SEMIGROUPS IN GENERALIZED FUNCTION ALGEBRAS. HEAT EQUATION WITH SINGULAR POTENTIAL AND SINGULAR DATA

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ABSTRACT. Nets of C_0-semigroups $(S_\varepsilon)_\varepsilon$ with polynomial growth in ε as $\varepsilon \to 0$ are used for solving Cauchy problems $(\partial_t - \Delta)u + Vu = f(t,u)$, $u(0,x) = u_0(x)$, in particular for $f = 0$, in suitable generalized function algebras, where V and u_0 are singular generalized functions and f satisfies Lipshitz type conditions. The existance of distribution solutions to some classes of such equations is given.

1. INTRODUCTION

In this paper nets of C_0-semigroups, with the controlled growth rate with respect to a parameter, are used in solving a class of heat equations with singular coefficients and data. The general idea is simple. It lies in the core of a construction of a generalized function space or algebra (cf. [9] and [24]). Singular coefficients (generalized functions) of a PDE are regularized to become nets of smooth functions depending on a small parameter ε. Regularized PDE is then solved using an appropriate net of semigroups. A net of solutions obtained in this way represents a generalized function solution. Moreover, apriori bounds imply that the nets of solutions contain convergent sequences in L^2 or L^1 space and this leads to distribution solutions of corresponding linear and semilinear equations with singular data or potential.

we will use different variants of Colombeau type generalized function algebras. They contain embedded distributions and with the notion of association in such algebras the notions of weak limit and equality in distribution theory are extended. We refer to [8], [24], [3] and the recent papers [12], [11], [13] for the properties of Colombeau type algebras (and distributions embedded therein) and their use in PDEs.

The first part of this paper is devoted to the construction and analysis of generalized semigroups which map algebras of generalized functions into themselves. They are defined after constructions of generalized function spaces. The concept of an associated solution for the equation $\partial_t G = AG$ is defined through the existence of the limit $\varepsilon^{-a} \sup_{t\in[0,T)} ||\partial_t G_\varepsilon - A_\varepsilon G_\varepsilon||_{L^2} \to 0$, $\varepsilon \to 0$, for every $a > 0$, where $(G_\varepsilon)_\varepsilon$ and $(A_\varepsilon)_\varepsilon$ are appropriate nets determining a generalized function G and a generalized infinitesimal generator A. Later, this concept is used for the definition of generalized solutions to Cauchy problems under considerations. We pose in Remark 1 an open problem concerning the Hille -Yosida type theorem for a generalized semigroup of operators.

Note that the analysis of families of semigroups and corresponding families of resolvents and infinitesimal generators dates back to Trotter [28] and has been used later by many authors.

In the second part of the paper, we use semigroups related to Schrödinger operators $\Delta - V_\varepsilon$, $\varepsilon \in (0,1)$, in solving a class of linear and semilinear parabolic equations $\partial_t u - (\Delta - V)u = f$, $u|_{t=0} = u_0$, with singular potential V and singular initial data u_0.

We refer to [1] and the references therein for the stationary case, $\partial_t u \equiv 0$, which concerns $-\Delta$ and its singular perturbations (for example, $\Delta u(x) - \alpha u(0)\delta(x) = 0$), Here we will give only some remarks related to this case.

Concerning semilinear parabolic equations with distributional singularities and potential $V = 0$, the work of Brézis and Friedman in [7] gave the stimulus for many papers in this direction. We mention Kato [14], Kato and Ponce [15], Kozono and Yamazaki [17], Biagioni, Cadeddu and Gramchev [4]. In general, in these papers conditions on a growth order of a nonlinear term $g(u) = u|u|^p$ and the order of singularity of initial data lead to a unique global solvability in an appropriate Kato type space. For instance, in the paper [4], Biagioni, Cadeddu and Gramchev considered the Cauchy problem for the semilinear parabolic equation $\partial_t u - \Delta u + g(u) = 0,\ t > 0,\ x \in \mathbb{R}^n$, where $g(u)$ is a locally Lipschitz real-valued function and the initial data are strongly singular, i.e. belong to the strong dual of the Banach space $C_b^k(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$ of functions with bounded derivatives up to the order k. Colombeau and Langlais studied in [10] this problem for semilinear parabolic equation in case $n = 1$ and $g(u) = u^3$ and recovered the classical solutions when the initial data are L^p-functions.

