GENERALIZED UNIFORMLY CONTINUOUS SEMIGROUPS AND SEMILINEAR HYPERBOLIC SYSTEMS WITH REGULARIZED DERIVATIVES

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Abstract. We adopt the theory of uniformly continuous operator semigroups for use in Colombeau generalized function spaces. The main objective is to find a unique solution to a class of semilinear hyperbolic systems with singularities. The idea of regularized derivatives is to transform unbounded differential operators into bounded, integral ones. This idea is used here to permit working with uniformly continuous operators.

1. Introduction

Generalized uniformly continuous semigroups are introduced in [5] in order to use the classical theory of semigroups in solving linear partial differential equations with singularities in Colombeau generalized function spaces. This paper is a continuation of that effort. Our aim is to solve some semilinear hyperbolic systems. Singularities in initial data or in coefficients are represented by generalized functions, so the theory of semigroups can be used efficiently.

Regularized derivatives (see [3]) are used for transforming differential into integral (bounded) operators. This was the main tool permitting a use of uniformly continuous semigroups for differential operators. Generalized functions used here are based on Sobolev spaces instead of distributions, and the use of Banach space methods is available.

The main results of the paper are assertions concerning existence and uniqueness of solutions to a class of semilinear hyperbolic systems. Cauchy problem for semilinear first-order hyperbolic system with smooth coefficients and Lipschitz nonlinearity is solved in the usual Colombeau generalized function space in [6]. In this paper, besides giving a solution to the problem above, we also investigate its relation with the solution from [6] in the case when the system has smooth coefficients with appropriate type of initial data.

The definitions of basic generalized function spaces suitable for work with generalized uniformly continuous semigroups are given in the second section. Let us note that similar procedure can be performed for generalized C_0-semigroups (see [4], for example).

In the third section, the definitions given previously by the authors in [5] are repeated, and followed with some assertions about uniformly continuous semigroups and their infinitesimal generators.

The existence and uniqueness theorem is given and proved in the fourth section.

The last, fifth, section is devoted to examination of the connection between the results from the previous section and the result in [6] for a class of systems described above and the L^2-association of the solutions is established.

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2. Basic spaces

Let $(E, \|\cdot\|)$ be a Banach space and $\mathcal{L}(E)$ the space of all linear continuous mappings from E into E.

$\mathcal{S}E_M([0,\infty):\mathcal{L}(E))$ is the space of nets

$$S_\varepsilon : [0,\infty) \to \mathcal{L}(E), \varepsilon \in (0,1)$$

differentiable with respect to $t \in [0,\infty)$, with the property that for every $T>0$ there exist $N \in \mathbb{N}$, $M>0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T)} \left\| \frac{d^\alpha}{dt^\alpha} S_\varepsilon(t) \right\|_{\mathcal{L}(E)} \leq M\varepsilon^{-N}, \varepsilon < \varepsilon_0, \alpha \in \{0,1\}. \tag{1}$$

It is an algebra with respect to composition of operators.

$\mathcal{S}N([0,\infty):\mathcal{L}(E))$ is the space of nets

$$N_\varepsilon : [0,\infty) \to \mathcal{L}(E), \varepsilon \in (0,1)$$

differentiable with respect to $t \in [0,\infty)$, with the property that for every $T>0$ and $a \in \mathbb{R}$ there exist $M>0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T)} \left\| \frac{d^\alpha}{dt^\alpha} N_\varepsilon(t) \right\|_{\mathcal{L}(E)} \leq M\varepsilon^{a}, \varepsilon < \varepsilon_0, \alpha \in \{0,1\}. \tag{2}$$

It is an ideal of $\mathcal{S}E_M$. Thus, we define Colombeau space as

$$\mathcal{S}G([0,\infty):\mathcal{L}(E)) = \frac{\mathcal{S}E_M([0,\infty):\mathcal{L}(E))}{\mathcal{S}N([0,\infty):\mathcal{L}(E))}.$$

Elements of $\mathcal{S}G([0,\infty):\mathcal{L}(E))$ will be denoted by $S=[S_\varepsilon]$ where S_ε is a representative of the class. Similarly, one can define following spaces:

$\mathcal{S}E_M(E)$ is the space of nets of linear continuous mappings

$$A_\varepsilon : E \to E,\ \varepsilon \in (0,1)$$

with the property that there exist constants $N \in \mathbb{N}$, $M>0$ and $\varepsilon_0 \in (0,1)$ such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)} \leq M\varepsilon^{-N}, \varepsilon < \varepsilon_0. \tag{3}$$

$\mathcal{S}N(E)$ is the space of nets of linear continuous mappings $A_\varepsilon : E \to E$, $\varepsilon \in (0,1)$ with the property that for every $a \in \mathbb{R}$ there exist $M>0$ and $\varepsilon_0>0$ such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)} \leq M\varepsilon^{a}, \varepsilon < \varepsilon_0. \tag{4}$$

Now, Colombeau space is defined by

$$\mathcal{S}G(E) = \frac{\mathcal{S}E_M(E)}{\mathcal{S}N(E)}.$$

Elements of $\mathcal{S}G(E)$ will be denoted by $A=[A_\varepsilon]$ where A_ε is a representative of the class.

