsilon(x,t) &= \begin{cases} u_0, & x < ct \\ u_m, & ct < x < \lambda_2(u_m,\rho_m) \\ R_{2,u}(x/t), & \lambda_2(u_m,\rho_m)t < x < \lambda_2(u_0,\rho_1) \\ u_1, & x > \lambda_2(u_0,\rho_1)t. \end{cases}
\end{aligned} \tag{22}$$

Here, $c = \dfrac{g(u_m)\rho_m - g(u_0)\rho_0}{\rho_m - \rho_0}$ is the speed of S_1 and $(R_{2,\rho}, R_{2,u})$ is a non-constant part of R_2.

It remains to find limits of (21) and (22) as $\varepsilon \to 0$. Since $\rho \in (\rho_0, \rho_1)$ or $\rho \in (\rho_1, \rho_0)$, ρ is bounded in ε. Also, (19) implies that u_m and ρ_m satisfy

$$(u_m - u_0)(g(u_m) - g(u_0)) = \varepsilon\left(\frac{1}{\rho_1} - \frac{1}{\rho_m}\right)(p(\rho_m) - p(\rho_1)),$$

and

$$\lim_{\varepsilon\to 0}(u_m - u_0)(g(u_m) - g(u_0)) = 0,$$

which implies $\lim_{\varepsilon\to 0} u_m = u_0$ since g is increasing. Boundedness of ρ implies

$$\lambda_i(\rho_m, u_m) \to g(u_0), \quad i = 1, 2, \text{ as } \varepsilon \to 0.$$

Thus the solution $(u_\varepsilon, \rho_\varepsilon)$ given by (21) or (22) tends to the contact discontinuity

$$\rho(x,t) = \begin{cases} \rho_0, & x < g(u_0) \\ \rho_1, & x > g(u_0) \end{cases}$$
$$u(x,t) = u_0$$

for system (1).

4.2. $u_0 < u_1$. In that case the rarefaction curves for problem (3) are given by the following formulas.

The 1-rarefaction curve starting from (ρ_0, u_0) is defined by

$$\frac{\mathrm{d}u_{R_1}}{\mathrm{d}\rho} = -\frac{\sqrt{\varepsilon p'(\rho)}}{\rho\sqrt{g'(u_{R_1}(\rho))}}, \quad u_{R_2}(\rho_0) = u_0,\ \rho < \rho_0,$$

so

$$u_{R_1}(\rho) = -\sqrt{\varepsilon}\int_{\rho_0}^{\rho} \frac{\sqrt{p'(\rho')}}{\rho'\sqrt{g'(u_{R_1}(\rho'))}}\mathrm{d}\rho' + u_0.$$

Since $\rho \in [0, \rho_0]$ is bounded and $g' > 0$ we have

$$u_{R1}(\rho) \to u_0,\ \varepsilon \to 0, \quad \rho \in [0, \rho_0]. \tag{23}$$

The 2-rarefaction curve with the starting point (ρ, u) and the end point (ρ_1, u_1) is a solution to

$$\frac{\mathrm{d}u_{R_2}}{\mathrm{d}\rho} = \frac{\sqrt{\varepsilon p'(\rho)}}{\rho\sqrt{g'(u_{R_2})}}, \quad u_{R_2}(\rho_1) = u_1,\ \rho < \rho_1,$$

i.e.

$$u_{R_2}(\rho) = \sqrt{\varepsilon}\int_{\rho}^{\rho_1} \frac{\sqrt{p'(\rho')}}{\rho'\sqrt{g'(u_{R_1}(\rho'))}}\mathrm{d}\rho' + u_1.$$

The variable ρ lies in $[0, \rho_1]$ and $g' > 0$, so

$$u_{R2}(\rho) \to u_1, \varepsilon \to 0, \rho \in [0, \rho_0]. \tag{24}$$

Since we have assumed that $u_1 > u_0$, relations (23) and (24) imply that

$$u_{R_1}(0) > u_{R_2}(0),$$

diminishing ε if necessary.

In this case, the solution is given as a combination of two rarefaction waves connected by the vacuum state:

$$\begin{gathered}
\rho_\varepsilon(x,t) = \begin{cases} \rho_0, \quad x < \lambda_1(u_0,\rho_0) \\ R_{1,\rho}(x/t), \quad \lambda_1(u_0,\rho_0)t \le x < \lambda_1(u_m^1, 0)t \\ 0, \quad \lambda_1(u_m^1,0)t \le x < \lambda_2(u_m^2,0)t \\ R_{2,\rho}(x/t), \quad \lambda_2(u_m^2,0)t \le x < \lambda_2(u_1,\rho_1)t \\ \rho_1, \lambda_2(u_1,\rho_1)t \end{cases} \\
u_\varepsilon(x,t) = \begin{cases} u_0, \quad x < \lambda_1(u_0,\rho_0) \\ R_{1,u}(x/t), \quad \lambda_1(u_0,\rho_0)t \le x < \lambda_1(u_m^1, 0)t \\ \text{const}, \quad \lambda_1(u_m^1,0)t \le x < \lambda_2(u_m^2,0)t \\ R_{2,u}(x/t), \quad \lambda_2(u_m^2,0)t \le x < \lambda_2(u_1,\rho_1)t \\ u_1, \lambda_2(u_1,\rho_1)t \end{cases}
\end{gathered} \tag{25}$$

We have to inspect limits of u_m^1 and u_m^2 as $\varepsilon \to 0$.

