

# GENERALIZED SOLUTIONS TO A SEMILINEAR WAVE EQUATION

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ABSTRACT. This paper is devoted to solving the Cauchy problem for a nonlinear wave equation in space dimensions  $n \leq 9$ . Solutions belong to the Colombeau-like multiplicative algebra  $\mathcal{G}_{L^2}$  (see [1]).

First, a nonlinear term is regularized with respect to a small parameter  $\varepsilon$  such that it becomes globally Lipschitz for each such  $\varepsilon$ . A net of solutions to the Cauchy problem determines an element in  $\mathcal{G}_{L^2}$  which is called the solution.

Next, we solve the equation without the regularization for certain growth conditions on the nonlinear term with respect to the space dimension  $n$ .

Finally, we show that a solution to the regularized equation is also a solution to the non-regularized one under some additional assumptions.

## INTRODUCTION

The aim of this paper is to bring together two areas:  $L^2$ -theory for the nonlinear wave equation

$$\partial_t^2 u - \Delta u + g(u) = 0, \quad g(0) = 0, \quad u = u(x, t), \quad x \in \mathbb{R}^n, \quad t \geq 0,$$

and theory of generalized functions where nonlinear operations makes sense for a large collection of singular objects.

We use the multiplicative algebra of generalized functions  $\mathcal{G}_{L^2}$  defined in [1] which is the version of Colombeau algebra  $\mathcal{G}$  (see [2]). A representative of a generalized functions is a net of smooth functions which permits a use of the classical energy estimates on each element of the above net. This was the main idea used in the paper.

Initial data are elements in  $\mathcal{G}_{L^2}(\mathbb{R}^n)$ . Also, classical singular initial data in classical functions spaces could be replaced with generalized functions and give a meaning to the above equation. The advantage of using generalized functions lies in the fact that we can multiply and make some other nonlinear operations within this space.

The above equation is very well known, especially the illustrative cases  $g(u) = |u|^p u$ . Good references on strong solutions can be found in [4], [9], [11]. Also, [3] and [5] contain different kinds of weak solutions to the above equation.

Previous results for generalized solutions of Colombeau type can be found in [2], Ch. VIII, where the 3D sub-cubic equation is solved without growth conditions on the initial data in  $\varepsilon$ . Solutions in  $\mathcal{G}$  were found by the modified Jorgens  $L^\infty$ -method.

In order to apply various energy estimates as well as Sobolev type estimates for the above equation, we use the algebra  $\mathcal{G}_{L^2}$  as a general framework, the special case of  $\mathcal{G}_{L^p, L^q}$ , which was constructed in [1].

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Essentially, the paper consists of two parts. In the first one, growth conditions on a nonlinear term  $g(u)$  are irrelevant. It is substituted by a net of globally Lipschitz functions  $g_\varepsilon$ . Then the equation, called the "regularized" one, is solved for each fixed  $\varepsilon$ . Afterwards one has to check whether a solution obtained in this way solves the original equation. In the second part,  $g$  is not regularized and the growth conditions on  $g$  are important similarly to the classical case.

The paper is organized as follows.

In Theorem 1 the regularized equation is solved. For  $n \leq 6$  it is solved in a standard manner, but for the higher dimensional equations,  $n = 7, 8, 9$  the precise and technically complex results of the papers [7,8] are needed. In general, a solution to the regularized equation is not a solution to the non-regularized one. We give a partial answer on a question of their relations. In Theorem 2 we prove that if  $u(t, \cdot) \in H^2(\mathbb{R}^3)$  is a classical solution to the equation with the initial data in  $(H^2, H^3)$  (notice that we need more regularity than usually), then the solution to the regularized equation, which is a net of solutions, converge to  $u$ . It was necessary to regularize the initial data  $(a, b)$  by a  $(a * \phi_{\varepsilon^s}, b * \phi_{\varepsilon^s})$ , where  $\phi_{\varepsilon^s} = \varepsilon^{-ns} \phi(\cdot/\varepsilon^s)$  is a delta net and  $s$  is large enough. Here,  $\int \phi(x) dx = 1$ ,  $\phi \in C_0^\infty$  and  $\phi \geq 0$ . In the 1D case the solution to the regularized equation is a solution to the non-regularized one if the initial data is of  $\log \varepsilon^{-1}$ -growth. If initial data belong to some Sobolev space  $H^s$ ,  $s \in \mathbb{R}$ , then they have to be regularized using the convolution with  $\phi_{h(\varepsilon)}$ , where  $h(\varepsilon) \sim (\log \varepsilon^{-1})^{1/m}$  for some  $m \in \mathbb{N}$ . In Theorem 3 the cubic 3D wave equation is solved in  $\mathcal{G}_{L^2}$  providing that representatives of the initial data are  $o((\log \varepsilon^{-1})^{1/2})$  as  $\varepsilon \rightarrow 0$ . The similar growth rate has to hold for the initial data belonging to the appropriate Sobolev space  $H^s$ ,  $s \in \mathbb{R}$ . In Theorems 4 and 5 the 3D and 4D equations with nonlinearities of order less than 5 in case of 3D and less than 3 in case of 4D are solved. The precise estimates from [7,8] are used for their proofs. Also, we suppose that the initial data have appropriately slow growth with respect to  $\varepsilon$ , similarly to the conditions of Theorem 3.

## 1. NOTATION

Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . We will denote by  $K \subset\subset \Omega$  a compact subset  $K$  of  $\Omega$ . The notation  $f_\varepsilon = \mathcal{O}(\varepsilon^b)$  means that a net of real or complex numbers satisfies  $|f_\varepsilon| \leq C\varepsilon^b$  for some  $C > 0$ ,  $b \in \mathbb{R}$  and  $\varepsilon$  small enough. In this case we shall say that  $f_\varepsilon$  satisfies moderate bounds or it is of moderate growth or simply moderate. A net of functions  $g_\varepsilon$  in some Banach space  $B$  is called moderate if  $\|g_\varepsilon\|_B$  is of the moderate growth. We use the following algebras of Colombeau's generalized

functions.

$$\begin{aligned}
\mathcal{E}(\Omega) &= \{G : (0, 1) \times \Omega \rightarrow \mathbb{C} : G(\varepsilon, \cdot) = G_\varepsilon \in C^\infty(\Omega) \text{ for every } \varepsilon > 0\}. \\
\mathcal{E}_M(\Omega) &= \{G_\varepsilon \in \mathcal{E}(\Omega) : \text{ for every } \alpha \in \mathbb{N}_0^n, K \subset\subset \Omega \text{ there exists } b \in \mathbb{R} \\
&\quad \text{such that } \sup_{x \in K} |\partial^\alpha G_\varepsilon(x)| = \mathcal{O}(\varepsilon^b)\}. \\
\mathcal{N}(\Omega) &= \{G_\varepsilon \in \mathcal{E}_M(\Omega) : \text{ for every } \alpha \in \mathbb{N}_0^n, K \subset\subset \Omega \text{ and } a \in \mathbb{R} \\
&\quad \sup_{x \in K} |\partial^\alpha G_\varepsilon(x)| = \mathcal{O}(\varepsilon^a)\}. \\
\mathcal{G}(\Omega) &= \mathcal{E}_M(\Omega) / \mathcal{N}(\Omega). \\
\mathcal{E}_{\mathbb{C}} &= \{S : (0, 1) \rightarrow \mathbb{C} : \text{ there exists } b \in \mathbb{R} \text{ such that} \\
&\quad |S_\varepsilon| = \mathcal{O}(\varepsilon^b)\}. \\
\mathcal{N} &= \{S \in \mathcal{E}_{\mathbb{C}} : |S_\varepsilon| = \mathcal{O}(\varepsilon^a) \text{ for every } a \in \mathbb{R}\}. \\
\overline{\mathcal{C}} &= \mathcal{E}_{\mathbb{C}} / \mathcal{N}.
\end{aligned}$$

Similarly, using the space  $\mathbb{R}$  instead of  $\mathbb{C}$ , we define  $\overline{\mathbb{R}}$ .