In this paper, there are basically two types of Cauchy problems which are solved in generalized function algebra, $\mathcal{G}_{C^1,H^2}([0,T) : \mathbb{R}^n)$, $T > 0$, $n \le 3$. Initial data u_0 are taken to be the elements of Colombeau type space $\mathcal{G}_{H^2}(\mathbb{R}^n)$. This involves singular initial data, embedded singular distributions of the form $u_0 = \sum_{i=0}^p f_i^{(i)}$, $f_i \in L^2$, $i = 0,1,...,p$, as well as distributions of the form $\sum_{i=0}^p \sum_{j=1}^s \delta^{(i)}(\cdot - x_j)$.

The first Cauchy problem is a linear heat equation given in Theorem 2,

$$(\partial_t - \Delta)u + Vu = 0,\ u(0,x) = u_0(x),\ x \in \mathbb{R}^n, \tag{1}$$

where potential V and initial data u_0 are singular distributions, for example, the delta distribution or its powers. In case $V = \delta^\alpha \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$, $\alpha \in (0,1)$, (this generalized function is not an embedded Schwartz distribution) equation (1) with $u_0 \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ has a unique solution $[u_\varepsilon(t,x)] \in \mathcal{G}_{C^1,L^2}([0,T) \times \mathbb{R}^n)$. If $u_0 \in L^2(\mathbb{R}^n)$, a representative of that solution has a subsequence $u_{\varepsilon_\nu}(t,x)$, $\nu \in \mathbb{N}$, converging in $\mathcal{D}'(\mathbb{R}^n)$ to $u(t,x) = e^{-\Delta t}u_0(x)$, the solution to equation (1) with $V = 0$. In case $V = \delta \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ and $u_0 \in L^2(\mathbb{R}^n)$, we also obtain converging subsequence $u_{\varepsilon_\nu}(t,x)$, $\nu \in \mathbb{N}$, but not a distributional solution. Propositions 3, 4 and 5 deal with the explanations of equations with such potentials and initial data.

Also, we consider the stationary case $\partial_t u \equiv 0$ with the singular potential $V = \delta$, essential in the analysis of singular point interactions ([1]). In case $n = 1$ our approximating procedure for the perturbation by δ leads to the solution semigroup for the Schrödinger equation $-iu_t = \Delta u - \delta u$ in an appropriate Hilbert space (cf. [1]). In cases when $n = 2, 3$, the corresponding results are mentioned and announced for further investigations.

Also we show that our method of approximating a Cauchy problem with a family of Cauchy problems leads to distributional solutions in case when V is a "good" potential and $u_0 = \delta$.

The second type of heat equation considered in this paper is the nonlinear Cauchy problem

$$(\partial_t - \triangle)u + Vu = f(t, u), \ u(0, x) = u_0(x),$$

where V and u_0 are singular generalized functions and a function f is sublinear with respect to u, with bounded first and second derivatives with respect to this variable. Constrains on f correspond to apriori L^2-bounds and can be relaxed with the change of Colombeau type algebras.This is shown in the last proposition where the usual assumptions on f with a potential $V \in H^{2,\infty}$ and $u_0 = \delta$ lead to the net of solutions having an L^1- convergent sequence, leading to a distribution solution. In Theorem 3 the existence and the uniqueness of a generalized function solution is proved with the general assumptions on V and u_0 and because of that, with necessary assumptions on f.

2. Generalized semigroups

2.1. **Spaces and algebras of generalized functions.** Let $H^{r,s}(\Omega)$ be the Sobolev space of functions in $L^s(\Omega)$ with all distributional derivatives of order $|\alpha| \leq r$ belonging to $L^s(\Omega)$, equipped with the usual norm. In case $s = 2$, we simply write $H^r(\Omega)$. We refer to [3], [9], [22] and [24] for general Colombeau algebras and to [5] and [21] for the Colombeau type algebras $\mathcal{G}_{L^p,L^q}$. Here, we make some necessary modifications depending on a Cauchy problem in question.