We have seen above that $u_m^1 = u_{R_1}(0) \to u_0$. In the same way we have $\lim_{\varepsilon\to 0} u_m^2 = u_1$.

So, in the case of $u_0 < u_1$ the solution to (3) given by (25) tends to the combination of two contact discontinuities joined by the vacuum state,

$$\rho(x,t) = \begin{cases} \rho_0, x < g(u_0)t, \\ 0, g(u_0)t < x < g(u_1)t \\ \rho_1, x > g(u_1)t \end{cases}$$

$$u(x,t) = \begin{cases} u_0, x < g(u_0)t \\ \text{const}, g(u_0)t < x < g(u_1)t \\ u_1, x > g(u_1)t. \end{cases}$$

5. Approximated solutions

Delta shock wave solution to (1) can be viewed as a net of piecewise constant functions $(\rho_\varepsilon, u_\varepsilon)$ satisfying initial conditions and the system in approximated sense, i.e..

$$\int \rho_\varepsilon \varphi_t + \rho_\varepsilon g(u_\varepsilon)\varphi_x \to 0$$

$$\int \rho_\varepsilon u_\varepsilon \psi_t + \rho_\varepsilon u_\varepsilon g(u_\varepsilon)\varphi_x \to 0,$$

as $\varepsilon \to 0$, for every pair of test functions φ and ψ in $C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$.

Using the results in the third section one can see that the functions defined by

$$\begin{aligned} u_\varepsilon(x,t) &= \begin{cases} u_0, & x < c_{1,\varepsilon}t \\ u_\varepsilon, & c_{1,\varepsilon}t < x < c_{2,\varepsilon}t \\ u_1, & x > c_{2,\varepsilon}t \end{cases} \\ \rho_\varepsilon(x,t) &= \begin{cases} \rho_0, & x < c_{1,\varepsilon}t \\ \rho_\varepsilon, & c_{1,\varepsilon}t < x < c_{2,\varepsilon}t \\ \rho_1, & x > c_{2,\varepsilon}t \end{cases} \end{aligned} \tag{26}$$

for $c_{1,\varepsilon}$, $c_{2,\varepsilon}$, u_s, ρ_ε determined there, is a solution to (3) if $u_1 = v_1/\rho_1 < u_0 = v_0/\rho_0$.

Our first task is to prove that (26) is the approximated solution to (1). Second, we shall show that (26) satisfies a kind of entropy inequality.

Theorem 2. *Functions given by (26) with $c_{1,\varepsilon}$, $c_{2,\varepsilon}$, u_ε, ρ_ε obtained in Section 3 defines an approximate solution to (1).*

Proof. We will calculate limits after substitution of step functions for every fixed ε into (1). Let us denote by Γ_1, Γ_2 and Γ the sets $\{x = c_{1,\varepsilon}t\}$, $\{x = c_{2,\varepsilon}t\}$ and $\{x = ct\}$, resp. Substitution of (26) into the first equation of (1) gives

$$\begin{aligned} A_\varepsilon =& - c_{1,\varepsilon}(\rho_\varepsilon - \rho_0)\delta_{\Gamma_1} - c_{2,\varepsilon}(\rho_1 - \rho_\varepsilon)\delta_{\Gamma_2} \\ &+ (\rho_\varepsilon g(u_\varepsilon) - \rho_0 g(u_0))\delta_{\Gamma_1} + (\rho_1 g(u_1) - \rho_\varepsilon g(u_\varepsilon))\delta_{\Gamma_2} \\ =& - (\rho_\varepsilon g(u_\varepsilon) - \rho_0 g(u_0))\delta_{\Gamma_1} - (\rho_1 g(u_1) - \rho_\varepsilon g(u_\varepsilon))\delta_{\Gamma_2} \\ &+ (\rho_\varepsilon g(u_\varepsilon) - \rho_0 g(u_0))\delta_{\Gamma_1} + (\rho_1 g(u_1) - \rho_\varepsilon g(u_\varepsilon))\delta_{\Gamma_2} = 0, \end{aligned}$$

where we have used that

$$c_{1,\varepsilon} = \frac{\rho_\varepsilon g(u_\varepsilon) - \rho_0 g(u_0)}{\rho_\varepsilon - \rho_0} \text{ and } c_{2,\varepsilon} = \frac{\rho_1 g(u_1) - \rho_\varepsilon g(u_\varepsilon)}{\rho_1 - \rho_\varepsilon}.$$