Let $H^m(\mathbb{R}^n)$ be the usual Sobolev space $H^{m,2}(\mathbb{R}^n)$. In the paper we shall use the following spaces:

$\mathcal{E}_M([0,\infty):H^m(\mathbb{R}^n))$ is the space of nets

$$G_\varepsilon : [0,\infty)\times\mathbb{R}^n \to \mathbb{C}, G_\varepsilon(t,\cdot) \in H^m(\mathbb{R}^n), \text{for every } t\in[0,\infty),$$

with the property that for every $T>0$ there exist $C>0$, $N\in\mathbb{N}$ and $\varepsilon_0>0$ such that

$$\sup_{t\in[0,T)}\|\partial_t^\alpha G_\varepsilon(t,\cdot)\|_{H^m}\leq C\varepsilon^{-N},\alpha\in\{0,1\},\varepsilon<\varepsilon_0.$$

$\mathcal{N}([0,\infty):H^m(\mathbb{R}^n))$ is the space of nets $G_\varepsilon\in\mathcal{E}_M([0,\infty):H^m(\mathbb{R}^n))$ with the property that for every $T>0$ and $a\in\mathbb{R}$ there exist $C>0$ and $\varepsilon_0>0$ such that

$$\sup_{t\in[0,T)}\|\partial_t^\alpha G_\varepsilon(t,\cdot)\|_{H^m}\leq C\varepsilon^{a},\alpha\in\{0,1\},\varepsilon<\varepsilon_0.$$

Define the quotient space

$$\mathcal{G}([0,\infty):H^m(\mathbb{R}^n))=\frac{\mathcal{E}_M([0,\infty):H^m(\mathbb{R}^n))}{\mathcal{N}([0,\infty):H^m(\mathbb{R}^n))}.$$

In similar way, by omitting t-variable, one can define spaces $\mathcal{E}_M(H^m(\mathbb{R}^n))$, $\mathcal{N}(H^m(\mathbb{R}^n))$ and $\mathcal{G}(H^m(\mathbb{R}^n))$.

Note that the above spaces are not algebras with respect to multiplication (which is the case for the original definition of generalized function spaces).

3. Generalized semigroups

Definition 1. *$S\in\mathcal{SG}([0,\infty):\mathcal{L}(E))$ is called a uniformly continuous Colombeau semigroup if it has a representative S_ε which is a uniformly continuous semigroup for every ε small enough, i.e.*
(i) $S_\varepsilon(0)=I$.
(ii) $S_\varepsilon(t_1+t_2)=S_\varepsilon(t_1)S_\varepsilon(t_2)$, for every $t_1,t_2\geq 0$.
(iii) $\lim_{t\to 0}\|S_\varepsilon(t)-I\|=0$.

The following proposition is proved in [5].

Proposition 1. *Let S_ε and $\widetilde{S_\varepsilon}$ be representatives of a uniformly continuous Colombeau semigroup S, with infinitesimal generators A_ε, and $\widetilde{A_\varepsilon}$, respectively, for ε small enough. Then $A_\varepsilon-\widetilde{A_\varepsilon}\in\mathcal{SN}(E)$.*

Definition 2. *$A\in\mathcal{SG}(E)$ is called the infinitesimal generator of a uniformly continuous Colombeau semigroup $S\in\mathcal{SG}([0,\infty):\mathcal{L}(E))$ if A_ε is the infinitesimal generator of the representative S_ε, for every ε small enough.*

Definition 3. *Let h_ε be a positive net satisfying $h_\varepsilon\leq\varepsilon^{-1}$. It is said that $A\in\mathcal{SG}(E)$ is of h_ε-type if it has a representative A_ε such that*

$$\|A_\varepsilon\|_{\mathcal{L}(E)}=\mathcal{O}(h_\varepsilon),\varepsilon\to 0. \tag{5}$$

$G\in\mathcal{G}([0,\infty):H^m(\mathbb{R}^n))$ is said to be of h_ε-type if it has a representative G_ε such that

$$\|G_\varepsilon\|_{H^m}=\mathcal{O}(h_\varepsilon),\varepsilon\to 0.$$

In the classical theory semigroups of bounded linear operators the following theorem holds.

Theorem 1. *([7], Theorem 1.2) A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

The following lemma holds for generalized operators.