Let  $Q$  denote  $[0, T) \times \Omega$  or  $\Omega$ . Then

$$\begin{aligned}
\mathcal{E}_{L^p}(Q) &= \{G_\varepsilon \in \mathcal{E}(Q) : \text{ for every } \alpha \in \mathbb{N}_0^n \text{ there exists } b \in \mathbb{R} \text{ such that} \\
&\quad \|\partial^\alpha G_\varepsilon\|_{L^p(Q)} = \mathcal{O}(\varepsilon^b)\}. \\
\mathcal{N}_{L^p}(Q) &= \{G_\varepsilon \in \mathcal{E}(Q) : \text{ for every } \alpha \in \mathbb{N}_0^n \text{ and } a \in \mathbb{R} \\
&\quad \|\partial^\alpha G_\varepsilon\|_{L^p(Q)} = \mathcal{O}(\varepsilon^a)\}. \\
\mathcal{G}_{L^p}(Q) &= \mathcal{E}_{L^p}(Q) / \mathcal{N}_{L^p}(Q).
\end{aligned}$$

If  $T$  is finite, then the  $L^2$  norm with respect to the time variable  $t$  can be changed by the  $L^\infty$ -norm (see [1]).

The algebra  $\mathcal{G}_{L^p}(Q)$  equals  $\mathcal{G}_{p,p}(Q)$  defined in [1]. The proof that  $\mathcal{N}_{L^p}(Q)$  is an ideal of  $\mathcal{E}_{L^p}(Q)$  and the properties of the above algebras are given in [1].

Let us note that  $\mathcal{N}_{L^2}(Q) \subset \mathcal{N}(Q)$  and  $\mathcal{E}_{L^2}(Q) \subset \mathcal{E}_M(Q)$ , because of the Sobolev embedding theorem. Therefore there exists a canonical mapping from  $\mathcal{G}_{L^2}(Q)$  into  $\mathcal{G}(Q)$ .

## 2. PRELIMINARY CONSTRUCTION

Consider a family of equations in  $\mathcal{E}_{L^2}([0, T) \times \Omega)$

$$(1) \quad (\partial_t^2 - \Delta)G_\varepsilon = -g(G_\varepsilon), \quad G_\varepsilon|_{t=0} = A_\varepsilon, \partial_t G_\varepsilon|_{t=0} = B_\varepsilon, \quad \varepsilon \in (0, 1),$$

where  $A_\varepsilon, B_\varepsilon \in \mathcal{E}_{L^2}(\mathbb{R}^n)$ ,  $\varepsilon \in (0, 1)$ , and  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  is smooth, polynomially bounded together with all its derivatives and  $g(0) = 0$ .

In Section 3 a function  $g$  will be substituted by a family of smooth functions  $g_\varepsilon$ ,  $\varepsilon \in (0, 1)$  which is called the regularization of  $g$ . One can choose the regularization  $g_\varepsilon$  such that there exists a suitable net  $a_\varepsilon$  of positive real numbers such that for every  $s \in \mathbb{N}_0$  there exist  $\varepsilon_0 > 0$  and  $m_s \in \mathbb{N}$  such that

$$(2) \quad g_\varepsilon(y) = g(y), \text{ for } |y| \leq a_\varepsilon, \quad \varepsilon < \varepsilon_0, \quad \left\| \frac{d^s}{dy^s} g_\varepsilon(y) \right\|_{L^\infty} = \mathcal{O}(a_\varepsilon^{m_s}).$$

Moreover, if the primitive of  $g$ , which equals zero at the origin, is non-negative, then one can choose  $g_\varepsilon$  such that the same is true for its primitive for  $\varepsilon$  small enough. Suppose that there exist a constant  $M > 0$  such that  $g(y) > 0$  for  $y > M$  and  $g(y) < 0$  for  $y < -M$ . Then we make the following regularization. Let  $\kappa_\varepsilon(y)$  be smooth function equals  $y$  for  $|y| \leq a_\varepsilon$  and equals some constant for  $|y| > 2a_\varepsilon$  such that for every  $\alpha \in \mathbb{N}_0$  there exists  $p_\alpha \in \mathbb{R}$  such that  $\sup_{y \in \mathbb{R}} |\kappa_\varepsilon^{(\alpha)}(y)| = \mathcal{O}(\varepsilon^{p_\alpha})$ . Then  $g_\varepsilon(y) = g(\kappa_\varepsilon(y))$ . The positivity (negativity) of  $g$  as  $y \rightarrow \infty$  ( $y \rightarrow -\infty$ ) gives the non negativity of a primitive of  $g_\varepsilon$  if  $g$  has a non-negative primitive.

In some interesting cases when  $g$  is not smooth there exist a regularization with the above properties. For example, let  $g(y) = |y|^{p-1}y$ . Then  $g_\varepsilon(y) = g(\kappa_\varepsilon(y)) * \phi_{a_\varepsilon}(y)$ , where  $\phi_{a_\varepsilon}$  is a non-negative delta net described in the introduction (one has to put  $a_\varepsilon$  instead of  $\varepsilon^s$  there) has the above properties. Let us note that in this case a solution to the equation can be at most associated to the classical one if it exists.

First, we shall solve

$$(3) \quad (\partial_t^2 - \Delta)G_\varepsilon = -g_\varepsilon(G_\varepsilon), \quad G_\varepsilon|_{t=0} = A_\varepsilon, \partial_t G_\varepsilon|_{t=0} = B_\varepsilon, \quad \varepsilon \in (0, 1),$$

where  $A_\varepsilon, B_\varepsilon \in \mathcal{E}_{L^2}(\mathbb{R}^n)$ , instead of (1).

Equation (3) will be called the regularized equation for Equation (1).

### 3. REGULARIZED WAVE EQUATION

In order to obtain the existence and the uniqueness of a generalized solution, we need to estimate  $L^2$ -norms of all derivatives of solution  $G_\varepsilon$  to (3) with respect to  $\varepsilon$ . The bounds depend on a space dimension because they are obtained by Sobolev type embedding and interpolation theorems. If  $n \leq 5$ , the bounds for  $a_\varepsilon$  are of  $(\log(\varepsilon^{-1}))^s$ -type, for some  $s$ . In the case  $n = 6$ , one needs  $a_\varepsilon$  to be estimated by a slower function net. Moreover, the initial data has to be of small enough growth rate with respect to  $\varepsilon$ . The cases  $n = 7, 8, 9$  will be briefly described in the remark after Theorem 1. Actually, the classical result for the existence of a global smooth solution to (3) for fixed  $\varepsilon$  it causes a lot of technical problems as well as a lot of additional assumptions on the initial data in order to obtain moderate bounds for a solution.