Substitution in the second equation gives

$$\begin{aligned}
B_\varepsilon =& - c_{1,\varepsilon}(\rho_\varepsilon u_\varepsilon - \rho_0 u_0)\delta_{\Gamma_1} - c_{2,\varepsilon}(\rho_1 u_1 - \rho_\varepsilon u_\varepsilon)\delta_{\Gamma_2} \\
&+ (\rho_\varepsilon u_\varepsilon g(u_\varepsilon) - \rho_0 u_0 g(u_0))\delta_{\Gamma_1} + (\rho_1 u_1 g(u_1) - \rho_\varepsilon u_\varepsilon g(u_\varepsilon))\delta_{\Gamma_2} \\
=& \frac{-(\rho_\varepsilon g(u_\varepsilon) - \rho_0 g(u_0))(\rho_\varepsilon u_\varepsilon - \rho_0 u_0) + (\rho_\varepsilon - \rho_0)(\rho_\varepsilon u_\varepsilon g(u_\varepsilon) - \rho_0 u_0 g(u_0))}{\rho_\varepsilon - \rho_0}\delta_{\Gamma_1} \\
&+ \frac{(\rho_\varepsilon g(u_\varepsilon) - \rho_1 g(u_1))(\rho_1 u_1 - \rho_\varepsilon u_\varepsilon) + (\rho_1 - \rho_\varepsilon)(\rho_1 u_1 g(u_1) - \rho_\varepsilon u_\varepsilon g(u_\varepsilon))}{\rho_1 - \rho_\varepsilon}\delta_{\Gamma_2} \\
=& \frac{\rho_\varepsilon \rho_0 u_\varepsilon g(u_0) + \rho_\varepsilon \rho_0 u_0 g(u_\varepsilon) - \rho_\varepsilon \rho_0 u_0 g(u_0) - \rho_\varepsilon \rho_0 u_\varepsilon g(u_\varepsilon)}{\rho_\varepsilon - \rho_0}\delta_{\Gamma_1} \\
&+ \frac{\rho_\varepsilon \rho_1 u_1 g(u_\varepsilon) + \rho_\varepsilon \rho_1 u_\varepsilon g(u_1) - \rho_\varepsilon \rho_1 u_\varepsilon g(u_\varepsilon) - \rho_\varepsilon \rho_1 u_1 g(u_1)}{\rho_1 - \rho_\varepsilon}\delta_{\Gamma_2}.
\end{aligned}$$

The coefficients in front of δ_{Γ_1} and δ_{Γ_2} converge as $\varepsilon \to 0$ (because $\rho_\varepsilon \to \infty$, as $\varepsilon \to 0$). Also, Γ_1 and Γ_2 tends to Γ as $\varepsilon \to 0$. Therefore, the distributional limit of B_ε is

$$\begin{aligned}
\lim_{\varepsilon \to 0} B_\varepsilon =& (\rho_0 u_s g(u_0) + \rho_0 u_0 g(u_s) - \rho_0 u_0 g(u_0) - \rho_0 u_s g(u_s))\delta_\Gamma \\
&- (\rho_1 u_1 g(u_s) + \rho_1 u_s g(u_1) - \rho_1 u_s g(u_s) - \rho_1 u_1 g(u_1))\delta_\Gamma \\
=& (\rho_0(u_s - u_0)(g(u_0) - g(u_s)) - \rho_1(u_1 - u_s)(g(u_s) - g(u_1)))\delta_\Gamma.
\end{aligned}$$

Taking the limit of (9) as $\varepsilon \to 0$ one gets

$$\frac{1}{\rho_0}\lim_{\varepsilon \to 0} \varepsilon p(\rho_\varepsilon) = (u_s - u_0)(g(u_s) - g(u_0)),$$

and repeating the procedure for (10),

$$\frac{1}{\rho_1}\lim_{\varepsilon \to 0} \varepsilon p(\rho_\varepsilon) = (u_1 - u_s)(g(u_1) - g(u_s)),$$

because $u_\varepsilon \to u_s$ as $\varepsilon \to 0$. Since $\lim_{\varepsilon \to 0} \varepsilon p(\rho_\varepsilon)$ exists,

$$\rho_0(u_s - u_0)(g(u_0) - g(u_s)) = \rho_1(u_1 - u_s)(g(u_s) - g(u_1)),$$

and $\lim_{\varepsilon \to 0} B_\varepsilon = 0$.

□

6. Entropy and entropy flux pairs

As it was expected, system (1) does not admit a convex entropy function, but one can construct semi-convex entropy function η ($D^2\eta$ is positive semi-definite).

We shall look for entropy and entropy-flux functions in the region where $\rho > 0$. Using (ρ, v) variables and (4), one can find that an entropy function is of the form $\eta(\rho, v) = \theta(v/\rho)\rho + \overline{\theta}(v/\rho)$. Semi-convex condition on η implies that θ has to be convex, $\theta'' > 0$, and $\overline{\theta} \equiv 0$. For such η, the entropy-flux is $q(\rho, v) = \theta(v/\rho)g(v/\rho)\rho$. Now, we shall return to (ρ, u) variables because the calculations has a simpler form in this case.

As above, denote by Γ_1, Γ_2 and Γ the sets $\{x = c_{1,\varepsilon}t\}$, $\{x = c_{2,\varepsilon}t\}$ and $\{x = ct\}$, resp, where $c = \lim_{\varepsilon \to 0} c_{i,\varepsilon}$, $i = 1, 2$. Substitution of (26) into $\eta_t + q_x$ gives

$$\begin{aligned}
\eta_t + q_x =& - c_{1,\varepsilon}\left(\theta(u_\varepsilon)\rho_\varepsilon - \theta(u_0)\rho_0\right)\delta_{\Gamma_1} - c_{1,\varepsilon}\left(\theta(u_1)\rho_1 - \theta(u_\varepsilon)\rho_\varepsilon\right)\delta_{\Gamma_2} \\
&+ \left(\theta(u_\varepsilon)g\left(u_\varepsilon\right)\rho_\varepsilon - \theta(u_0)g\left(u_0\right)\rho_0\right)\delta_{\Gamma_1} \\
&+ \left(\theta(u_1)g\left(u_1\right)\rho_1 - \theta(u_\varepsilon)g\left(u_\varepsilon\right)\rho_\varepsilon\right)\delta_{\Gamma_2}.
\end{aligned}$$