Notation $f_\varepsilon = \mathcal{O}(\varepsilon^a)$ means that $|f_\varepsilon| \leq C\varepsilon^a$, $0 < \varepsilon < \varepsilon_0$, for some constants $C > 0$ and $\varepsilon_0 \in (0, 1)$. In that case, we say that $(f_\varepsilon)_\varepsilon$ has the moderate bound, or it is of moderate growth, or simply moderate. A net of functions $(g_\varepsilon)_\varepsilon$ in some Banach space B is called moderate, or of moderate growth, if this holds for $(\|g_\varepsilon\|_B)_\varepsilon$.

Definition 1. $\mathcal{E}_{C^1,H^2}([0, T) : \mathbb{R}^n)$ (respectively $\mathcal{N}_{C^1,H^2}([0, T) : \mathbb{R}^n)$), $T > 0$, is the vector space of nets $(G_\varepsilon)_\varepsilon$ of functions

$$G_\varepsilon \in C^0\left([0, T) : H^2(\mathbb{R}^n)\right) \cap C^1\left((0, T) : L^2(\mathbb{R}^n)\right), \ \varepsilon \in (0, 1)$$

with the property: for every $T_1 \in (0, T)$ there exists $a \in \mathbb{R}$, (respectively, for every $a \in \mathbb{R}$) such that

$$\max\left\{\sup_{t\in[0,T)} \|G_\varepsilon(t)\|_{H^2}, \sup_{t\in[T_1,T)} \|\partial_t G_\varepsilon(t)\|_{L^2}\right\} = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \to 0. \tag{2}$$

The quotient space

$$\mathcal{G}_{C^1,H^2}([0, T) : \mathbb{R}^n) = \frac{\mathcal{E}_{C^1,H^2}([0, T) : \mathbb{R}^n)}{\mathcal{N}_{C^1,H^2}([0, T) : \mathbb{R}^n)}$$

is a Colombeau type vector space.

Dropping the conditions on $\partial_t G_\varepsilon$ in (2) we obtain spaces $\mathcal{E}_{C^0,H^2}([0,T):\mathbb{R}^n)$, $\mathcal{N}_{C^0,H^2}([0,T):\mathbb{R}^n)$ and $\mathcal{G}_{C^0,H^2}([0,T):\mathbb{R}^n)$.

The assertions given in the following lemma are consequences of Sobolev type inequalities, i.e. of the fact that only for $n \le 3$, $H^2(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$.

Lemma 1. *If $n \le 3$, then $\mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$ is an algebra with the multiplication and $\mathcal{N}_{C^1,H^2}([0,T):\mathbb{R}^n)$ is an ideal of $\mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$. Therefore, $\mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ is an algebra with the multiplication. The same holds for $\mathcal{E}_{C^0,H^2}([0,T):\mathbb{R}^n)$, $\mathcal{N}_{C^0,H^2}([0,T):\mathbb{R}^n)$ and $\mathcal{G}_{C^0,H^2}([0,T):\mathbb{R}^n)$.*

Substituting H^2-norm with L^2-norm in (2) we obtain vector spaces

$$\mathcal{E}_{C^1,L^2}([0,T):\mathbb{R}^n)\,,\ \mathcal{N}_{C^1,L^2}([0,T):\mathbb{R}^n) \text{ and } \mathcal{G}_{C^1,L^2}([0,T):\mathbb{R}^n)\,.$$

Canonical mapping $\iota_{L^2}:\mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)\to\mathcal{G}_{C^1,L^2}([0,T):\mathbb{R}^n)$ is defined by $\iota_{L^2}(G)=G$, where $G=[G_\varepsilon]$.

Space $\mathcal{G}_{H^2}(\mathbb{R}^n)$ is defined in a similar way as $\mathcal{G}_{C^1,H^2}(\mathbb{R}^n)$, but with representatives independent of time variable t. This space is also an algebra in case $n\le 3$. We give more explanations for space $\mathcal{G}_{H^{2,\infty}}([0,T):\mathbb{R}^n)$.