**Theorem 1.** *a) Let  $n \leq 5$ . Then there exists a net  $a_\varepsilon$  such that for every  $T > 0$  there exists a unique solution to (3) in  $\mathcal{G}_{L^2}([0, T) \times \mathbb{R}^n)$ .*

*b) Let  $n = 6$  and let  $\|A_\varepsilon\|_{H^{3,2}}$  and  $\|B_\varepsilon\|_{H^{2,2}}$  be bounded by  $(\log(\log(\varepsilon^{-1})))^s$ , as  $\varepsilon \rightarrow 0$ , where  $s < 1$ . Then there exists a net  $a_\varepsilon$  such that for every  $T > 0$  there exists a unique solution to (3) in  $\mathcal{G}_{L^2}([0, T) \times \mathbb{R}^n)$ .*

*Proof.*

*Existence*

For every fixed  $\varepsilon$  there exists a unique smooth solution to equation (3) (see [7, Theorem 4.8]) since  $g_\varepsilon$  is globally Lipschitz. Energy inequality (cf. [11, (2.5)]) and (2) give

$$(4) \quad \begin{aligned} \|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(t)\|_{L^2} &\leq \|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(0)\|_{L^2} + \int_0^t \|g_\varepsilon(G_\varepsilon(s))\|_{L^2} ds \\ &\leq \|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(0)\|_{L^2} + \int_0^t a_\varepsilon^{m_1} \|G_\varepsilon(s)\|_{L^2} ds. \end{aligned}$$

Since

$$(5) \quad \|G_\varepsilon(t)\|_{L^2} \leq C_T \|\nabla G_\varepsilon\|_{L^2},$$

Gronwall type inequality gives

$$\|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(t)\|_{L^2} \leq \|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(0)\|_{L^2} \exp(T C_T a_\varepsilon^{m_1})$$

for  $\varepsilon$  small enough. Now we chose a net  $a_\varepsilon$  such that  $a_\varepsilon = O((\log \varepsilon^{-1})^{1/m_1})$ . Then

$$\|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(t)\|_{L^2} \leq C \varepsilon^{-N}, \text{ as } \varepsilon \rightarrow 0, t \in [0, T], \text{ for some } N > 0, C > 0.$$

By (5) this is also true for  $\|G_\varepsilon(t)\|_{L^2}$ ,  $t \in [0, T]$ .

We differentiate equation (3) with respect to a spatial variable  $y$  in order to obtain moderate bounds for higher order derivatives. Then

$$(\partial_t^2 - \Delta) \partial_y G_\varepsilon = -g'_\varepsilon(G_\varepsilon) \partial_y G_\varepsilon.$$

By (4) we have

$$(6) \quad \begin{aligned} & \|(\partial_{ty} G_\varepsilon, \nabla \partial_y G_\varepsilon)(t)\|_{L^2} \\ & \leq \|(\partial_{ty} G_\varepsilon, \nabla \partial_y G_\varepsilon)(0)\|_{L^2} + \int_0^t \|g'_\varepsilon(G_\varepsilon(s)) \partial_y G_\varepsilon(s)\|_{L^2} ds \\ & \leq \|(\partial_{ty} G_\varepsilon, \nabla \partial_y G_\varepsilon)(0)\|_{L^2} + \int_0^t a_\varepsilon^{m_1} \|\partial_y G_\varepsilon(s)\|_{L^2} ds. \end{aligned}$$

Since  $\|\partial_y G_\varepsilon(s)\|_{L^2}$  is of the moderate growth,  $\sup_{t \in [0, T]} \|(\partial_{ty} G_\varepsilon, \nabla \partial_y G_\varepsilon)(t)\|_{L^2}$  is also of the moderate growth. Therefore the second derivative with respect to the time variable is also moderate, because of (3).

Another differentiation with respect to a spatial variable  $y$  gives

$$(\partial_t^2 - \Delta) \partial_{yz} G_\varepsilon = -g'_\varepsilon(G_\varepsilon) \partial_{yz} G_\varepsilon - g''_\varepsilon(G_\varepsilon) \partial_y G_\varepsilon \partial_z G_\varepsilon$$

and by (4) we have

$$(7) \quad \begin{aligned} & \|(\partial_{tz} \nabla G_\varepsilon, \nabla^2 \partial_z G_\varepsilon)(t)\|_{L^2} \leq \|(\partial_{tz} \nabla G_\varepsilon, \nabla^2 \partial_z G_\varepsilon)(0)\|_{L^2} \\ & + \int_0^t \|g'_\varepsilon(G_\varepsilon(s)) \partial_z \nabla G_\varepsilon(s) + g''_\varepsilon(G_\varepsilon(s)) \nabla G_\varepsilon(s) \partial_z G_\varepsilon(s)\|_{L^2} ds. \end{aligned}$$

(Here  $\nabla^2 G_\varepsilon = [\partial^2 G_\varepsilon / \partial x_i \partial x_j]_{i,j=1, \dots, n}$ .)

*Case  $n \leq 4$*

Since  $\|\nabla G_\varepsilon\|_{L^2}^2 \leq \|\nabla G_\varepsilon\|_{L^4}^2 \leq \|\nabla G_\varepsilon\|_{H^1}^2$ , (7) and (3) imply that  $(\partial_t \nabla G_\varepsilon, \nabla^2 G_\varepsilon)$  and  $\partial_{tt} G_\varepsilon$  are moderate. Other derivatives can be approximated in a similar manner.

*Case  $n = 5$*

Note that  $H^{9/4,2}(\mathbb{R}^5) \hookrightarrow H^{1,4}(\mathbb{R}^5)$  and  $H^{9/4,2} = (H^{2,2}(\mathbb{R}^5), H^{3,2}(\mathbb{R}^5))_{[1/4]}$ , where  $(\cdot, \cdot)_{[\theta]}$  denotes a complex interpolation space. This implies

$$\|u\|_{H^{3,2}} \leq C_2 \|u\|_{H^{2,2}}^{3/4} \|u\|_{H^{3,2}}^{1/4}.$$

Inequality (7) implies

$$\begin{aligned} \|G_\varepsilon(t)\|_{H^{3,2}} &\leq \|G_\varepsilon(0)\|_{H^{3,2}} + \int_0^t a_\varepsilon^{m_2} \|G_\varepsilon(s)\|_{H^{2,2}} ds \\ &\quad + \int_0^t a_\varepsilon^{m_2} \|G_\varepsilon(s)\|_{H^{2,2}}^{3/2} \|G_\varepsilon(s)\|_{H^{3,2}}^{1/2} ds. \end{aligned}$$

Writing  $u_\varepsilon(t) = \|G_\varepsilon(s)\|_{H^{3,2}}^{1/2}$ , the last inequality implies

$$\begin{aligned} \frac{d}{dt} u_\varepsilon^2(t) &\leq a_\varepsilon^{m_1} \|G_\varepsilon(t)\|_{H^{2,2}} + \|G_\varepsilon(t)\|_{H^{2,2}}^{3/2} u_\varepsilon(t) \\ u_\varepsilon(0) &= \|G_\varepsilon(0)\|_{H^{3,2}}^{1/2}. \end{aligned}$$