Lemma 1. *Every $A\in\mathcal{SG}(E)$ of h_ε-type, where $h_\varepsilon\leq C\log\frac{1}{\varepsilon}$, is the infinitesimal generator of some $T\in\mathcal{SG}([0,\infty):\mathcal{L}(E))$.*

Proof. According to Theorem 1 every bounded operator A_ε is the infinitesimal generator of the uniformly continuous semigroup

$$T_\varepsilon(t) = e^{tA_\varepsilon} = \sum_{n=0}^{\infty} \frac{(tA_\varepsilon)^n}{n!}.$$

Let us show that $T_\varepsilon \in \mathcal{S}E_M\left([0,\infty) : \mathcal{L}(E)\right)$. We have that

$$\|T_\varepsilon(t)\| \leq \sum_{n=0}^{\infty} \frac{\|tA_\varepsilon\|^n}{n!} \leq M \sum_{n=0}^{\infty} \frac{1}{n!}(th_\varepsilon)^n = Me^{th_\varepsilon}.$$

Since $h_\varepsilon \leq C \log\frac{1}{\varepsilon}$ we have $\sup_{t\in[0,T)} \|T_\varepsilon(t)\| \leq M\varepsilon^{-TC}$, for ε small enough. Since $\frac{d}{dt}T_\varepsilon(t) = A_\varepsilon$, for every ε small enough, we have

$$\left|\frac{d}{dt}T_\varepsilon(t)\right| = \|A_\varepsilon\| \leq C \log\frac{1}{\varepsilon} \leq C\varepsilon^{-1},$$

for every such ε, i.e., $T_\varepsilon \in \mathcal{S}E_M\left([0,\infty) : \mathcal{L}(E)\right)$. Thus, the proof is completed. □

Proposition 2. *Let A be the infinitesimal generator of a uniformly continuous Colombeau semigroup S, and B be the infinitesimal generator of a uniformly continuous Colombeau semigroup T. If $A = B$, then $S = T$.*

Proof. Let $N_\varepsilon = A_\varepsilon - B_\varepsilon \in \mathcal{SN}(E)$. We have

$$\frac{d}{dt}(S_\varepsilon(t) - T_\varepsilon(t))x = A_\varepsilon(S_\varepsilon(t) - T_\varepsilon(t))x + N_\varepsilon T_\varepsilon(t)x.$$

Duhamel principle and $S_\varepsilon(0) = T_\varepsilon(0) = I$ imply

$$(S_\varepsilon(t) - T_\varepsilon(t))x = \int_0^t S_\varepsilon(t-s)N_\varepsilon T_\varepsilon(s)x ds.$$

One can easily show that $\|S_\varepsilon(t) - T_\varepsilon(t)\| \leq C\varepsilon^a$, for every real a, because $N_\varepsilon \in \mathcal{SN}(E)$. The same bounds for t-derivative of $S_\varepsilon(t) - T_\varepsilon(t)$ can be obtained by a successive differentiation of the above term. □

4. Semilinear hyperbolic systems with regularized derivatives

Definition 4. *Let $\alpha \in \mathbb{N}_0^n$. Regularized α-th derivative of a generalized function G is defined by the representative*

$$\tilde{\partial}^{\alpha}_{h_\varepsilon} G_\varepsilon = G_\varepsilon * \partial^\alpha \phi_{h_\varepsilon}, \tag{6}$$

where $\phi_{h_\varepsilon}(x) = h_\varepsilon^n \phi(xh_\varepsilon)$, $\phi(y) = \phi_1(y_1)\cdot \ldots \cdot \phi_1(y_n)$, $\phi_1 \in C_0^\infty(\mathbb{R})$, $\phi_1(\xi) \geq 0$, ϕ is symmetric function with $\int \phi_1(\xi)d\xi = 1$.

For definition and some basic properties of regularized derivatives we refer to [3] and [8]. In the sequel, we shall use the symbol ∂_{h_ε} for the first order derivative in one-dimensional case.

Theorem 2. *Let $u_0 \in \left(\mathcal{G}\left(H^1(\mathbb{R})\right)\right)^n$ and h_ε be a net satisfying $h_\varepsilon = \mathcal{O}\left((\log 1/\varepsilon)^{1/2}\right)$, as $\varepsilon \to 0$. Let function $f(x,t,u(t)) = (f_1(x,t,u(t)),\ldots,f_n(x,t,u(t)))$ be globally Lipschitz with respect to x and u with bounded second order derivative with respect to u and $f(x,t,0) = 0$. Also, suppose that $\partial_x f(x,t,u(t))$ is globally Lipschitz function with respect to u.*

Let operator $A \in \mathcal{SG}\left(\left(H^1(\mathbb{R})\right)^n\right)$ *be represented by the nets of operators*

$$A_\varepsilon : \left(H^1(\mathbb{R})\right)^n \to \left(H^1(\mathbb{R})\right)^n, \quad A_\varepsilon u = -\lambda_\varepsilon \tilde{\partial}_{h_\varepsilon} u, \tag{7}$$

where $\lambda_\varepsilon = \operatorname{diag}(\lambda^1_\varepsilon, \ldots, \lambda^n_\varepsilon) \in \left(H^{1,\infty}(\mathbb{R})\right)^{n\times n}$ *and* $\|\lambda_\varepsilon\|_{L^\infty(\mathbb{R})} = \mathcal{O}\left((\log 1/\varepsilon)^{1/2}\right)$.