$\mathcal{E}_{H^{2,\infty}}(\mathbb{R}^n)$, (respectively, $\mathcal{N}_{H^{2,\infty}}(\mathbb{R}^n)$) is the space of nets $(G_\varepsilon)_\varepsilon$ of functions $G_\varepsilon\in H^{2,\infty}(\mathbb{R}^n)$, $\varepsilon\in(0,1)$, with the property: there exists $a\in\mathbb{R}$ (respectively, for every $a\in\mathbb{R}$) such that

$$\|G_\varepsilon\|_{H^{2,\infty}(\mathbb{R}^n)}=\mathcal{O}(\varepsilon^a),\quad \text{as } \varepsilon\to 0.$$

Both spaces are algebras with the usual multiplication and $\mathcal{N}_{H^{2,\infty}}(\mathbb{R}^n)$ is an ideal. Colombeau type algebra is defined by

$$\mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)=\frac{\mathcal{E}_{H^{2,\infty}}(\mathbb{R}^n)}{\mathcal{N}_{H^{2,\infty}}(\mathbb{R}^n)}.$$

Definition 2. An element $V\in\mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ is of logarithmic type if it has a representative $(V_\varepsilon)_\varepsilon\in\mathcal{E}_{H^{2,\infty}}(\mathbb{R}^n)$ with the property

$$\|V_\varepsilon\|_{H^{2,\infty}(\mathbb{R}^n)}=\mathcal{O}\left(\log\varepsilon^{-1}\right),\quad \text{as } \varepsilon\to 0.$$

An element $V\in\mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ is said to be of log-log type if it has a representative $(V_\varepsilon)_\varepsilon\in\mathcal{E}_{H^{2,\infty}}(\mathbb{R}^n)$ such that, for every $a\in(0,1)$,

$$\|V_\varepsilon\|_{H^{2,\infty}(\mathbb{R}^n)}=\mathcal{O}(\log^a(\log\varepsilon^{-1})),\quad \text{as } \varepsilon\to 0.$$

2.2. **Generalized semigroups.** Let $(E,\|\cdot\|)$ be a Banach space and let $\mathcal{L}(E)$ be the space of all linear continuous mappings $E\to E$.

Definition 3. $\mathcal{S}E_M([0,\infty):\mathcal{L}(E))$ is the space of nets $(S_\varepsilon)_\varepsilon$ of strongly continuous mappings $S_\varepsilon:[0,\infty)\to\mathcal{L}(E)$, $\varepsilon\in(0,1)$ with the property that for every $T>0$ there exists $a\in\mathbb{R}$ such that

$$\sup_{t\in[0,T)}\|S_\varepsilon(t)\|=\mathcal{O}\left(\varepsilon^a\right),\quad \text{as } \varepsilon\to 0. \tag{3}$$

$\mathcal{S}N([0,\infty):\mathcal{L}(E))$ is the space of nets $(N_\varepsilon)_\varepsilon$ of strongly continuous mappings $N_\varepsilon:[0,\infty)\to\mathcal{L}(E)$, $\varepsilon\in(0,1)$, with the properties:

For every $b \in \mathbb{R}$ and $T > 0$

$$\sup_{t\in[0,T)} \|N_\varepsilon(t)\| = \mathcal{O}(\varepsilon^b), \quad \text{as } \varepsilon \to 0. \tag{4}$$

There exist $t_0 > 0$ and $a \in \mathbb{R}$ such that

$$\sup_{t<t_0} \left\| \frac{N_\varepsilon(t)}{t} \right\| = \mathcal{O}(\varepsilon^a). \tag{5}$$

There exists a net $(H_\varepsilon)_\varepsilon$ in $\mathcal{L}(E)$ and $\varepsilon_0 \in (0,1)$ such that

$$\lim_{t\to 0} \frac{N_\varepsilon(t)}{t} x = H_\varepsilon x,\ x \in E,\ \varepsilon < \varepsilon_0. \tag{6}$$

For every $b > 0$,

$$\|H_\varepsilon\| = \mathcal{O}(\varepsilon^b), \quad \text{as } \varepsilon \to 0. \tag{7}$$

Let us remark that, because of 5, it is enough that (6) holds for all $x \in D$, where D is a dense subspace of E.

