# INVERSE OF A SPLIT DELTA SHOCK WITH APPLICATIONS 

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#### Abstract

A split delta shock is a variant of so called delta shock solution types. We define a notion of a split delta shock inverse and use it for solving some conservation law systems in a general form. We give a way of calculating such formal solutions with some examples. They can be used in cases of systems that does not posses a bounded solution for each Riemann initial data.


## 1. Introduction

Split delta shock are introduced in order to solve some conservation laws systems with unbounded solutions (see [5]). The main idea is to split an entire domain $\mathbb{R} \times \mathbb{R}_{+}$into pieces. In the interior of each such piece unknowns are chosen to be classical solutions to the system while a boundary could contain a signed delta measure. It is called split delta shock. After performing all necessary operations we join these pieces back and use the distributional derivatives in the original system. This procedure works well if the system is linear in one of unknowns. Here, we expand the above procedure for systems that involves division by a split delta shock and use it in calculations.

In the paper, one will find a formal calculation in a fairly general case of systems. One can look for specific models in in [3], [8] and [9] for a chromatography model and in [1] and [4] for the Chaplygin gas dynamics model.

## 2. The definition of split delta shocks

Let $\Omega_{i} \neq \emptyset, i=1, \ldots, n$ be a finite family of disjoint open sets with piecewise smooth boundary curves $\Gamma_{i}, i=1, \ldots, m: \Omega_{i} \cap \Omega_{j}=\emptyset, \bigcup_{i=1}^{n} \bar{\Omega}_{i}=\overline{R_{+}^{2}}$ where $\bar{\Omega}_{i}$ denotes the closure of $\Omega_{i}$. Denote by $\mathcal{C}\left(\bar{\Omega}_{i}\right)$ the space of bounded and continuous real-valued functions on $\bar{\Omega}_{i}$, equipped with the $L^{\infty}$-norm. Let $\mathcal{M}\left(\bar{\Omega}_{i}\right)$, be the space of measures on $\bar{\Omega}_{i}$.

Define

$$
\mathcal{C}_{\Gamma}=\prod_{i=1}^{n} \mathcal{C}\left(\bar{\Omega}_{i}\right), \quad \mathcal{M}_{\Gamma}=\prod_{i=1}^{n} \mathcal{M}\left(\bar{\Omega}_{i}\right) .
$$

The multiplication of $G=\left(G_{1}, \ldots, G_{n}\right) \in \mathcal{C}_{\Gamma}$ and $D=\left(D_{1}, \ldots, D_{n}\right) \in \mathcal{M}_{\Gamma}$ is defined to be an element $D \cdot G=\left(D_{1} G_{1}, \ldots, D_{n} G_{n}\right) \in \mathcal{M}_{\Gamma}$, where each component is defined as the usual product of a continuous function and a measure.

Every measure on $\bar{\Omega}_{i}$ can be identified with a measure defined on $\overline{\mathbb{R}_{+}^{2}}$ with support in $\bar{\Omega}_{i}$. Thus one can define the mapping $m$ in the following way

$$
m: \mathcal{M}_{\Gamma} \rightarrow \mathcal{M}\left(\overline{\mathbb{R}_{+}^{2}}\right), m(D)=D_{1}+D_{2}+\ldots+D_{n}
$$

A typical example is obtained when $\overline{\mathbb{R}_{+}^{2}}$ is divided into two regions $\Omega_{1}, \Omega_{2}$ by a piecewise smooth curve $x=\gamma(t)$. The delta function $\delta(x-\gamma(t)) \in \mathcal{M}\left(\overline{\mathbb{R}_{+}^{2}}\right)$ along