Using the formulas for the speeds (13) one gets

$$\begin{aligned}
-c_{1,\varepsilon}\theta(u_\varepsilon)\rho_\varepsilon+\theta(u_\varepsilon)g\left(u_\varepsilon\right)\rho_\varepsilon=&\rho_\varepsilon\theta(u_\varepsilon)(g(u_\varepsilon)-\frac{\rho_\varepsilon g(u_\varepsilon)-\rho_0 g(u_0)}{\rho_\varepsilon-\rho_0})\\
=&\rho_0\rho_\varepsilon\theta(u_\varepsilon)\frac{g(u_0)-g(u_\varepsilon)}{\rho_\varepsilon-\rho_0}\\
\to&\rho_0\theta(u_s)(g(u_0)-g(u_s)),\\
c_{2,\varepsilon}\theta(u_\varepsilon)\rho_\varepsilon-\theta(u_\varepsilon)g\left(u_\varepsilon\right)\rho_\varepsilon=&\rho_\varepsilon\theta(u_\varepsilon)(\frac{\rho_1 g(u_1)-\rho_\varepsilon g(u_\varepsilon)}{\rho_1-\rho_\varepsilon}-g(u_\varepsilon))\\
=&\rho_1\rho_\varepsilon\theta(u_\varepsilon)\frac{g(u_\varepsilon)-g(u_1)}{\rho_\varepsilon-\rho_1}\\
\to&\rho_1\theta(u_s)(g(u_s)-g(u_1)),\\
c_{1,\varepsilon}\theta(u_0)\rho_0-\theta(u_0)g\left(u_0\right)\rho_0\to&-\rho_0\theta(u_0)(g(u_0)-g(u_s)),\\
-c_{2,\varepsilon}\theta(u_1)\rho_1+\theta(u_1)g\left(u_1\right)\rho_1\to&-\rho_1\theta(u_1)(g(u_s)-g(u_1)).
\end{aligned}$$

Since all of these limits are finite,

$$\begin{aligned}
\eta_t+q_x\to&(\rho_0\theta(u_s)(g(u_0)-g(u_s))+\rho_1\theta(u_s)(g(u_s)-g(u_1))\\
&-\rho_0\theta(u_0)(g(u_0)-g(u_s))-\rho_1\theta(u_1)(g(u_s)-g(u_1)))\delta_\Gamma.
\end{aligned}$$

Thus the entropy condition is

$$\rho_0(\theta(u_s)-\theta(u_0))(g(u_0)-g(u_s))+\rho_1(\theta(u_s)-\theta(u_1))(g(u_s)-g(u_1))\leq 0.$$

This is true if

$$\begin{aligned}
&\rho_0(g(u_0)-g(u_s))\theta(u_0)+\rho_1(g(u_s)-g(u_1))\theta(u_1)\\
\geq&(\rho_0(g(u_0)-g(u_s))+\rho_1(g(u_s)-g(u_1)))\theta(u_s),
\end{aligned}$$

i.e. if

$$\begin{aligned}
\theta(u_s)\leq&\frac{\rho_0(g(u_0)-g(u_s))}{\rho_0(g(u_0)-g(u_s))+\rho_1(g(u_s)-g(u_1))}\theta(u_0)\\
&+\frac{\rho_1(g(u_s)-g(u_1))}{\rho_0(g(u_0)-g(u_s))+\rho_1(g(u_s)-g(u_1))}\theta(u_1).
\end{aligned}$$

But, since

$$\rho_0(u_0-u_s)(g(u_0)-g(u_s))-\rho_1(u_1-u_s)(g(u_1)-g(u_s))=0$$

(which is true by definition of u_s, see (12)) is equivalent to

$$\begin{aligned}
u_s=&\frac{\rho_0(g(u_0)-g(u_s))}{\rho_0(g(u_0)-g(u_s))+\rho_1(g(u_s)-g(u_1))}u_0\\
&+\frac{\rho_1(g(u_s)-g(u_1))}{\rho_0(g(u_0)-g(u_s))+\rho_1(g(u_s)-g(u_1))}u_1,
\end{aligned}$$

the entropy follows directly from the fact that θ is a convex function.

7. Interaction of δ shock waves

Previous considerations we will use to describe δ shock waves interaction. Consider system (3) with the initial condition:

$$\rho|_{t=0}=\begin{cases}\rho_0, & x<a_1\\ \rho_1, a_1<x<a_2\\ \rho_2, & x>a_2\end{cases},\ v|_{t=0}=\begin{cases}u_0, & x<a_1\\ u_1, a_1<x<a_2\\ u_2, & x>a_2\end{cases},$$

where $u_0 > u_1 > u_2$ and $a_1 < a_2$. According to the results of the previous section, the states (u_0,ρ_0) and (u_1,ρ_1) as well as (u_1,ρ_1) and (u_2,ρ_2) are connected by the combination of S_1 and S_2 with the intermediate states $(u^0_{1,\varepsilon},\rho^0_{1,\varepsilon})$ and $(u^0_{2,\varepsilon},\rho^0_{2,\varepsilon})$, respectively. Also, $p(\rho^0_{i,\varepsilon}) = \mathcal{O}(1/\varepsilon)$, $u^0_{i,\varepsilon} \to u^0_{i,s}$ as $\varepsilon \to 0$, $i = 1,2$.