Then there exists the generalized function $u \in \left(\mathcal{G}([0,\infty) : (H^1(\mathbb{R}))\right)^n$*, which uniquely solves the Cauchy problem*

$$\frac{d}{dt}u(t) = Au(t) + f(\cdot, t, u(t)), \; u(0) = u_0. \tag{8}$$

That solution is represented by

$$u^i_\varepsilon(t) = S_\varepsilon(t)u^i_{0\varepsilon} + \int_0^t S_\varepsilon(t-s)f^i(\cdot, s, u^i(s))ds, \; i = 1, \ldots, n, \tag{9}$$

where $S \in \mathcal{SG}\left([0,\infty) : \left(H^1(\mathbb{R})\right)^n\right)$ *is the uniformly continuous Colombeau semigroup generated by* A*.*

Proof. Take $u \in (H^1(\mathbb{R}))^n$. Then

$$\begin{aligned}
\|A_\varepsilon u\|_{L^2} \leq& \sum_{i=1}^n \|\lambda^i_\varepsilon \tilde{\partial}_{h_\varepsilon} u^i\|_{L^2} \leq \sum_{i=1}^n \|\lambda^i_\varepsilon\|_{L^\infty} \|u^i * \tilde{\partial}_{h_\varepsilon}\phi_{h_\varepsilon}\|_{L^2} \\
\leq& \sum_{i=1}^n \|\lambda^i_\varepsilon\|_{L^\infty}\|u^i\|_{L^2}\|\tilde{\partial}_{h_\varepsilon}\phi_{h_\varepsilon}\|_{L^1} \leq \sum_{i=1}^n C_1 \left(\log 1/\varepsilon\right)^{1/2} \cdot C_2 \left(\log 1/\varepsilon\right)^{1/2} \|u^i\|_{L^2} \\
\leq& C \log 1/\varepsilon \sum_{i=1}^n \|u^i\|_{L^2}
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_x(A_\varepsilon u)\|_{L^2} \leq& \sum_{i=1}^n \|\partial_x(\lambda^i_\varepsilon)\tilde{\partial}_{h_\varepsilon}u^i\|_{L^2} + \sum_{i=1}^n \|\lambda^i_\varepsilon \partial_x(\tilde{\partial}_{h_\varepsilon}u^i)\|_{L^2} \\
\leq& \sum_{i=1}^n \|\partial_x \lambda^i_\varepsilon\|_{L^\infty}\|\tilde{\partial}_{h_\varepsilon}u^i\|_{L^2} + \sum_{i=1}^n \|\lambda^i_\varepsilon\|_{L^\infty}\|\tilde{\partial}_{h_\varepsilon}(\partial_x u^i)\|_{L^2} \\
\leq& \sum_{i=1}^n \|\partial_x \lambda^i_\varepsilon\|_{L^\infty}\|\tilde{\partial}_{h_\varepsilon}\|_{L^1}\|u^i\|_{L^2} + \sum_{i=1}^n \|\lambda^i_\varepsilon\|_{L^\infty}\|\tilde{\partial}_{h_\varepsilon}\|_{L^1}\|(\partial_x u^i)\|_{L^2} \\
\leq& C \log\frac{1}{\varepsilon} \sum_{i=1}^n \left(\|u^i\|_{L^2} + \|\partial_x u^i\|_{L^2}\right).
\end{aligned}$$

Since $u \in \left(H^1(\mathbb{R})\right)^n$ it follows that operator A is of $\log\frac{1}{\varepsilon}$-type. Therefore, it is the infinitesimal generator of a uniformly continuous Colombeau semigroup $S \in \left(\mathcal{SG}([0,\infty) : (H^1(\mathbb{R}))\right)^n$. By well known classical result (see [7]) we know that (9) represents a solution to (8).

Let us show that this solution is an element of $\left(\mathcal{G}([0,\infty) : (H^1(\mathbb{R}))\right)^n$. We have

$$\|u_\varepsilon(t)\|_{L^2} \leq \|S_\varepsilon(t)u_{0\varepsilon}\|_{L^2} + \int_0^t \|S_\varepsilon(t-s)f(\cdot, s, u_\varepsilon(s))\|_{L^2}ds$$

and

$$\|\frac{d}{dt}u_\varepsilon(t)\Big\|_{L^2} \leq \|A_\varepsilon(t)u_\varepsilon\|_{L^2} + \|f(x,t,u_\varepsilon(t))\|_{L^2}.$$

Since $f(x,t,u)$ is globally Lipschitz with respect to u and $f(x,t,0)=0$, one gets $f(\cdot,t,u(t)) \in (H^1(\mathbb{R}))^n$ if $u(t) \in (H^1(\mathbb{R}))^n$. Thus, the moderate bounds for $\|u_\varepsilon(t)\|_{L^2}$ and $\left\|\frac{d}{dt}u_\varepsilon(t)\right\|_{L^2}$ immediately follow.