[^0]the line $x=\gamma(t)$ can be split in a non unique way into a left-hand side $D^{-} \in \mathcal{M}\left(\bar{\Omega}_{1}\right)$ and the right-hand component $D^{+} \in \mathcal{M}\left(\bar{\Omega}_{2}\right)$ such that
$$
\delta(x-\gamma(t))=\alpha_{0}(t) D^{-}+\alpha_{1}(t) D^{+}=m\left(\alpha_{0}(t) D^{-}+\alpha_{1}(t) D^{+}\right)
$$
with $\alpha_{0}(t)+\alpha_{1}(t)=1$. The solution concept which allows to incorporate such two sided delta functions as well as shock waves is modeled along the lines of the classical weak solution concept and proceeds as follows:
Step 1: Perform all nonlinear operations of functions in the space $\mathcal{C}_{\Gamma}$.
Step 2: Perform multiplications with measures in the space $\mathcal{M}_{\Gamma}$.
Step 3: Map the space $\mathcal{M}_{\Gamma}$ into $\mathcal{M}\left(\overline{\mathbb{R}_{+}^{2}}\right)$ by means of the map $m$ and embed it into the space of distributions.
Step 4: Perform the differentiation in the sense of distributions and require that the equation is satisfied in this sense.

Let us define an inverse of a split delta function now. Suppose that

$$
u=\left\{\begin{array}{ll}
u_{0}, & x \leq c t \\
u_{1}, & x \geq c t
\end{array}+\alpha_{0} \delta^{-}(x-c t)+\delta^{+}(x-c t)\right.
$$

We define $\frac{1}{u} \in \mathcal{C}_{\Gamma}, \Gamma=\{(x, t): x=c t\}$, to be a function satisfying $\frac{1}{u} u=1$ in the $\mathcal{M}_{\Gamma}$ sense. Using the above definition that means

$$
\begin{aligned}
& \frac{1}{u} \cdot\left(\left\{\begin{array}{ll}
u_{0}, & x \leq c t \\
u_{1}, & x \geq c t
\end{array}+\alpha_{0}(t) \delta^{-}(x-c t)+\alpha_{1}(t) \delta^{+}(x-c t)\right)\right. \\
= & 1+\frac{\alpha_{0}(t)}{u_{0}} \delta^{-}(x-c t)+\frac{\alpha_{1}(t)}{u_{1}} \delta^{+}(x-c t) \stackrel{m}{\mapsto} 1+\left(\frac{\alpha_{0}(t)}{u_{0}}+\frac{\alpha_{1}(t)}{u_{1}}\right) \delta(x-c t) .
\end{aligned}
$$

Thus, $\alpha_{0}(t) / u_{0}+\alpha_{1}(t) / u_{1}=0$ should hold.

## 3. Systems given in a general form

Let us consider Riemann problem
(1) ?sys?

$$
\begin{aligned}
& u_{t}+\left(\frac{a_{0}+a_{1} u}{v}+\frac{b_{0}+b_{1} v}{u}\right)_{x}=0, u(x, 0)= \begin{cases}u_{0}, & x<0 \\
u_{1}, & x>0\end{cases} \\
& v_{t}+\left(\frac{\bar{a}_{0}+\bar{a}_{1} u}{v}+\frac{\bar{b}_{0}+\bar{b}_{1} v}{u}\right)_{x}=0, v(x, 0)= \begin{cases}v_{0}, & x<0 \\
v_{1}, & x>0\end{cases}
\end{aligned}
$$

We assume that $(u, v) \in \Omega$, where $\Omega \subset \mathbb{R}^{2}$ is a physical domain, i.e. a set of all possible values for $(u, v)$. Let us looking for a solution in the form of two component split delta shock
$u(x, t)=\underbrace{ \begin{cases}u_{0}, & x \leq c t \\ u_{1}, & x \geq c t\end{cases} }_{=: \hat{u}}+\alpha_{0} t \delta^{-}+\alpha_{1} t \delta^{+}, \quad v(x, t)=\underbrace{ \begin{cases}v_{0}, & x \leq c t \\ v_{1}, & x \geq c t\end{cases} }_{=: \hat{v}}+\beta_{0} t \delta^{-}+\beta_{1} t \delta^{+}$,
In the sequel, notation $[u]$ is used for a jump in $\hat{u}$. For a given point $\left(u_{0}, v_{0}\right)$ in a physical domain $\Omega$ for (1), a set of all $\left(u_{1}, v_{1}\right)$ in the domain such that there exists a split delta shock connecting these states is called split delta locus denoted by $L\left(\left(u_{0}, v_{0}\right)\right)$.