If  $u_\varepsilon(t) \leq 1$ ,  $t \in [0, T]$ , then  $\|G_\varepsilon(s)\|_{H^{3,2}}$  is obviously moderate. Otherwise,  $u_\varepsilon(t) \leq v_\varepsilon(t)$ , where  $v_\varepsilon$  is a solution to

$$\frac{d}{dt} v_\varepsilon(t) \leq \frac{1}{2} \left( a_\varepsilon^{m_1} \|G_\varepsilon(t)\|_{H^{2,2}} + \|G_\varepsilon(t)\|_{H^{2,2}}^{3/2} \right), \quad v_\varepsilon(t_0) = \max\{1, \|G_\varepsilon(0)\|_{H^{3,2}}^{1/2}\}$$

and  $t \in [0, T]$ . Since  $\|G_\varepsilon(t)\|_{H^{2,2}}$  and  $\|G_\varepsilon(0)\|_{H^{3,2}}$  are of the moderate growth, this is true for  $v_\varepsilon(t)$  also and hence for  $u_\varepsilon(t)$ .

Let us prove that  $(\partial_t \nabla^3 G_\varepsilon(t), \nabla^4 G_\varepsilon(t))$  is of the moderate growth. After another differentiation in a space direction, (2) implies

$$\begin{aligned} (8) \quad \|(\partial_t \nabla^3 G_\varepsilon(t), \nabla^4 G_\varepsilon(t))\|_{L^2} &\leq \|\nabla^4 G_\varepsilon(0)\|_{L^2} + \int_0^t a_\varepsilon^m \|\nabla^3 G_\varepsilon(s)\|_{L^2} ds \\ &\quad + 2 \int_0^t a_\varepsilon^{m_2} \|\nabla G_\varepsilon(s) \nabla^2 G_\varepsilon(s)\|_{L^2} ds \\ &\quad + \int_0^t a_\varepsilon^{m_3} \| |\nabla G_\varepsilon(s)|^3 \|_{L^2} ds, \end{aligned}$$

where components of  $\nabla G_\varepsilon \nabla^2 G_\varepsilon$  are products of the first and the second order spatial derivatives of  $G_\varepsilon$ .  $\|\nabla^4 G_\varepsilon(0)\|_{L^2}$  is moderate, since the initial data are. We have already proved that  $\|\nabla^3 G_\varepsilon(s)\|_{L^2}$  is moderate. The other two terms on the right-hand side of (8) can be estimated in the following way.

$$\|\nabla G_\varepsilon(t) \nabla^2 G_\varepsilon(t)\|_{L^2} \leq \|\nabla G_\varepsilon(t)\|_{L^6} \|\nabla^2 G_\varepsilon(t)\|_{L^3} \leq \|G_\varepsilon(t)\|_{H^{1,6}} \|G_\varepsilon(t)\|_{H^{2,3}}.$$

By the Sobolev embedding theorem  $H^{3,2}(\mathbb{R}^n) \hookrightarrow H^{2,3}(\mathbb{R}^n)$ ,  $H^{3,2}(\mathbb{R}^n) \hookrightarrow H^{1,6}(\mathbb{R}^n)$ , for  $n \leq 6$ . Then

$$\|\nabla G_\varepsilon(t) \nabla^2 G_\varepsilon(t)\|_{L^2} \leq \|G_\varepsilon(t)\|_{H^{3,2}},$$

And

$$\| |\nabla G_\varepsilon(t)|^3 \|_{L^2} \leq \|\nabla G_\varepsilon(t)\|_{L^6}^3 \leq \|G_\varepsilon(t)\|_{H^{3,2}}^3.$$

In this way we obtain the moderateness of the right-hand side of (8).

All the other derivatives can be more easily estimated in the similar way.

*Case  $n = 6$*

Similarly to the case  $n \leq 5$ , by (4), (5) and the assumption on the initial data, one can deduce that  $a_\varepsilon$  has to be  $\mathcal{O}((\log(\log(\log \varepsilon^{-1})))^r)$ , where  $r + s < 1$ . Then  $\|G_\varepsilon\|_{H^{2,2}} = \mathcal{O}((\log \varepsilon^{-1})^{r+s})$ .

Since  $\|\nabla G_\varepsilon(s)\|_{L^2}^2 \leq \|\nabla G_\varepsilon(s)\|_{L^4}^2$ ,  $H^{5/2}(\mathbb{R}^6) \hookrightarrow H^{1,4}(\mathbb{R}^6)$ , and  $H^{5/2,2}(\mathbb{R}^6) = (H^{2,2}(\mathbb{R}^6), H^{3,2}(\mathbb{R}^6))_{1/2}$ ,

$$\begin{aligned} \|G_\varepsilon(t)\|_{H^{3,2}} &\leq \|G_\varepsilon(0)\|_{H^{3,2}} + \int_0^t a_\varepsilon^m \|G_\varepsilon(s)\|_{H^{2,2}} ds \\ &\quad + \int_0^t a_\varepsilon^{m_2} \|G_\varepsilon(s)\|_{H^{2,2}} \|G_\varepsilon(s)\|_{H^{3,2}} ds. \end{aligned}$$

By the Gronwall type inequality, the moderate growth of  $\|G_\varepsilon(t)\|_{H^{3,2}}$  follows from the assumptions of the theorem. The higher order derivatives can be estimated in the same manner as in the previous case.

*Uniqueness*

Now, let us show the uniqueness of the solution in  $\mathcal{G}_{L^2}([0, T] \times \mathbb{R}^n)$ . Let  $\tilde{G}_\varepsilon$  be another solution. Then

$$(\partial_t^2 - \Delta)(G_\varepsilon - \tilde{G}_\varepsilon) = -g_\varepsilon(G_\varepsilon) + g_\varepsilon(\tilde{G}_\varepsilon) + N_\varepsilon$$

and  $(G_\varepsilon - \tilde{G}_\varepsilon)|_{t=0} = N_{1\varepsilon}$ ,  $\partial_t(G_\varepsilon - \tilde{G}_\varepsilon)|_{t=0} = N_{2\varepsilon}$ , where  $N_\varepsilon \in \mathcal{N}_{L^2}([0, T] \times \mathbb{R}^n)$  and  $N_{i\varepsilon} \in \mathcal{N}_{L^2}(\mathbb{R}^n)$ ,  $i = 1, 2$ . Let  $\bar{G}_\varepsilon = G_\varepsilon - \tilde{G}_\varepsilon$ . Then (4) implies

$$\begin{aligned} (9) \quad \|(\partial_t \bar{G}_\varepsilon, \nabla \bar{G}_\varepsilon)(t)\|_{L^2} &\leq \|(N_{2\varepsilon}, \nabla N_{1\varepsilon})\|_{L^2} + \|N_\varepsilon(t)\|_{L^2} \\ &\quad + \int_0^t \|g_\varepsilon'(G_\varepsilon(s))\|_{L^\infty} \|\bar{G}_\varepsilon(s)\|_{L^2} ds. \end{aligned}$$

Since  $\|g_\varepsilon'(G_\varepsilon(s))\|_{L^\infty} \leq a_\varepsilon^{m_1}$  and  $a_\varepsilon = \mathcal{O}((\log \varepsilon^{-1})^{1/m_1})$  as  $\varepsilon \rightarrow 0$ , for every  $N > 0$  there exists  $C > 0$  such that

$$\|(\partial_t \bar{G}_\varepsilon, \nabla \bar{G}_\varepsilon)(t)\|_{L^2} \leq C\varepsilon^N.$$

From (5) the same follows for  $\|\bar{G}_\varepsilon(t)\|_{L^2}$ .