Proposition 1. $\mathcal{SE}_M\left([0,\infty) : \mathcal{L}(E)\right)$ *is an algebra with respect to composition and* $\mathcal{SN}\left([0,\infty) : \mathcal{L}(E)\right)$ *is an ideal of* $\mathcal{SE}_M\left([0,\infty) : \mathcal{L}(E)\right)$.

Proof. Let

$$(S_\varepsilon(t))_\varepsilon \in \mathcal{SE}_M\left([0,\infty) : \mathcal{L}(E)\right) \quad \text{and} \quad (N_\varepsilon(t))_\varepsilon \in \mathcal{SN}\left([0,\infty) : \mathcal{L}(E)\right).$$

We will prove only the second assertion, i.e., that

$$\left(S_\varepsilon(t)N_\varepsilon(t)\right)_\varepsilon,\ \left(N_\varepsilon(t)S_\varepsilon(t)\right)_\varepsilon \in \mathcal{SN}\left([0,\infty) : \mathcal{L}(E)\right),$$

where $S_\varepsilon(t)N_\varepsilon(t)$ denotes the composition.

Let $\varepsilon \in (0,1)$. By (3) and (4), for some $a \in \mathbb{R}$ and every $b \in \mathbb{R}$,

$$\|S_\varepsilon(t)N_\varepsilon(t)\| \leq \|S_\varepsilon(t)\| \cdot \|N_\varepsilon(t)\| = \mathcal{O}(\varepsilon^{a+b}), \quad \text{as } \varepsilon \to 0.$$

The same holds for $\|N_\varepsilon(t)S_\varepsilon(t)\|$. Further, (3) and (6) yield

$$\sup_{t<t_0} \left\| \frac{S_\varepsilon(t)N_\varepsilon(t)}{t} \right\| \leq \sup_{t<t_0} \|S_\varepsilon(t)\| \ \sup_{t<t_0} \left\| \frac{N_\varepsilon(t)}{t} \right\| = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \to 0,$$

for some $t_0 > 0$ and $a \in \mathbb{R}$. Also,

$$\sup_{t<t_0} \left\| \frac{N_\varepsilon(t)S_\varepsilon(t)}{t} \right\| = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \to 0,$$

for some $t_0 > 0$ and $a \in \mathbb{R}$.

Let now $\varepsilon \in (0,1)$ be fixed. We have

$$\begin{aligned}
&\left\| \frac{S_\varepsilon(t)N_\varepsilon(t)}{t} x - S_\varepsilon(0)H_\varepsilon x \right\| \\
&= \left\| S_\varepsilon(t)\frac{N_\varepsilon(t)}{t} x - S_\varepsilon(t)H_\varepsilon x + S_\varepsilon(t)H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x \right\| \\
&\leq \|S_\varepsilon(t)\| \cdot \left\| \frac{N_\varepsilon(t)}{t} x - H_\varepsilon x \right\| + \|S_\varepsilon(t)H_\varepsilon x - S_\varepsilon(0)H_\varepsilon x\|.
\end{aligned}$$

By (3) and (6) as well as by the continuity of $t \mapsto S_\varepsilon(t)(H_\varepsilon x)$ at zero, it follows that the last expression tends to zero as $t \to 0$. Similarly, we have

$$\begin{aligned}
&\left\| \frac{N_\varepsilon(t) S_\varepsilon(t)}{t} x - H_\varepsilon S_\varepsilon(0) x \right\| \\
&= \left\| \frac{N_\varepsilon(t)}{t} S_\varepsilon(t) x - \frac{N_\varepsilon(t)}{t} S_\varepsilon(0) x + \frac{N_\varepsilon(t)}{t} S_\varepsilon(0) x - H_\varepsilon S_\varepsilon(0) x \right\| \\
&\leq \left\| \frac{N_\varepsilon(t)}{t} \right\| \| S_\varepsilon(t) x - H_\varepsilon(t) S_\varepsilon(0) x \| + \left\| \frac{N_\varepsilon(t)}{t} (S_\varepsilon(0) x) - H_\varepsilon(S_\varepsilon(0) x) \right\| .
\end{aligned}$$