The definition of the inverses of $u$ and $v$ gives the following equations
(2) ? $\mathfrak{i 1 2}$ ?

$$
\alpha_{0} / u_{0}+\alpha_{1} / u_{1}=0, \beta_{0} / v_{0}+\beta_{1} / v_{1}=0
$$

Using the procedure for split delta shock calculations, from the first equation in (1) one gets

$$
\begin{aligned}
& -c[u] \delta+\left[\frac{a_{0}+a_{1} u}{v}+\frac{b_{0}+b_{1} v}{u}\right] \delta+\left(\alpha_{0}+\alpha_{1}\right) \delta \\
& -c t\left(\alpha_{0}+\alpha_{1}\right) t \delta^{\prime}+\left(\frac{a_{1}}{v_{0}} \alpha_{0}+\frac{a_{1}}{v_{1}} \alpha_{1}+\frac{b_{1}}{u_{0}} \beta_{0}+\frac{b_{1}}{u_{1}} \beta_{1}\right) t \delta^{\prime}=0
\end{aligned}
$$

where the support of $\delta$ and $\delta^{\prime}$ is the line $x=c t$.
The above equality is true if and only if
(3) ? ${ }^{e 11}$ ?

$$
\alpha_{0}+\alpha_{1}=c[u]-\left[\frac{a_{0}+a_{1} u}{v}+\frac{b_{0}+b_{1} v}{u}\right]=: \kappa_{1}
$$

(4) ? $e 12$ ?

$$
c\left(\alpha_{0}+\alpha_{1}\right)=\frac{a_{1}}{v_{0}} \alpha_{0}+\frac{a_{1}}{v_{1}} \alpha_{1}+\frac{b_{1}}{u_{0}} \beta_{0}+\frac{b_{1}}{u_{1}} \beta_{1} .
$$

With the same arguments, one gets
(5) $? \mathrm{e} 21$ ?

$$
\beta_{0}+\beta_{1}=c[v]-\left[\frac{\bar{a}_{0}+\bar{a}_{1} u}{v}+\frac{\bar{b}_{0}+\bar{b}_{1} v}{u}\right]=: \kappa_{2}
$$

(6) $? \mathrm{e} 22$ ?

$$
c\left(\beta_{0}+\beta_{1}\right)=\frac{\bar{a}_{1}}{v_{0}} \alpha_{0}+\frac{\bar{a}_{1}}{v_{1}} \alpha_{1}+\frac{\bar{b}_{1}}{u_{0}} \beta_{0}+\frac{\bar{b}_{1}}{u_{1}} \beta_{1},
$$

from the second equation in (1).
3.1. A general algorithm. If $u_{0} \neq u_{1}$ and $v_{0} \neq v_{1}$ then the variables $\alpha_{0}, \alpha_{1}, \beta_{0}, \beta_{1}$ are uniquely determined by the following systems
(7) ? ab?

$$
\begin{array}{ll}
\alpha_{0}+\alpha_{1}=\kappa_{1} & \beta_{0}+\beta_{1}=\kappa_{2} \\
\alpha_{0} / u_{0}+\alpha_{1} / u_{1}=0 & \beta_{0} / v_{0}+\beta_{1} / v_{1}=0
\end{array}
$$