Differentiating (10) with respect to a spatial variable  $y$ , we obtain

$$(\partial_t^2 - \Delta)\partial_y \bar{G}_\varepsilon = -g_\varepsilon'(G_\varepsilon)\partial_y G_\varepsilon - g_\varepsilon'(\tilde{G}_\varepsilon)\partial_y \tilde{G}_\varepsilon + \partial_y N_\varepsilon.$$

Using (4) gives

$$\begin{aligned} \|(\partial_{ty} \bar{G}_\varepsilon, \nabla \partial_y \bar{G}_\varepsilon)(t)\|_{L^2} &\leq \|(\partial_y N_{2\varepsilon}, \nabla \partial_y N_{1\varepsilon})\|_{L^2} + \|\partial_y N_\varepsilon(t)\|_{L^2} \\ &\quad + \int_0^t \|(g_\varepsilon'(G_\varepsilon(s)) - g_\varepsilon'(\tilde{G}_\varepsilon(s)))\partial_y G_\varepsilon(s)\|_{L^2} \\ &\quad + \|g_\varepsilon'(\tilde{G}_\varepsilon(s))(\partial_y G_\varepsilon(s) - \partial_y \tilde{G}_\varepsilon(s))\|_{L^2} ds \\ &\leq \|(\partial_y N_{2\varepsilon}, \nabla \partial_y N_{1\varepsilon})(t)\|_{L^2} + \|\partial_y N_\varepsilon(t)\|_{L^2} \\ &\quad + \int_0^t a_\varepsilon^m \|G_\varepsilon(s) - \tilde{G}_\varepsilon(s)\|_{L^2} \|\partial_y G_\varepsilon(s)\|_{L^\infty} \\ &\quad + a_\varepsilon^m \|\partial_y G_\varepsilon(s) - \partial_y \tilde{G}_\varepsilon(s)\|_{L^2} ds. \end{aligned}$$

The Sobolev embedding theorem implies that all  $L^\infty$  norms can be estimated by the corresponding  $L^2$ -norms of higher order derivatives. The above estimates implies the moderate growth for  $\|(\partial_{ty}\bar{G}_\varepsilon, \nabla\partial_y\bar{G}_\varepsilon)(t)\|_{L^2}$ .

Other derivatives can be approximated in the similar manner. This finishes the proof.

*Remark 1.* Let  $n = 7$ . In order to obtain the existence of a unique solution with the moderate growth of all its derivatives, we need that  $H^{3,2}$ -norms of the initial data are bounded by  $\underbrace{\log(\log \dots (\log \varepsilon^{-1}) \dots)}_q$  with respect to  $\varepsilon$  for some  $s$  and  $q$ .

This follows from [8, Theorem 4.8]. The cases  $n = 8, 9$  can be handled out using the procedure and Lemmas 2.1-2.20 in the same paper as well as a composition of the logarithmic functions sufficiently many times.

*Remark 2.* In the case  $n = 1$ , when the initial data have compact supports, the above calculations are simpler since in that case we can use

$$\|G_\varepsilon(t)\|_{L^\infty} \leq C_T \|\nabla G_\varepsilon(t)\|_{L^2}, \quad t \leq T.$$

Also, for the existence of a generalized solution we do not need that the cut-off constants  $a_\varepsilon$  are logarithmically bounded. But we need it for the proof of uniqueness.

**Corollary.** *Let  $g(y)$  be globally Lipschitz. Then*

- a) *If  $n \leq 5$  and  $T > 0$ , then there exists a solution to (1) in  $\mathcal{G}_{L^2}([0, T] \times \mathbb{R}^n)$ .*
- b) *If  $n = 6$ ,  $\|A_\varepsilon\|_{H^3}$  and  $\|B_\varepsilon\|_{H^2}$  are bounded by  $|\log(\log \varepsilon^{-1})|^s$  as  $\varepsilon \rightarrow 0$ ,  $s < 1$ , then for every  $T > 0$  there exists a unique solution to (1) in  $\mathcal{G}_{L^2}([0, T] \times \mathbb{R}^n)$ .*

#### 4. NON REGULARIZED WAVE EQUATIONS

**Theorem 2.** *The equation*

$$(\partial_t^2 - \Delta)G = -G^3, \quad G|_{t=0} = A, \quad \partial_t G|_{t=0} = B,$$

*where  $A, B \in \mathcal{G}_{L^2}(\mathbb{R}^3)$ , has a unique solution in  $\mathcal{G}_{L^2}([0, T] \times \mathbb{R}^3)$  for every  $T > 0$  if there exist representatives of initial data such that*

$$(10) \quad \|(\nabla^2 A_\varepsilon, \nabla B_\varepsilon)\|_{L^2} = o((\log \varepsilon^{-1})^{1/2}).$$

*Proof.* Equation

$$(11) \quad (\partial_t^2 - \Delta)G_\varepsilon = -G_\varepsilon^3, \quad G_\varepsilon|_{t=0} = A_\varepsilon, \quad \partial_t G_\varepsilon|_{t=0} = B_\varepsilon$$

has a unique smooth solution in every strip  $[0, T] \times \mathbb{R}^3$  as one can see in [9]. We have to prove the moderate growth of the solution. Energy inequality immediately gives

$$\|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(t)\|_{L^2} \leq \|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(0)\|_{L^2}$$

and from (5) it follows

$$\|G_\varepsilon(t)\|_{L^2} \leq C_T \|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(t)\|_{L^2} \leq C_T \|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(0)\|_{L^2}.$$



After differentiating equation (11) with respect to the space variables we obtain (using (4))

$$\begin{aligned} \|(\partial_t \nabla G_\varepsilon, \nabla^2 G_\varepsilon)(t)\|_{L^2} &\leq \|\nabla^2 G_\varepsilon(0)\|_{L^2} + 3 \int_0^t \|G_\varepsilon^2(s) \nabla G_\varepsilon(s)\|_{L^2} ds \\ &\leq \|\nabla^2 G_\varepsilon(0)\|_{L^2} + 3 \int_0^t \|G_\varepsilon(s)\|_{L^6}^2 \|\nabla G_\varepsilon(s)\|_{L^6} ds \\ &\leq \|\nabla^2 G_\varepsilon(0)\|_{L^2} + 3 \int_0^t \|G_\varepsilon(s)\|_{H^1}^2 \|\nabla G_\varepsilon(s)\|_{H^1} ds \end{aligned}$$

which gives the moderate growth of the second derivatives of  $G_\varepsilon$  if the initial energy satisfies assumption (10). The moderate growth of  $\partial_t^2 G$  follows from the equation.