Assumptions (5), (6) and (3) imply that the last expression tends to zero as $t \to 0$. Thus (6) is proved in both cases. □

Now we define Colombeau type algebra as the factor algebra

$$\mathcal{S}G\left([0,\infty) : \mathcal{L}(E)\right) = \frac{\mathcal{S}E_M\left([0,\infty) : \mathcal{L}(E)\right)}{\mathcal{S}N\left([0,\infty) : \mathcal{L}(E)\right)}.$$

Elements of $\mathcal{S}G\left([0,\infty) : \mathcal{L}(E)\right)$ will be denoted by $S = [S_\varepsilon]$, where $(S_\varepsilon)_\varepsilon$ is a representative of the above class.

Definition 4. $S \in \mathcal{S}G\left([0,\infty) : \mathcal{L}(E)\right)$ is called a Colombeau C_0-semigroup if it has a representative $(S_\varepsilon)_\varepsilon$ such that, for some $\varepsilon_0 > 0$, S_ε is a C_0-semigroup, for every $\varepsilon < \varepsilon_0$.

In the sequel we will use only representatives $(S_\varepsilon)_\varepsilon$ of a Colombeau C_0-semigroup S which are C_0-semigroups, for ε small enough.

Proposition 2. *Let $(S_\varepsilon)_\varepsilon$ and $(\tilde{S}_\varepsilon)_\varepsilon$ be representatives of a Colombeau C_0-semigroup S, with the infinitesimal generators A_ε, $\varepsilon < \varepsilon_0$, and $\tilde{A}_\varepsilon$, $\varepsilon < \tilde{\varepsilon}_0$, respectively, where ε_0 and $\tilde{\varepsilon}_0$ correspond (in the sense of Definition 4) to $(S_\varepsilon)_\varepsilon$ and $(\tilde{S}_\varepsilon)_\varepsilon$, respectively.*

Then, $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \bar{\varepsilon}_0 = \min\{\varepsilon_0, \tilde{\varepsilon}_0\}$ and $A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to an element of $\mathcal{L}(E)$, denoted again by $A_\varepsilon - \tilde{A}_\varepsilon$.

Moreover, for every $a \in \mathbb{R}$,

$$\| A_\varepsilon - \tilde{A}_\varepsilon \| = \mathcal{O}\left(\varepsilon^a\right), \quad \text{as } \varepsilon \to 0. \tag{8}$$

Proof. Denote $(N_\varepsilon)_\varepsilon = (S_\varepsilon - \tilde{S}_\varepsilon)_\varepsilon \in \mathcal{S}N\left([0,\infty) : \mathcal{L}(E)\right)$. Let $\varepsilon < \bar{\varepsilon}_0$ be fixed and $x \in E$. We have

$$\frac{S_\varepsilon(t)x - x}{t} - \frac{\tilde{S}_\varepsilon(t)x - x}{t} = \frac{N_\varepsilon(t)}{t} x.$$

This implies, letting $t \to 0$, that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$. Now we have

$$\begin{aligned}
(A_\varepsilon - \tilde{A}_\varepsilon)x &= \lim_{t\to 0} \frac{S_\varepsilon(t)x - x}{t} - \lim_{t \to 0} \frac{\tilde{S}_\varepsilon(t)x - x}{t} \\
&= \lim_{t \to 0} \frac{N_\varepsilon(t)}{t} x = H_\varepsilon x, \ x \in D(A_\varepsilon).
\end{aligned} \tag{9}$$

Since $D(A_\varepsilon)$ is dense in E, properties (6), (7) and (9) imply that for every $a \in \mathbb{R}$, $\left\| A_\varepsilon - \tilde{A}_\varepsilon \right\| = \mathcal{O}(\varepsilon^a)$, as $\varepsilon \to 0$. □

Now we define the infinitesimal generator of a Colombeau C_0-semigroup S. Denote by $\mathcal{A}$ the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$ where A_ε is a closed linear operator on E with the dense domain $D(A_\varepsilon) \subset E$, for every $\varepsilon \in (0,1)$. We introduce an equivalence relation in $\mathcal{A}$:

$$((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon) \sim ((\tilde{A}_\varepsilon)_\varepsilon, (D(\tilde{A}_\varepsilon))_\varepsilon)$$

if there exist $\varepsilon_0 \in (0,1)$ such that $D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$, for every $\varepsilon < \varepsilon_0$, and for every $a \in \mathbf{R}$ there exist $C > 0$ and $\varepsilon_a \leq \varepsilon_0$ such that, for $x \in D(A_\varepsilon)$, $\|(A_\varepsilon - \tilde{A}_\varepsilon)x\| \leq C\varepsilon^a \|x\|$, $x \in D(A_\varepsilon)$, $\varepsilon \leq \varepsilon_a$.