After a differentiation of (9) with respect to x we have

$$\begin{aligned}\|\partial_x u_\varepsilon(t)\|_{L^2} \leq & \|S_\varepsilon(t)\partial_x u_{0\varepsilon}\|_{L^2} + \int_0^t \|S_\varepsilon(t-s)\left(\nabla_u f(x,s,u_\varepsilon(s))\partial_x u_\varepsilon(s)\right)\|_{L^2} ds \\ & + \int_0^t \|S_\varepsilon(t-s)\partial_x f(x,s,u_\varepsilon(s))u_\varepsilon(s)\|_{L^2} ds \\ \leq & \|S_\varepsilon(t)\partial_x u_{0\varepsilon}\|_{L^2} + \int_0^t \|S_\varepsilon(t-s)\| \cdot \|\nabla_u f\|_{L^\infty} \cdot \|\partial_x u_\varepsilon(s)\|_{L^2} ds \\ & + \int_0^t \|S_\varepsilon(t-s)\| \cdot \|\partial_x f\|_{L^\infty} \cdot \|u_\varepsilon(s)\|_{L^2} ds.\end{aligned}$$

Since f is Lipschitz with respect to u and x the moderate bound for $\|\partial_x u_\varepsilon\|_{L^2}$ follows. Thus, $u \in \left(\mathcal{G}([0,\infty):(H^1(\mathbb{R}))\right)^n$.

To show that the solution is unique in $\left(\mathcal{G}([0,\infty):(H^1(\mathbb{R}^n))\right)^n$, suppose that there exist two solutions u and v to equation (8) and set $w_\varepsilon = u_\varepsilon - v_\varepsilon$. This difference satisfies

$$\frac{d}{dt}w_\varepsilon(t) = A_\varepsilon(t)w_\varepsilon(t) + f(\cdot,t,u_\varepsilon(t)) - f(\cdot,t,v_\varepsilon(t)) + N_\varepsilon(t),\ w_\varepsilon(0) = w_{0\varepsilon}, \tag{10}$$

where $N_\varepsilon(t) \in \mathcal{N}([0,\infty):(H^1(\mathbb{R}))^n)$ and $w_{0\varepsilon} \in \mathcal{N}((H^1(\mathbb{R})^n)$. Then

$$\begin{aligned}w^i_\varepsilon(t) = & S_\varepsilon(t)w^i_{0\varepsilon} + \int_0^t S_\varepsilon(t-s)(f_i(\cdot,s,u^i_\varepsilon(s)) - f_i(\cdot,s,v^i_\varepsilon(s)))ds \\ & + \int_0^t S_\varepsilon(t-s)N^i_\varepsilon(s)ds,\end{aligned} \tag{11}$$

and

$$\begin{aligned}\|w_\varepsilon(t)\|_{L^2} \leq & \|S_\varepsilon(t)w_{0\varepsilon}\|_{L^2} + \int_0^t \|S_\varepsilon(t-s)\left(f(\cdot,s,u_\varepsilon(s)) - f(\cdot,s,v_\varepsilon(s))\right)\|_{L^2} ds \\ & + \int_0^t \|S_\varepsilon(t-s)N_\varepsilon(s)\|_{L^2} ds.\end{aligned}$$

Using the fact that $\|\partial_{u_\varepsilon} f\|_{L^\infty} \leq C < \infty$ and

$$\|f(x,s,u_\varepsilon(s)) - f(x,s,v_\varepsilon(s))\|_{L^2} \leq \|\nabla_u f\|_{L^\infty} \cdot \|u_\varepsilon(s) - v_\varepsilon(s)\|_{L^2}$$

we obtain $\mathcal{N}$-bound for $\|w_\varepsilon(t)\|_{L^2}$.

Equation (10) implies

$$\left\|\frac{d}{dt}w_\varepsilon(t)\right\|_{L^2} \leq \|A_\varepsilon w_\varepsilon(t)\|_{L^2} + \|f(x,s,u_\varepsilon(s)) - f(x,s,v_\varepsilon(s))\|_{L^2} + \|N_\varepsilon(t)\|_{L^2}.$$

Since, as we showed in previous step, $\|w_\varepsilon(t)\|_{H^1}$ has the $\mathcal{N}$-bound and $N_\varepsilon(t) \in \mathcal{N}([0,T):(H^1(\mathbb{R}))^n)$ we obtain that $\|\frac{d}{dt}w_\varepsilon(t)\|_{L^2}$ has the $\mathcal{N}$-bound, too.