The speeds of 1-shock connecting (u_0,ρ_0) and $(u^0_{1,\varepsilon},\rho^0_{1,\varepsilon})$ and 2-shock connecting $(u^0_{1,\varepsilon},\rho^0_{1,\varepsilon})$ and (u_1,ρ_1) are, respectively,

$$c_{11,\varepsilon} = \frac{\rho_0 g(u_0) - \rho^0_{1,\varepsilon} g(u^0_{1,\varepsilon})}{\rho_0 - \rho^0_{1,\varepsilon}} \text{ and } c_{12,\varepsilon} = \frac{\rho^0_{1,\varepsilon} g(u^0_{1,\varepsilon}) - \rho_1 g(u_1)}{\rho^0_{1,\varepsilon} - \rho_1}$$

and the speeds of 1-shock connecting (u_1,ρ_1) and $(u^0_{2,\varepsilon},\rho^0_{2,\varepsilon})$ and 2-shock connecting $(u^0_{2,\varepsilon},\rho^0_{2,\varepsilon})$ and (u_2,ρ_2) are, respectively,

$$c_{21,\varepsilon} = \frac{\rho^0_{2,\varepsilon} g(u^0_{2,\varepsilon}) - \rho_1 g(u_1)}{\rho^0_{2,\varepsilon} - \rho_1} \text{ and } c_{22,\varepsilon} = \frac{\rho_2 g(u_2) - \rho^0_{2,\varepsilon} g(u^0_{2,\varepsilon})}{\rho_2 - \rho^0_{2,\varepsilon}}.$$

Lax admissibility conditions imply $c_{12,\varepsilon} \geq g(u^0_{1,\varepsilon}) + \sqrt{\varepsilon g'(u^0_{1,\varepsilon}) p'(\rho^0_{1,\varepsilon})}$ and $c_{21,\varepsilon} \leq g(u^0_{2,\varepsilon}) - \sqrt{\varepsilon g(u^0_{2,\varepsilon}) p'(\rho^0_{2,\varepsilon})}$, where g is increasing function and $u^0_{1,\varepsilon} > u^0_{2,\varepsilon}$. That implies $c_{12,\varepsilon} > c_{21,\varepsilon}$. So, there exists a moment $t^*_\varepsilon > 0$ such that

$$x^*_\varepsilon := c_{12,\varepsilon} t^*_\varepsilon + a_1 = c_{21,\varepsilon} t^*_\varepsilon + a_2,$$

and

$$t^*_\varepsilon = \frac{a_2 - a_1}{c_{12,\varepsilon} - c_{21,\varepsilon}},\, x^*_\varepsilon = \frac{a_2 c_{12,\varepsilon} - a_1 c_{21,\varepsilon}}{c_{12,\varepsilon} - c_{21,\varepsilon}}.$$

This is the moment of the first interaction which happens between the 2-shock wave connecting $(u^0_{1,\varepsilon},\rho^0_{1,\varepsilon})$ and (u_1,ρ_1), and the 1-shock wave connecting $(u^0_{1,\varepsilon},\rho^0_{1,\varepsilon})$ and (u_1,ρ_1).

So, in the moment t^*_ε the functions u_ε and ρ_ε representing the solution of the considered problem have the form

$$\rho_\varepsilon(x,t^*_\varepsilon) = \begin{cases} \rho_0, & x < c_{11,\varepsilon} t^*_\varepsilon + a_1 \\ \rho^0_{1,\varepsilon}, & c_{11,\varepsilon} t^*_\varepsilon + a_1 < x < c_{12,\varepsilon} t^*_\varepsilon + a_1 \\ \rho^0_{2,\varepsilon}, & c_{21,\varepsilon} t^*_\varepsilon + a_2 < x < c_{22,\varepsilon} t^*_\varepsilon + a_2 \\ \rho_2, & x > c_{22,\varepsilon} t^*_\varepsilon + a_2 \end{cases}$$

$$u_\varepsilon(x,t^*_\varepsilon) = \begin{cases} u_0, & x < c_{11,\varepsilon} t^*_\varepsilon + a_1 \\ u^0_{1,\varepsilon}, & c_{11,\varepsilon} t^*_\varepsilon + a_1 < x < c_{12,\varepsilon} t^*_\varepsilon + a_1 \\ u^0_{2,\varepsilon}, & c_{21,\varepsilon} t^*_\varepsilon + a_2 < x < c_{22,\varepsilon} t^*_\varepsilon + a_2 \\ u_2, & x > c_{22,\varepsilon} t^*_\varepsilon + a_2 \end{cases}$$

The "normal" procedure would be to track all wave interactions after that time t^*_ε. At least isentropic gas dynamics system (2) has a bounded global solution (see [3] and [9]). But the existence of a solution was not enough for us to conclude something about its limit as $\varepsilon \to 0$. So, the complete result of delta shock waves interaction is left open. There are some numerical results which suggest that such a result would not be (always) a single delta shock as expected in sticky particle model ([1] or [7]) and paper [8]. We propose a different procedure described bellow, but we did not succeed to solve the complete problem as one could see.