We have used (2), (3) and (5). All possible values for $c$ and a relation between leftand right-hand initial data are determined combining (3) and (4) as well as (3) and (4) and solving the following system of equations (quadratic in $c$ )

$$
\begin{aligned}
& a_{1}\left(\alpha_{0} / v_{0}+\alpha_{1} / v_{1}\right)+b_{1}\left(\beta_{0} / u_{0}+\beta_{1} / u_{1}\right)=c \kappa_{1} \\
& \bar{a}_{1}\left(\alpha_{0} / v_{0}+\alpha_{1} / v_{1}\right)+\bar{b}_{1}\left(\beta_{0} / u_{0}+\beta_{1} / u_{1}\right)=c \kappa_{2}
\end{aligned}
$$

After solving (7) and inserting a solution in the above system one gets
(8) ?eq?

$$
\begin{aligned}
& a_{1}\left[\frac{u}{v}\right] \kappa_{1} /[u]+b_{1}\left[\frac{v}{u}\right] \kappa_{2} /[v]=c \kappa_{1} \\
& \bar{a}_{1}\left[\frac{u}{v}\right] \kappa_{1} /[u]+\bar{b}_{1}\left[\frac{v}{u}\right] \kappa_{2} /[v]=c \kappa_{2}
\end{aligned}
$$

In general, we expect that one could get a value(s) for $c$ and a curve with possible right-hand states that could be connected ny the left-hand ones by a split delta shock. Of course, there are a lot of specific situations. We will look at some of them in this paper.

For a real model one has to check whether $\left(u_{1}, v_{1}\right) \in \Omega$ and an admissibility condition for split delta shocks, too. The most usual admissibility condition is that split delta shocks are required to be overcompressive, i.e. all characteristics should run into the shock curve. Another admissible solution is delta shock that propagates along a characteristics. It is called a delta contact discontinuity (see [5] or [7]). That is possible for systems having a linearly degenerate field.

## 4. Some special cases

4.1. $b_{0}=b_{1}=\bar{a}_{0}=\bar{a}_{1}=0$. In this case, (8) reduces to $a_{1}\left[\frac{u}{v}\right]=c[u], \bar{b}_{1}\left[\frac{v}{u}\right]=c[v]$. That is, a speed $c$ is uniquely determined with split delta locus given by the relation

$$
L\left(\left(u_{0}, v_{0}\right)\right)=\left\{\left(u_{1}, v_{1}\right) \in \Omega: a_{1}\left(\frac{u_{1}}{v_{1}}-\frac{u_{0}}{v_{0}}\right)\left(v_{1}-v_{0}\right)=\bar{b}_{1}\left(\frac{v_{1}}{u_{1}}-\frac{v_{0}}{u_{0}}\right)\left(u_{1}-u_{0}\right)\right\} .
$$

Note that this relation can be easily solved now (quadratic equation for $v_{1}$ or $u_{1}$ ), contrary to the general case given in (8).
4.2. $b_{0}=b_{1}=\bar{b}_{0}=\bar{b}_{1}=0$. Now, there is only one condition for an inverse $1 / v$, relation (2). Equations (3-6) are reduced to
(9) $? \underline{f 11}$ ?

$$
\begin{aligned}
\alpha_{0}+\alpha_{1} & =c[u]-\left[\frac{a_{0}+a_{1} u}{v}\right]=: \kappa_{1}, \\
c\left(\alpha_{0}+\alpha_{1}\right) & =\frac{a_{1}}{v_{0}} \alpha_{0}+\frac{a_{1}}{v_{1}} \alpha_{1} . \\
\beta_{0}+\beta_{1} & =c[v]-\left[\frac{\bar{a}_{0}+\bar{a}_{1} u}{v}\right]=: \kappa_{2}, \\
c\left(\beta_{0}+\beta_{1}\right) & =\frac{\bar{a}_{1}}{v_{0}} \alpha_{0}+\frac{\bar{a}_{1}}{v_{1}} \alpha_{1} .
\end{aligned}
$$

(10) ? $f 12$ ?
(11) ? f 21 ?
(12) ? $\underline{f 22}$ ?

One could easily see that the above equations imply $\kappa_{1}=\frac{a_{1}}{a_{1}} \kappa_{2}$ and that relation uniquely determined a speed $c$ of a split delta shock and (12) is satisfied.