Finally,

$$\begin{aligned}\|\partial_x w_\varepsilon\|_{L^2} \leq &\|S_\varepsilon(t)\partial_x w_{0\varepsilon}\|_{L^2} + \|\partial_x N_\varepsilon\|_{L^2} \\ &+ \int_0^t \|S_\varepsilon(t-s)\left((\nabla_u f(\cdot,s,u_\varepsilon(s))\partial_x u_\varepsilon(s) - \nabla_u f(\cdot,s,v_\varepsilon(s))\partial_x v_\varepsilon(s))\right)\|_{L^2} ds \\ &+ \int_0^t \|S_\varepsilon(t-s)\left((\partial_x f(\cdot,s,u_\varepsilon(s))u_\varepsilon(s) - \partial_x f(\cdot,s,v_\varepsilon(s))v_\varepsilon(s))\right)\|_{L^2} ds.\end{aligned}$$

But

$$\begin{aligned}&\|(\nabla_u f(\cdot,s,u_\varepsilon(s))\partial_x u_\varepsilon(s) - \nabla_u f(\cdot,s,v_\varepsilon(s))\partial_x v_\varepsilon(s)\|_{L^2} \\ \leq &\|(\nabla_u f(\cdot,s,u_\varepsilon(s))\partial_x u_\varepsilon(s) - \nabla_u f(\cdot,s,v_\varepsilon(s))\partial_x u_\varepsilon(s)\|_{L^2} \\ &+ \|\nabla_u f(\cdot,s,v_\varepsilon(s))\partial_x u_\varepsilon(s) - \nabla_u f(\cdot,s,v_\varepsilon(s))\partial_x v_\varepsilon(s)\|_{L^2} \\ \leq &\|\partial_x u_\varepsilon(s)\|_{L^2} \cdot \|f_{uu}(\cdot,s,\tilde{y}(s))\|_{L^\infty} \cdot \|u_\varepsilon(s) - v_\varepsilon(s)\|_{L^\infty} \\ &+ \|\nabla_u f(\cdot,s,v_\varepsilon(s))\|_{L^\infty} \cdot \|\partial_x u_\varepsilon(s) - \partial_x v_\varepsilon(s)\|_{L^2} \\ \leq &C_1\|\partial_x u_\varepsilon(s)\|_{L^2} \cdot \|w_\varepsilon(s)\|_{H^1} + C_2 \cdot \|\partial_x w_\varepsilon(s)\|_{L^2},\end{aligned}$$

and

$$\begin{aligned}&\|\partial_x f(\cdot,s,u_\varepsilon(s))u_\varepsilon(s) - \partial_x f(\cdot,s,v_\varepsilon(s))v_\varepsilon(s)\|_{L^2} \\ \leq &\|\partial_x f(\cdot,s,u_\varepsilon(s))u_\varepsilon(s) - \partial_x f(\cdot,s,u_\varepsilon(s))v_\varepsilon(s)\|_{L^2} \\ &+ \|\partial_x f(\cdot,s,u_\varepsilon(s))v_\varepsilon(s)\partial_x f(\cdot,s,v_\varepsilon(s))v_\varepsilon(s)\|_{L^2} \\ \leq &\|\partial_x f(\cdot,s,u_s)\|_{L^\infty}\|u_\varepsilon(s) - v_\varepsilon(s)\|_{L^2} \\ &+ \|v_\varepsilon(s)\|_{L^\infty}\|\nabla_u^2 f(\cdot,s,\tilde{y}(s))\|_{L^\infty}\|u_\varepsilon(s) - v_\varepsilon(s)\|_{L^2}.\end{aligned}$$

for some functions $\tilde{y}(s) \in (H^1(\mathbb{R}))^n$. Since $w_{0\varepsilon} \in \left(\mathcal{N}((H^1(\mathbb{R}))\right)^n$ and $N_\varepsilon \in \left(\mathcal{N}([0,T) : (H^1(\mathbb{R}))\right)^n$, Gronwall inequality gives the $\mathcal{N}$-bound for $\|\partial_x w_\varepsilon(t)\|_{L^2}$. Thus, $w_\varepsilon := u_\varepsilon - v_\varepsilon \in \left(\mathcal{N}([0,\infty) : (H^1(\mathbb{R}))\right)^n$, i.e. the solution is unique. □

Definition 5. *The solution u of the problem (8) introduced in Theorem 2 is called generalized solution of the equation*

$$\frac{d}{dt}u(t) = -\lambda\partial_x u(t) + f(\cdot,t,u(t))$$

with regularized derivatives.

5. Relations with the previous results

In [6] Oberguggenberger gives the following theorem:

Theorem 3. *([6],Theorem 16.1) Let $\lambda(x,t)$ be a smooth, real valued diagonal matrix. Let f be smooth and $u \mapsto f(x,t,u)$ be polynomially bounded together with all derivatives, uniformly for (x,t) varying in compact subsets of $\mathbb{R}^2$. Let $u \mapsto \bigtriangledown_u f(x,t,u)$ be globally bounded, uniformly with respect to (x,t) varying in compact subsets of $\mathbb{R}^2$.*

Then, for given $u_0 \in \mathcal{G}(\mathbb{R})$, the problem

$$(\partial_t + \lambda(x,t)\partial_x)u = f(x,t,u),\ u|_{t=0} = u_0 \tag{12}$$

has a unique solution $u \in \mathcal{G}(\mathbb{R}^2)$.

Let $K_T = K \cap \{(x,t) : t \in [0,T]\}$, K is a region bounded by the external characteristics emerging from the end points of $K_0 = [A,B] \supset \mathrm{supp} U_{0\varepsilon}$. The following Lemma will be useful.