In the moment t^*_ε we stop the clock and re-approximate the solution. According to the results of the previous section, distributional limits of the functions $\rho_\varepsilon(x,t^*_\varepsilon)$

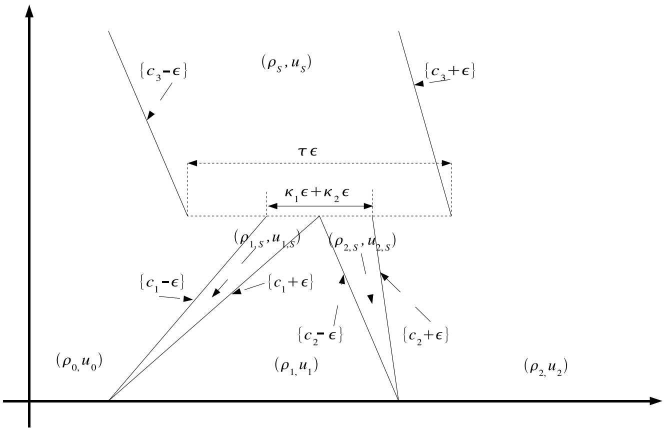

FIGURE 1

and $u_\varepsilon(x,t^*_\varepsilon)$ are given by

$$\begin{aligned}\rho_\varepsilon(x,t^*_\varepsilon) \to & G^0(x-a_1-c_1t^*)\\ & + t^*\left(\rho_0 g(u_0) + g(u_{s1})(\rho_1-\rho_0) - \rho_2 g(u_2),\right.\\ & \left.+ g(u_{s2})(\rho_2-\rho_1)\right)\delta(x-a_1-c_1t^*)\\ u_\varepsilon(x,t^*_\varepsilon) \to & H^0(x-a_1-c_1t^*),\end{aligned} \tag{27}$$

where

$$G^0(x) = \left\{\begin{array}{ll}\rho_0, & x<0\\ \rho_1, & x>0\end{array}\right., H^0(x) = \left\{\begin{array}{ll}u_0, & x<0\\ u_1, & x>0\end{array}\right.,$$

$$\begin{aligned}c_1 &= \lim_{\varepsilon\to0} c_{11,\varepsilon} = \lim_{\varepsilon\to0} c_{12,\varepsilon} = g(u_{s1}), u_{s1}\in(u_1,u_0),\\ c_2 &= \lim_{\varepsilon\to0} c_{21,\varepsilon} = \lim_{\varepsilon\to0} c_{22,\varepsilon} = g(u_{s2}), u_{s2}\in(u_2,u_1),\\ t^* &= \lim_{\varepsilon\to0} t^*_\varepsilon = \frac{a_1-a_2}{c_1-c_2},\end{aligned}$$

and (u_{s1}, u_{s2}) is the solution to the system

$$\begin{aligned}\rho_0(u_0-u_{1s})(g(u_0)-g(u_{1s})) - \rho_1(u_1-u_{1s})(g(u_1)-g(u_{1s}) = 0,\\ \rho_1(u_1-u_{2s})(g(u_1)-g(u_{2s})) - \rho_2(u_2-u_{2s})(g(u_2)-g(u_{2s}) = 0.\end{aligned}$$

Now, we approximate distributions on the right-hand sides of (27) by piecewise constant families of functions

$$\begin{aligned}\tilde\rho_\varepsilon(x,t^*) &= \left\{\begin{array}{ll}\rho_0, & x-x^*_\varepsilon < -a(\varepsilon)\\ \rho^1_\varepsilon, & -a(\varepsilon) < x-x^*_\varepsilon < a(\varepsilon)\\ \rho_2, & x-x^*_\varepsilon > a(\varepsilon)\end{array}\right.\\ \tilde u_\varepsilon(x,t^*) &= \left\{\begin{array}{ll}u_0, & x-x^*_\varepsilon < -a(\varepsilon)\\ u^1_\varepsilon, & -a(\varepsilon) < x-x^*_\varepsilon < a(\varepsilon)\\ u_2, & x-x^*_\varepsilon > a(\varepsilon)\end{array}\right.\end{aligned} \tag{28}$$

in the way described later. The idea is to solve (3) in the region $t > t^*$ with the new initial data (28) using the elementary waves and to connect such a solution to the already obtained in the region $t < t^*$. (The connected solution will be only approximated solution to (1)). In order to get such connection it is necessary to ensure ρ and ρu to be continuous in the distributional sense across the line $t = t^*$. (One has to take care only about the terms with t-derivative in (1) across the line). Denote by $\Gamma_1 := \{(x,t^*) : c_{11,\varepsilon}t^*_\varepsilon + a_1 < x < c_{12,\varepsilon}t^*_\varepsilon + a_1\}$, $\Gamma_2 := \{(x,t^*) : c_{21,\varepsilon}t^*_\varepsilon + a_1 < x < c_{22,\varepsilon}t^*_\varepsilon + a_2\}$ and $\Gamma := \{(x,t^*) : -a(\varepsilon) < x - x^*_\varepsilon < a(\varepsilon)\}$. Distributional continuity of ρ and ρu implies