By another differentiation of equation with respect to spatial variables we obtain

$$\begin{aligned} \|(\partial_t \nabla G_\varepsilon, \nabla^3 G_\varepsilon)(t)\|_{L^2} &\leq \|\nabla^2 G_\varepsilon(0)\|_{L^2} \\ &\quad + \int_0^t \|3G_\varepsilon^2(s) \nabla^2 G_\varepsilon(s) + 6G_\varepsilon(s) |\nabla G_\varepsilon(s)|^2\|_{L^2} ds \\ &\leq \|\nabla^2 G_\varepsilon(0)\|_{L^2} + 3 \int_0^t \|G_\varepsilon(s)\|_{H^1} \|\nabla^2 G_\varepsilon(s)\|_{H^1} \\ &\quad + 2\|G_\varepsilon(s)\|_{H^1} \|\nabla G_\varepsilon(s)\|_{H^1}^2 ds. \end{aligned}$$

This implies the existence of desired bounds. Other derivatives can be handled in a similar way.

Let us show the uniqueness of the solution. Let  $\tilde{G}_\varepsilon$  be another solution. Then, by taking the notation from the proof of Theorem 1

$$(\partial_t^2 - \Delta)\bar{G}_\varepsilon = -G_\varepsilon^3 + \tilde{G}_\varepsilon^3 + N_\varepsilon$$

and

$$\begin{aligned} \|(\partial_t \bar{G}_\varepsilon, \nabla \bar{G}_\varepsilon)(t)\|_{L^2} &\leq \|(N_{2\varepsilon}, \nabla N_{1\varepsilon})\|_{L^2} + \|N_\varepsilon(t)\|_{L^2} \\ &\quad + \int_0^t \|(G_\varepsilon^2(s) + G_\varepsilon(s)\tilde{G}_\varepsilon(s) + \tilde{G}_\varepsilon^2(s))\bar{G}_\varepsilon(s)\|_{L^2} ds. \end{aligned}$$

Like in the previous case the integral can be estimated by

$$\int_0^t (\|G_\varepsilon(s)\|_{L^6}^2 + \|G_\varepsilon(s)\|_{L^6} \|\tilde{G}_\varepsilon(s)\|_{L^6} + \|\tilde{G}_\varepsilon^2(s)\|_{L^6}) \|\bar{G}_\varepsilon(s)\|_{L^6} ds$$

and the Gronwall type inequality gives the moderate estimate like in the existence proof. Also, this gives the moderate estimate for  $\|G_\varepsilon\|_{L^2}$ . The usual estimations

$$\begin{aligned} \|\nabla(G_\varepsilon - \tilde{G}_\varepsilon)(s)\|_{L^2} &\leq \|3G_\varepsilon^2(s) \nabla G_\varepsilon(s) - 3\tilde{G}_\varepsilon^2(s) \nabla \tilde{G}_\varepsilon(s)\|_{L^2} \\ &\leq \|3(G_\varepsilon^2(s) - \tilde{G}_\varepsilon^2(s)) \nabla G_\varepsilon(s)\|_{L^2} \\ &\quad + \|3(\nabla G_\varepsilon^2(s) - \nabla \tilde{G}_\varepsilon^2(s)) \tilde{G}_\varepsilon(s)\|_{L^2} \\ &\leq 3\|G_\varepsilon^2(s) - \tilde{G}_\varepsilon^2(s)\|_{L^6} \|G_\varepsilon^2(s) + \tilde{G}_\varepsilon^2(s)\|_{L^6} \|\nabla G_\varepsilon(s)\|_{L^6} \\ &\quad + 3\|\nabla G_\varepsilon^2(s)\|_{L^6} \|\nabla G_\varepsilon^2(s) - \nabla \tilde{G}_\varepsilon^2(s)\|_{L^6} \\ &\leq 3\|G_\varepsilon^2(s) - \tilde{G}_\varepsilon^2(s)\|_{H^1} \|G_\varepsilon^2(s) + \tilde{G}_\varepsilon^2(s)\|_{H^1} \|\nabla G_\varepsilon(s)\|_{H^1} \\ &\quad + 3\|\nabla G_\varepsilon(s)\|_{H^1}^2 \|\nabla G_\varepsilon^2(s) - \nabla \tilde{G}_\varepsilon^2(s)\|_{H^1} \end{aligned}$$

give the desired result for the first order derivatives of  $G_\varepsilon$ . All the other derivatives can be estimated in a similar manner.

In order to compose generalized functions we use the algebra of Colombeau tempered generalized functions (cf. [2])

$$\mathcal{G}_t(\mathbb{R}) = \mathcal{E}_t(\mathbb{R})/\mathcal{N}_t(\mathbb{R}),$$

where

$$\mathcal{E}_t(\mathbb{R}) = \{G_\varepsilon \in \mathcal{E}(\mathbb{R}) : \text{for every } \alpha \in \mathbb{N}_0^n \text{ there exist } M \in \mathbb{N}, b \in \mathbb{R} \text{ such that}$$

$$\sup_{x \in \mathbb{R}} |\partial^\alpha G_\varepsilon(x)| (1 + |x|)^{-M} = \mathcal{O}(\varepsilon^b)\}.$$

$$\mathcal{N}_t(\mathbb{R}) = \{G_\varepsilon \in \mathcal{E}(\mathbb{R}) : \text{for every } \alpha \in \mathbb{N}_0^n \text{ and } p \in \mathbb{R} \text{ there exist } M \in \mathbb{N} \text{ such that}$$

$$\sup_{x \in \mathbb{R}} |\partial^\alpha G_\varepsilon(x)| (1 + |x|)^{-M} = \mathcal{O}(\varepsilon^p)\}.$$

One can easily prove that the composition  $F(G) \in \mathcal{G}_{L^p}(\Omega)$  makes sense if  $F \in \mathcal{G}_t(\mathbb{R})$  and  $G \in \mathcal{G}_{L^p}(\Omega)$ .

The following theorem is a generalized version of the 3D wave equation with sub-critical nonlinearity ( $p < 5$ ).

**Theorem 3.** *Let a representative of  $F \in \mathcal{E}_t(\mathbb{R})$  satisfy*

$$|F'_\varepsilon(y)| \leq C(\log(\varepsilon^{-1}))^r (|y|^{\rho-1} + |y|^{\min(\rho-1,1)}), \quad r < 5, \quad y \in \mathbb{R}, \quad \varepsilon \text{ is small enough},$$

where  $\rho < 5$ ,  $F_\varepsilon(0) = 0$  and let its primitive be non negative. Let  $A, B \in \mathcal{G}_{L^2}(\mathbb{R}^3)$ . Then the equation

$$(12) \quad G_{tt} - \Delta G + F_\varepsilon(G) = 0, \quad G|_{t=0} = A, \quad G_t|_{t=0} = B$$

has a unique solution in  $\mathcal{G}_{L^2}([0, T] \times \mathbb{R}^3)$  for every  $T > 0$  if  $\|A_\varepsilon\|_{H^{1,2}}$  and  $\|B_\varepsilon\|_{H^{1,2}}$  are bounded by  $(\log(\varepsilon^{-1}))^s$ ,  $r + 4s < 1$  as  $\varepsilon \rightarrow 0$ .

*Proof.*

*Existence*

For every  $\varepsilon$  small enough there exists a unique solution  $G_\varepsilon$  to (12) (cf. [8, Theorem 4.8]). We have to prove that it belongs to  $\mathcal{E}_{L^2}([0, T] \times \mathbb{R}^3)$  for every  $T > 0$ .