Lemma 2. *([6], Lemma 16.2) Let v be a smooth solution to the linear problem*

$$(\partial_t + \lambda(x,t)\partial_x)v(x,t) = f(x,t)v(x,t) + g(x,t), v(x,0) = b(x),$$

where f is a smooth matrix and g, b are smooth. Then

$$\|v\|_{L^\infty(K_T)} \leqslant \|b\|_{L^\infty(K_0)} + T\|g\|_{L^\infty(K_T)} \exp(nT\|f\|_{L^\infty(K_T)}).$$

In order to compare solutions to regularized and non-regularized equations we need some additional a priori bounds for solution to (12).

Lemma 3. *Let u be a solution to (12), where all derivatives of f with respect to u^i, $i = 1, ..., n$, and x of order less or equal to three are bounded when (x,t) belongs to some compact set. Suppose that $\|\partial^i u_0\|_{L^\infty(K_0)} \leqslant b_\varepsilon$, $i = 0,1,2$, where $b_\varepsilon \geqslant 1$. Then*

$$\|u\|_{L^\infty(K_T)} \leqslant \text{const} b_\varepsilon,\ \|\partial_x u\|_{L^\infty(K_T)} \leqslant \text{const} b_\varepsilon,$$
$$\textit{and } \|\partial_x^2 u\|_{L^\infty(K_T)} \leqslant \text{const} b_\varepsilon^2.$$

Proof. We differentiate system in (12) three times with respect to x and use the Lemma 2. The bounds for u and $\partial_x u$ directly follows from the proof of Theorem 3 given in [6], so we shall give the estimate for the second derivative.

After differentiating the system two times with respect to x we have

$$\begin{aligned}(\partial_t + \lambda\partial_x)w =&(\nabla_u f - 2\partial_x\lambda)w + \partial_x^2 f\\ &(\nabla_u^2 f v + \nabla_u \partial_x f - \partial_x^2\lambda)v,\end{aligned}$$

where $v = \partial_x u$ and $w = \partial_x^2 u$. Now, the above lemma gives the estimate

$$\begin{aligned}\|w\|_{L^\infty(K_T)} \leqslant&\|w_0\|_{L^\infty(K_0)} + \|\partial_x^2 f\|_{L^\infty(K_T)} + \exp(nT\|\nabla_u f - 2\partial_x\lambda\|_{L^\infty(K_T)})\\ &\cdot T(\|\nabla_u^2 f v^2\|_{L^\infty(K_T)} + \|\nabla_u\partial_x f v\|_{L^\infty(K_T)} + \|\partial_x^2\lambda v\|_{L^\infty(K_T)}).\end{aligned}$$

Using the assumptions and the bounds for u and v one can see that there exists a constant C such that $\|\partial_x^2 u\|_{L^\infty(K_T)} \leqslant Cb_\varepsilon^2$. □

Theorem 4. *Let u be a solution of (12), with $\lambda = \lambda(x)$ from Theorem 3. Let the initial data be compactly supported and satisfy*

$$\|\partial_x^i u_{0\varepsilon}\|_{L^\infty} \leq g_\varepsilon^{-M/3}, i = 0,1,2, \tag{13}$$

for some $M > 0$ and $g_\varepsilon = \log h_\varepsilon$, $h_\varepsilon < \log\frac{1}{\varepsilon}$.

Then, u and the solution v to (8) with the initial data $v(0) = u_0$ given in Theorem 2 are L^2-associated.

Proof. In Lemma 3 is proved that

$$\|\partial_x^i u_\varepsilon\|_{L^\infty} \leq g_\varepsilon^{-M}, \text{as } \varepsilon \to 0, i = 0,1,2, \tag{14}$$

where L^∞-norm can taken over all $\mathbb{R} \times [0,T]$, since $\mathrm{supp}(U_\varepsilon) \subset K_T$.

Also, the assumption on compact supports for the initial data and finite propagation speed ensures that $u(\cdot,t) \in L^2(\mathbb{R})$, and (14) holds for L^2-norm, too for every $t \leqslant T$.

The following equations are satisfied

$$\partial_t u_\varepsilon(t) + \lambda\partial_x u_\varepsilon(t) = f(\cdot,t,u_\varepsilon(t))$$
$$\partial_t v_\varepsilon(t) + \lambda\partial_x v_\varepsilon * \phi_{h_\varepsilon}(t) = f(\cdot,t,v_\varepsilon(t)).$$

Difference of these two equalities gives

$$\begin{aligned}&\partial_t(u_\varepsilon(t)-v_\varepsilon(t))+\lambda\partial_x(u_\varepsilon(t)-v_\varepsilon(t))*\phi_{h_\varepsilon}\\ =&\lambda(\partial_x u_\varepsilon(t)-\partial_x u_\varepsilon*\phi_{h_\varepsilon}(t))+f(\cdot,t,u_\varepsilon(t))-f(\cdot,t,v_\varepsilon(t)),\end{aligned}\tag{15}$$

for every $t<T$.