$$\begin{aligned}
&\rho^1_\varepsilon|\Gamma| \approx \rho^0_{1,\varepsilon}|\Gamma_1| + \rho^0_{2,\varepsilon}|\Gamma_2| \\
&\rho^1_\varepsilon|\Gamma|u^1_\varepsilon \approx \rho^0_{1,\varepsilon}|\Gamma_1|u^0_{1,\varepsilon} + \rho^0_{2,\varepsilon}|\Gamma_2|u^0_{2,\varepsilon},
\end{aligned}$$

where $|\cdot|$ is one dimensional Lebesgue measure. The values $\alpha_1 := \rho^0_{1,\varepsilon}|\Gamma_1|$ and $\alpha_2 := \rho^0_{2,\varepsilon}|\Gamma_2|$ are strengths of delta shock waves at the time $t = t^*_\varepsilon$, in fact. From the above system it follows that u^1_ε is convex combination of $u^0_{1,\varepsilon}$ and $u^0_{12,\varepsilon}$:

$$u^1_\varepsilon = \frac{\alpha_1}{\alpha_1+\alpha_2}u^0_{1,\varepsilon} + \frac{\alpha_2}{\alpha_1+\alpha_2}u^0_{2,\varepsilon},$$

while we are still in position to chose ρ^1_ε providing $|\Gamma| \to 0$ as $\varepsilon \to 0$. The situation is similar when delta shock wave interacts with a rarefaction wave. One just has to split the rarefaction wave in a fan of non-entropic shock waves and apply the same procedure.

Assumption 1. If possible, let ρ^1_ε and u^1_ε be solutions to the system

$$\begin{aligned}
(u^1_\varepsilon - u_0)(g(u^1_\varepsilon) - g(u_0)) &= \varepsilon\left(\frac{1}{\rho_0} - \frac{1}{\rho^1_\varepsilon}\right)\left(p(\rho^1_\varepsilon) - p(\rho_0)\right) \\
(u^1_\varepsilon - u_2)(g(u^1_\varepsilon) - g(u_2)) &= \varepsilon\left(\frac{1}{\rho_2} - \frac{1}{\rho^1_\varepsilon}\right)\left(p(\rho^1_\varepsilon) - p(\rho_2)\right).
\end{aligned}$$

for every $\varepsilon \in (0,1)$.

The solution $(\rho^1_\varepsilon, u^1_\varepsilon)$ to the above system need not to exist, obviously. One can assume that one of the values (ρ_1, u_1) or (ρ_2, u_2) has to satisfy an a priori condition (belong to so called "second delta locus" – see [11] for that notion). In the present case there is no real use of that value, but in principle it could provide a possibility for adding some other wave on a side of the delta shock wave. Using the assumption, the states (u_0, ρ_0) and $(u^1_\varepsilon, \rho^1_\varepsilon)$ are connected by 1-shock of the speed

$$c^1_{1,\varepsilon} = \frac{\rho^1_\varepsilon g(u^1_\varepsilon) - \rho_0 g(u_0)}{\rho^1_\varepsilon - \rho_0},$$

and the states $(u^1_\varepsilon, \rho^1_\varepsilon)$ and (u_2, ρ_2) are connected by 2-shock of the speed

$$c^1_{2,\varepsilon} = \frac{\rho^1_\varepsilon g(u^1_\varepsilon) - \rho_2 g(u_2)}{\rho^1_\varepsilon - \rho_2}.$$

In order to have continuity of distributional limits from both sides of the line $t = t^*_\varepsilon$, put

$$a(\varepsilon) = \frac{t^*_\varepsilon\left(\rho_0 g(u_0) + g(u^1_\varepsilon)(\rho^1_\varepsilon - \rho_0) - \rho_2 g(u_2) + g(u^2_\varepsilon)(\rho_2 - \rho^1_\varepsilon)\right)}{2\rho^1_\varepsilon}. \tag{29}$$

Then

$$\begin{aligned}\tilde{\rho}_\varepsilon(x,t^*)\to& G_{02}(x-a_1-c^1t^*)\\&+t^*\left(\rho_0 g(u_0)+g(u_{s1})(\rho_1-\rho_0)-\rho_2 g(u_2),\right.\\&\left.+g(u_{s2})*(\rho_2-\rho_1)\right)*\delta(x-a_1-c^1*t^*)\\ \tilde{u}_\varepsilon(x,t^*_\varepsilon)\to& H_{02}(x-a_1-c^1*t^*),\end{aligned}$$

where $c^1:=\lim_{\varepsilon\to 0}c^1_{1,\varepsilon}=\lim_{\varepsilon\to 0}c^1_{2,\varepsilon}$.