If  $\rho \leq 0$ , then the proof is the same as in Theorem 2. Thus, let us assume  $\rho > 0$ . The energy inequality implies  $\|(\partial_t G_\varepsilon, \nabla G_\varepsilon)(t)\|_{H^1} = \mathcal{O}((\log(\varepsilon^{-1}))^s)$  as  $\varepsilon \rightarrow 0$ . Lemma 4.1 in [8] implies that for every  $\delta > 0$

$$(13) \quad \begin{aligned} & \|G_\varepsilon(t)\|_{H^{2\delta, 2/\delta}} + \|\partial_t G_\varepsilon(t)\|_{H^{\delta, 2/\delta}} \leq C(\|G_\varepsilon(0)\|_{H^{4,2}} + \|\partial_t G_\varepsilon(0)\|_{H^{3,2}}) \\ & + C_1(\log(\varepsilon^{-1}))^{1/r} \int_0^t (1+t-s)|t-s|^{-\alpha} (\|G_\varepsilon(s)\|_{H^{1,2}}^{\rho-1} + 1) \|G_\varepsilon(s)\|_{H^{2,2}} ds, \end{aligned}$$

where  $C, C_1, N > 0$  and  $\alpha \in (0, 1)$  are suitable constants. Thus, putting  $\delta = 1$  and using suitable constants, we have

$$\begin{aligned} & \|G_\varepsilon(t)\|_{H^{2,2}} + \|\partial_t G_\varepsilon(t)\|_{H^{1,2}} \leq C(\|G_\varepsilon(0)\|_{H^{4,2}} + \|\partial_t G_\varepsilon(0)\|_{H^{3,2}}) \\ & \cdot \exp((C_1 \log \varepsilon^{-1})^r C_2 \int_0^t (1+t-s)|t-s|^{-\alpha} ds (\|G_\varepsilon(s)\|_{H^{1,2}}^{\rho-1} + 1)) \\ & \leq C\varepsilon^{-N_1} \exp(C_T(\log \varepsilon^{-1})^{r+4s}) \leq C\varepsilon^{-N_1-C_T}, \quad \varepsilon \rightarrow 0, \quad t \in [0, T]. \end{aligned}$$

Since the space dimension equals 3,  $\|G_\varepsilon(t)\|_{L^\infty} = \mathcal{O}(\varepsilon^{-M})$  as  $\varepsilon \rightarrow 0$  for some  $M > 0$ . Now, differentiating (12) with respect to the spatial variables and the energy inequality one obtain the moderate growth of all derivatives of  $G_\varepsilon$ .

*Uniqueness*

Let  $G_\varepsilon$  and  $\tilde{G}_\varepsilon$  be solutions to (12). Denote  $\overline{G}_\varepsilon = G_\varepsilon - \tilde{G}_\varepsilon$ . Then

$$\begin{aligned} \|\overline{G}_\varepsilon(t)\|_{H^{1,2}} &\leq \|\overline{G}_\varepsilon(0)\|_{H^{1,2}} \\ &+ C(\log \varepsilon^{-1})^r \int_0^t (\|G_\varepsilon(s)\|_{L^2} + \|\tilde{G}_\varepsilon(s)\|_{L^2}) \|\overline{G}_\varepsilon(s)\|_{L^2} ds. \end{aligned}$$

Using (13), the Gronwall type inequality and growth assumptions of the initial data with respect to  $\varepsilon$ , one can see that  $\|\overline{G}_\varepsilon(t)\|_{H^{1,2}} = \mathcal{O}(\varepsilon^M)$  for every  $M \in \mathbb{R}$ . The higher derivatives can be estimated as above.

By the similar procedure it is possible to prove the following theorem for  $n = 4$ .

**Theorem 4.** *Let a representative of  $F \in \mathcal{G}_t(\mathbb{R})$  satisfy*

$$\begin{aligned} |F'_\varepsilon(y)| &\leq C(\log(\log(\log \varepsilon^{-1}))) (|y|^{\rho-1} + |y|^{\min(\rho-1,1)}), \\ |F''_\varepsilon(y)| &\leq C(\log(\log \varepsilon^{-1})) (|y|^{\beta-1} + |y|^{\min(\rho-1,1)}), \quad y \in \mathbb{R}, \quad \varepsilon \text{ is small enough,} \end{aligned}$$

where  $\rho < 3$ ,  $\beta \geq 1$ ,  $F_\varepsilon(0) = 0$  and let its primitive be non negative. Let  $A, B \in \mathcal{G}_{L^2(\mathbb{R}^3)}$ . Then the equation

$$G_{tt} - \Delta G + F_\varepsilon(G) = 0, \quad G|_{t=0} = A, \quad G_t|_{t=0} = B$$

has a unique solution in  $\mathcal{G}_{L^2}([0, T] \times \mathbb{R}^4)$ , for every  $T > 0$ , if  $\|A_\varepsilon\|_{H^{1,2}}$  and  $\|B_\varepsilon\|_{H^{1,2}}$  are bounded by  $\log(\log \varepsilon^{-1})$ , as  $\varepsilon \rightarrow 0$ .

*Remark.* It is possible to prove the existence and uniqueness for (12) for the space dimensions  $n = 5, 6, 7, 8, 9$ , when  $F_\varepsilon(y)$  satisfies appropriate growth conditions with respect to both  $\varepsilon$  and  $y$ . Like in Remark 1 after Theorem 1, one has to iterate the composition of logarithmic functions sufficiently many times in order to obtain the moderate bounds with respect to  $\varepsilon$ .

## 5. COMPARISON OF THE SOLUTIONS TO THE REGULARIZED AND THE NON-REGULARIZED EQUATIONS

**Theorem 5.** *Let  $u$  be a classical  $H^2$  solution to the  $n$ -dimensional semilinear wave equation ( $n \leq 3$ )*

$$(\partial_t^2 - \Delta)u + g(u), \quad u|_{t=0} = a, \quad \partial_t u|_{t=0} = b$$

where  $a \in H^3(\mathbb{R})$  and  $b \in H^2(\mathbb{R})$  have compact supports, a primitive function of  $g$  is positive and  $g$  satisfies the assumptions of Section 2. Then for every  $T > 0$  there exist  $s \in \mathbb{R}$  and a net  $a_\varepsilon \rightarrow \infty$  as  $\varepsilon \rightarrow 0$ , such that the solution  $G_\varepsilon$  to the regularized equation (3) with the initial data  $A_\varepsilon = a * \phi_{\varepsilon^s}$ ,  $B_\varepsilon = b * \phi_{\varepsilon^s}$  converges to  $u$  in  $H^2(\mathbb{R}^3)$ -norm for every  $t < T$ .