Provided $u_{0} \neq u_{1}$ and $v_{0} \neq v_{1}$, one could also see that $\beta_{0}$ and $\beta_{1}$ are determined from (2) and (11) while $\alpha_{0}$ and $\alpha_{1}$ are determined from (9) and (10). That means there are no restriction on $\left(u_{1}, v_{1}\right)$ and $L\left(\left(u_{0}, v_{0}\right)\right)=\Omega$. Of course, one should exclude all non-physical and non-admissible points, but that depends on a concrete model.

### 4.3. Chromatography system.

$$
\begin{equation*}
) ? \underline{\operatorname{chrom}}\left(\left(1+\frac{A}{1-u+v}\right) u\right)_{t}+u_{x}=0,\left(\left(1+\frac{B}{1-u+v}\right) v\right)_{t}+v_{x}=0 . \tag{13}
\end{equation*}
$$

Physical domain for solutions is defined by $1-u+v>$, or $v-u>-1$ and $A<B$. In [3] and [9] one can find all relevant things about that system. Let us note that the real model has determined values for $(x, 0), x>0$ and for $(0, t), t>0$ instead of the standard initial data, as we have assumed above. We use this system with simplifications just as an illustration. One can also look in [2] about the model but with $A=B=1$. That version is much simpler than (13). In order to simplify notation, we use the new variable $w=1-u+v>0$ instead of $v$. Also, with a change of variables $t \mapsto t-x$ system (13) becomes

$$
\left(\frac{A u}{w}\right)_{t}+u_{x}=0,\left(\frac{(B-A) u-1}{w}\right)_{t}+w_{x}=0 .
$$

Let us try with a split delta shock solution of the form

$$
\begin{aligned}
& u(x, t)=\left\{\begin{array}{ll}
u_{0}, & x \leq c t \\
u_{1}, & x \geq c t
\end{array}+\alpha_{0}(t) D^{-}+\alpha_{1}(t) D^{+}\right. \\
& w(x, t)=\left\{\begin{array}{ll}
w_{0}, & x \leq c t \\
w_{1}, & x \geq c t
\end{array}+\beta_{0}(t) D^{-}+\beta_{1}(t) D^{+}\right.
\end{aligned}
$$

even the system is not in form (1) we were using above. After some direct calculations, we have the following equations analogous to (3-6)

$$
\begin{aligned}
\frac{A}{w_{0}} \alpha_{0}+\frac{A}{w_{1}} \alpha_{1} & =c\left[\frac{A v}{w}\right]-[v] \\
c\left(\frac{A}{w_{0}} \alpha_{0}+\frac{A}{w_{1}} \alpha_{1}\right) & =\alpha_{0}+\alpha_{1} \\
\frac{B-A}{w_{0}} \alpha_{0}+\frac{B-A}{w_{1}} \alpha_{1} & =c\left[\frac{(B-A) v-1}{w}\right]-[w] \\
c\left(\frac{B-A}{w_{0}} \alpha_{0}+\frac{B-A}{w_{1}} \alpha_{1}\right) & =\beta_{0}+\beta_{1} .
\end{aligned}
$$

Also, the inverse condition $\frac{\beta_{0}}{w_{0}}+\frac{\beta_{1}}{w_{1}}=0$ should hold. Assume that $w_{0} \neq w_{1}$ (otherwise we do not expect a split delta shock). Using the first and the third equation above, one could se that the necessary condition is $\frac{\kappa_{1}}{A}=\frac{\kappa_{2}}{B-A}$. That condition determines a speed of the wave. Then $\alpha_{0}$ and $\alpha_{1}$ can be calculated from the first and the second equation, while $\beta_{0}$ and $\beta_{1}$ from the fourth equation and the inverse condition.

That was only a demonstration of split delta shock calculation, one could find all physically relevant solutions in [3], [6] or [9].

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