According to the choice of ρ^1_ε and u^1_ε, the states (u_0,ρ_0) and $(u^1_\varepsilon,\rho^1_\varepsilon)$ can be connected by 1-shock, while the states $(u^1_\varepsilon,\rho^1_\varepsilon)$ and (u_2,ρ_2) can be connected by 2-shock. Therefore, the solution to (3), (27) for $t>t^*$ is given by

$$\tilde{\rho}_\varepsilon(x,t)=\begin{cases}\rho_0, & x<c^1_{1,\varepsilon}t-a(\varepsilon)\\ \rho^1_\varepsilon, & c^1_{1,\varepsilon}t-a(\varepsilon)<x<c^1_{2,\varepsilon}t+a(\varepsilon)\\ \rho_2, & x>c^1_{2,\varepsilon}t+a(\varepsilon)\end{cases}$$

$$\tilde{u}_\varepsilon(x,t)=\begin{cases}u_0, & x<c^1_{1,\varepsilon}t-a(\varepsilon)\\ u^1_\varepsilon, & c^1_{1,\varepsilon}t-a(\varepsilon)<x<c^1_{2,\varepsilon}t+a(\varepsilon),\\ u_2, & x>c^1_{2,\varepsilon}t+a(\varepsilon)\end{cases}$$

Also, it is not difficult to see that for $t>t^*$ we have

$$\begin{aligned}\tilde{\rho}_\varepsilon(x,t)\to& G_{02}(x-a_1-c^1t)\\&+\left[t^*\left(\rho_0 g(u_0)+g(u_{s1})(\rho_1-\rho_0)-\rho_2 g(u_2)+g(u_{s2})(\rho_2-\rho_1)\right)\right.\\&\left.+(t-t^*)\left(\rho_0 g(u_0)-\rho_2 g(u_2)+g(u_s)(\rho_2-\rho_0)\right)\right]\delta(x-a_1-c^1t),\\ \tilde{u}_\varepsilon(x,t)\to& H_{02}(x-a_1-c^1t).\end{aligned}$$

The constant $u^1_s\in(u_2,u_0)$ is the solution of the equation:

$$\rho_0(u_0-u^1_s)(g(u_0)-g(u^1_s))-\rho_2(u_2-u^1_s)(g(u_2)-g(u^1_s))=0.$$

Now, the global approximate solution $(\rho^G_\varepsilon,u^G_\varepsilon)$ of problem (3), (27) is given by

$$\rho^G_\varepsilon(x,t)=\begin{cases}\rho_\varepsilon(x,t), & t<t^*_\varepsilon\\ \tilde{\rho}_\varepsilon(x,t), & t>t^*_\varepsilon\end{cases}\ \text{and}\ u^G_\varepsilon(x,t)=\begin{cases}u_\varepsilon(x,t), & t<t^*_\varepsilon\\ \tilde{u}_\varepsilon(x,t), & t>t^*_\varepsilon\end{cases}. \tag{30}$$

7.1. **Admissibility of re-approximated solution.** Let us first observe that both below and above the line $t=t^*_\varepsilon$, the solution (30) satisfies the entropy condition

$$\eta_t+q_x\le 0,\eta(\rho,u)=f(u)\rho,q(\rho,u)=g(u)f(u)\rho,\rho>0,$$

where f is a convex function, $f''>0$.

If one proves that $\eta_t\le 0$ across the line $\gamma:t=t^*_\varepsilon$, then the solution (30) satisfies the complete entropy condition. Thus we need

$$\begin{aligned}\langle\eta_t,\varphi\rangle&=-\int_{t<t^*_\varepsilon}\eta\varphi_t-\int_{t>t^*_\varepsilon}\eta\varphi_t\\&=-\int_{\mathbb{R}}\eta|_{t=t^*_\varepsilon-0}\varphi\mathrm{d}x+\int_{\mathbb{R}}\eta|_{t=t^*_\varepsilon+0}\varphi\mathrm{d}x\le 0.\end{aligned}\tag{31}$$

to hold on the line γ. The integrals in (31) over $\gamma\backslash(\Gamma\cup\Gamma_1\cup\Gamma_2)$ from upper and lower side annihilates, and (31) is now

$$2a(\varepsilon)\rho^1_\varepsilon\theta(u^1_\varepsilon)-|\Gamma_1|\rho^0_{1,\varepsilon}\theta(u^0_{1\varepsilon})-|\Gamma_2|\rho^0_{2,\varepsilon}\theta(u^0_{2,\varepsilon})\le 0.$$

From the previous section, one knows that $2a(\varepsilon)\rho^1_\varepsilon = |\Gamma_1|\rho^0_{1,\varepsilon} + |\Gamma_2|\rho^0_{2,\varepsilon}$ ($a(\varepsilon)$ was chosen to satisfy this relation, see (29)), and that

$$A_1 := |\Gamma_1|\rho^0_{1,\varepsilon} \to t^* \left(\rho_0 g(u_0) + g(u_{s1})(\rho_1 - \rho_0) - \rho_1 g(u_1)\right)$$
$$A_2 := |\Gamma_2|\rho^0_{2,\varepsilon} \to t^* \left(\rho_1 g(u_1) + g(u_{s2})(\rho_2 - \rho_1) - \rho_2 g(u_2)\right).$$

With such a notation, entropy inequality reads

$$\theta(u^1_s) \leq \frac{A_1}{A_1 + A_2}\theta(u_{s1}) + \frac{A_2}{A_1 + A_2}\theta(u_{s2}).$$

Since $u^1_s = \frac{A_1}{A_1+A_2}u_{s1} + \frac{A_2}{A_1+A_2}u_{s2}$ and θ is convex, we are in position to tell that it always holds if Assumption 1 is satisfied.

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