Let us fix t for a moment. Then

$$\begin{aligned}&\|\partial_x u_\varepsilon(\cdot,t)-\phi_{h_\varepsilon}*\partial_x u_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R})}\\ =&\sup_{x\in\mathbb{R}}\left|\int_{|y|\le h_\varepsilon}\phi_{h_\varepsilon}(y)\,(\partial_x u_\varepsilon(x,t)-\partial_x u_\varepsilon(x-y,t))\,dy\right|\\ =&\sup_{x\in\mathbb{R}}\left|\int_{|y|\le h_\varepsilon}\phi_{h_\varepsilon}(y)\left(\int_0^1\partial_x^2 u_\varepsilon(x-\sigma y,t)d\sigma\right)ydy\right|\le Cg_\varepsilon^{-M}h_\varepsilon^2\to 0,\varepsilon\to 0,\end{aligned}$$

since $g_\varepsilon=\log h_\varepsilon$, and (14) holds.

Since $K_T\cap\{(x,\tau):\tau=t\}$ is a bounded set with a finite Lebesgue measure, say $l(t)$, one can immediately see that

$$\begin{aligned}&\|\lambda(t)(\partial_x^i u_\varepsilon(\cdot,t)-\phi_{h_\varepsilon}*\partial_x^i u_\varepsilon(\cdot,t))\|_{L^2(\mathbb{R})}\\ \leqslant&\mathrm{const}(\lambda)\cdot l(t)\cdot\|\partial_x^i u_\varepsilon(\cdot,t)-\phi_{h_\varepsilon}*\partial_x^i u_\varepsilon(\cdot,t)\|_{L^\infty(\mathbb{R})}\to 0,\varepsilon\to 0,i=0,1\end{aligned}\tag{16}$$

by the previous estimate on L^∞-norm. Put $w_\varepsilon=u_\varepsilon-v_\varepsilon$. One can write (15) in the form

$$\frac{d}{dt}w_\varepsilon(t)=A_\varepsilon(t)w_\varepsilon(t)+f(\cdot,t,u_\varepsilon(t))-f(\cdot,t,v_\varepsilon(t))+N_\varepsilon(t),\ w_\varepsilon(0)=0,\tag{17}$$

where $A_\varepsilon u=-\lambda\tilde{\partial}_{h_\varepsilon}u$ and $N_\varepsilon(t)=\lambda(t)(\partial_x u_\varepsilon(\cdot,t)-\phi_{h_\varepsilon}*\partial_x u_\varepsilon(\cdot,t))$. Inequality (16) means that $\|N_\varepsilon\|_{H^1}\to 0$ as $\varepsilon\to 0$. Then, each component of w_ε satisfies

$$\begin{aligned}w_\varepsilon^i(t)=&\int_0^t S_\varepsilon(t-s)(f^i(\cdot,s,u_\varepsilon^i(s))-f^i(\cdot,s,v_\varepsilon^i(s)))ds\\ &+\int_0^t S_\varepsilon(t-s)N_\varepsilon^i(s)ds,i=1,\dots,n,\end{aligned}\tag{18}$$

and

$$\begin{aligned}\|w_\varepsilon(t)\|_{L^2}\le&\int_0^t\|S_\varepsilon(t-s)\left(f(\cdot,s,u_\varepsilon(s))-f(\cdot,s,v_\varepsilon(s))\right)\|_{L^2}ds\\ &+\int_0^t\|S_\varepsilon(t-s)N_\varepsilon(s)\|_{L^2}ds.\end{aligned}$$

If $u\in H^1$, then $\|A_\varepsilon u\|_{L^2}\le M\|u\|_{H^1}$, where M does not depend on ε. Since $S_\varepsilon(t)=e^{tA_\varepsilon}=\sum_{n=0}^\infty\frac{(tA_\varepsilon)^n}{n!}$, one can see that $\|S_\varepsilon u\|_{L^2}\leqslant e^{tM}\|u\|_{H^1}$. Particularly, $\|S_\varepsilon(t-s)N_\varepsilon(s)\|_{L^2}\to 0$, as $\varepsilon\to 0$. Using the fact that $\|\nabla_u f\|_{L^\infty}\le C<\infty$ and the estimate

$$\|f(x,s,u_\varepsilon(s))-f(x,s,v_\varepsilon(s))\|_{L^2}\le\|\nabla_u f\|_{L^\infty}\cdot\|u_\varepsilon(s)-v_\varepsilon(s)\|_{L^2}\leqslant C\cdot w_\varepsilon(s),$$

use of Gronwall lemma gives $\sup_{t\in(0,T)}\|w_\varepsilon(t)\|_{L^2}\to 0$, as $\varepsilon\to 0$. □

Immediate consequence of the above theorem is the following corollary.

Corollary 1. *If there exists a L^∞ classical solution to problem (12) with L^∞-initial data then it can be regularized so that it is L^∞-associated with the solution to system (8), when A given by (7).*

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