*Proof.* Since  $u \in H^2(\mathbb{R}^n)$ ,  $n \leq 3$ ,  $\|u(\cdot, t)\|_{L^\infty} \leq C$  for every  $t > 0$ . Therefore  $g_\varepsilon(u) = g(u)$  for  $\varepsilon$  small enough. Let  $D_\varepsilon = (A_\varepsilon - a, B_\varepsilon - b)$ . Since the initial data

$a$  and  $b$  are smooth enough,  $\|D_\varepsilon\|_{H^2} = \mathcal{O}(\varepsilon^s)$  as  $\varepsilon \rightarrow 0$ . Using inequality (4) we obtain

$$\begin{aligned} \|(\partial_t - \Delta)(G_\varepsilon - u)(t)\|_{L^2} &\leq \|D_\varepsilon\|_{H^1} + \int_0^t \|g_\varepsilon(G_\varepsilon(s)) - g_\varepsilon(u(s))\|_{L^2} ds \\ &\leq \|D_\varepsilon\|_{H^1} + a_\varepsilon^m \int_0^t \|G_\varepsilon(s) - u(s)\|_{L^2} ds \\ &\leq \|D_\varepsilon\|_{H^1} + C a_\varepsilon^m \int_0^t \|\nabla(G_\varepsilon(s)) - \nabla u(s)\|_{L^2} ds. \end{aligned}$$

Using the Gronwall type lemma and the construction of the regularized solution gives

$$\begin{aligned} \|\nabla(G_\varepsilon(t) - u(t))\|_{L^2} &\leq \|D_\varepsilon\|_{H^1} \exp(C_T a_\varepsilon^m t) \\ &\leq C \varepsilon^s \exp(C_T a_\varepsilon^{m_1} t) \rightarrow 0, \quad \varepsilon \rightarrow 0. \end{aligned}$$

Differentiating the equation with respect to a space variable  $y$  gives

$$(\partial_t^2 - \Delta)(\partial_y G_\varepsilon - \partial_y u) = g_\varepsilon'(u) \partial_y u - g_\varepsilon'(G_\varepsilon) \partial_y G_\varepsilon.$$

We have

$$\begin{aligned} &\|g_\varepsilon'(u(t)) \nabla u(t) - g_\varepsilon'(G_\varepsilon(t)) \nabla G_\varepsilon(t)\|_{L^2} \\ &\leq \|g_\varepsilon'(G_\varepsilon(t)) (\nabla G_\varepsilon(t) - \nabla u(t) + \nabla u(t) (g_\varepsilon'(G_\varepsilon(t)) - g_\varepsilon'(u(t)))\|_{L^2} \\ &\leq a_\varepsilon^m \|\nabla G_\varepsilon(t) - \nabla u(t)\|_{L^2} + a_\varepsilon^{m_2} \|\nabla u(t)\|_{L^2} \|G_\varepsilon(t) - u(t)\|_{L^\infty} \\ &\leq a_\varepsilon^m \|\nabla G_\varepsilon(t) - \nabla u(t)\|_{L^2} + a_\varepsilon^{m_2} \|\nabla u(t)\|_{L^2} \|\nabla^2 G_\varepsilon(t) - \nabla^2 u(t)\|_{L^2}. \end{aligned}$$

Therefore

$$\begin{aligned} \|\nabla(\nabla G_\varepsilon(t) - \nabla u(t))\|_{L^2} &\leq \|D_\varepsilon\|_{H^2} + a_\varepsilon^m \int_0^t \|\nabla G_\varepsilon(s) - \nabla u(s)\|_{L^2} ds \\ &\quad + a_\varepsilon^m \int_0^t \|\nabla u(s)\|_{L^2} \|\nabla^2 G_\varepsilon(s) - \nabla^2 u(s)\|_{L^2} ds. \end{aligned}$$

Using the Gronwall type lemma yields

$$\begin{aligned} \|\nabla(\nabla G_\varepsilon(t) - \nabla u(t))\|_{L^2} &\leq \|D_\varepsilon\|_{H^2} \exp(a_\varepsilon^m C_T T + a_\varepsilon^{m_2} T C_{1,T}) \\ &\leq C \varepsilon^s \exp(a_\varepsilon^m C_T T + a_\varepsilon^{m_2} T C_{1,T}), \end{aligned}$$

where  $C_{1,T}$  is the constant from the proof of Theorem 1. If we choose  $s$  large enough and  $a_\varepsilon \rightarrow \infty$  slowly enough, then the right-hand side tends to zero as  $\varepsilon \rightarrow 0$ .

**Theorem 6.** *Let  $n = 1$  and let the primitive function of  $g_\varepsilon$ ,  $G_\varepsilon$ , satisfies  $G_\varepsilon(0) = 0$  and  $G_\varepsilon(u) \geq 0$ ,  $u \in \mathbb{R}$   $\varepsilon \in (0, 1)$ . If*

$$\|(B_\varepsilon, \partial_x A_\varepsilon)\|_{L^2} = o(a_\varepsilon) \text{ as } \varepsilon \rightarrow 0,$$

then the solution to the regularized equation (3) is also the solution to the non-regularized one (1).

*Proof.* The energy inequality gives

$$\|(\partial_t G_\varepsilon, \partial_x G_\varepsilon)(t)\|_{L^2} \leq \|(B_\varepsilon, \partial_x A_\varepsilon)\|_{L^2}, \quad t \in [0, T],$$

and the Sobolev embedding theorem gives

$$\|G_\varepsilon(t, \cdot)\|_{L^\infty} \leq C \|\partial_x G_\varepsilon(t, \cdot)\|_{L^2}, \quad t \in [0, T]$$

Therefore,  $\|G_\varepsilon\|_{L^\infty} \leq a_\varepsilon$  and the result follows from the construction of  $g_\varepsilon$ .

*Remark 1.* If we regularize the 1-dimensional wave equation with an arbitrary moderate net of positive numbers  $a_\varepsilon$  (when the proof of uniqueness does not hold), so that  $a_\varepsilon > \|(B_\varepsilon, \partial_x A_\varepsilon)\|_{L^2}$ , then every solution to the regularized equation is the solution to non regularized one.

The following example illustrates that logarithmic bounds of initial data are needed even in a linear case.

*Example.* Consider the linear Klein-Gordon equation

$$u_{tt} - u_{xx} = \varepsilon^{-2}u, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = \delta.$$

Its solution, for every fixed  $\varepsilon$ , is  $u(x, t) = \frac{1}{2}H(t^2 - x^2)I_0(\sqrt{t^2 - x^2}/2)$ . Net of functions  $u_\varepsilon(x, t) = \frac{1}{2}H(t^2 - x^2)I_0(\sqrt{t^2 - x^2}/2) * \phi_\varepsilon(x, t)$  satisfies equation

$$u_{\varepsilon tt} - u_{\varepsilon xx} = \varepsilon^{-2}u_\varepsilon, \quad u_\varepsilon|_{t=0} = 0, \quad u_{\varepsilon t}|_{t=0} = \phi_\varepsilon(x).$$

This net does not belong to  $\mathcal{E}_M([0, T] \times \mathbb{R})$ . Let us prove this. It is known that

$$I_0(z/\varepsilon) \sim C \exp(z/\varepsilon)/\sqrt{z/\varepsilon}, \quad z \rightarrow \infty,$$

for suitable  $C > 0$ . Hence, if  $(x, t)$  varies in some compact  $K$  such that  $t^2 - x^2 \geq z_0^2$ , for some  $z_0 > 0$ , then  $u_\varepsilon$  does not satisfy a moderate growth.

*Remark 2.* In the case  $n \leq 3$  one can use formulas for the fundamental solution and work directly with  $L^\infty$  norms and obtain the generalized solution. This is the case of 3-dimensional sub-cubic wave equation in [2].

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