Since A_ε has a dense domain in E, $R_\varepsilon := A_\varepsilon - \tilde{A}_\varepsilon$ can be extended to be an operator in $\mathcal{L}(E)$ satisfying $\|A_\varepsilon - \tilde{A}_\varepsilon\| = \mathcal{O}(\varepsilon^a)$, $\varepsilon \to 0$, for every $a \in \mathbb{R}$. Such an operator R_ε is called the zero operator.

We denote by A the corresponding element of the quotient space $\mathcal{A}/\sim$. Due to Proposition 2, the following definition makes sense.

Definition 5. $A \in \mathcal{A}/\sim$ is the infinitesimal generator of a Colombeau C_0-semigroup S if there exists a representative $(A_\varepsilon)_\varepsilon$ of A such that A_ε is the infinitesimal generator of S_ε, for ε small enough.

We collect some obvious properties in the following proposition (cf. [25]).

Proposition 3. *Let S be a Colombeau C_0-semigroup with the infinitesimal generator A. Then there exists $\varepsilon_0 \in (0,1)$ such that:*

(a) *Mapping $t \mapsto S_\varepsilon(t)x : [0,\infty) \to E$ is continuous for every $x \in E$ and $\varepsilon < \varepsilon_0$.*

(b)
$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S_\varepsilon(s)x\, ds = S_\varepsilon(t)x,\ \varepsilon < \varepsilon_0,\ x \in E.$$

(c)
$$\int_0^t S_\varepsilon(s)x\, ds \in D(A_\varepsilon),\ \varepsilon < \varepsilon_0,\ x \in E.$$

(d) *For every $x \in D(A_\varepsilon)$ and $t \geq 0$, $S_\varepsilon(t)x \in D(A_\varepsilon)$ and*
$$\frac{d}{dt} S_\varepsilon(t)x = A_\varepsilon S_\varepsilon(t)x = S_\varepsilon(t)A_\varepsilon x,\ \varepsilon < \varepsilon_0. \tag{10}$$

(e) *Let $(S_\varepsilon)_\varepsilon$ and $(\tilde{S}_\varepsilon)_\varepsilon$ be representatives of Colombeau C_0-semigroup S, with infinitesimal generators A_ε and $\tilde{A}_\varepsilon$, $\varepsilon < \varepsilon_0$, respectively. Then, for every $a \in \mathbb{R}$ and $t \geq 0$*
$$\left\| \frac{d}{dt} S_\varepsilon(t) - \tilde{A}_\varepsilon S_\varepsilon(t) \right\| = \mathcal{O}\left(\varepsilon^a\right), \quad \text{as } \varepsilon \to 0. \tag{11}$$

(f) *For every $x \in D(A_\varepsilon)$ and every $t, s \geq 0$,*
$$S_\varepsilon(t)x - S_\varepsilon(s)x = \int_s^t S_\varepsilon(\tau)A_\varepsilon x\, d\tau = \int_s^t A_\varepsilon S_\varepsilon(\tau)x\, d\tau,\ \varepsilon < \varepsilon_0.$$

Theorem 1. *Let S and $\tilde{S}$ be Colombeau C_0-semigroups with infinitesimal generators A and $\tilde{A}$, respectively. If $A = \tilde{A}$ then $S = \tilde{S}$.*

Proof. Let ε be small enough and $x \in D(A_\varepsilon) = D(\tilde{A}_\varepsilon)$. Proposition 3 (d) implies that for $t \geq 0$, the mapping $s \mapsto \tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x$, $t \geq s \geq 0$ is differentiable and

$$\frac{d}{ds}\left(\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x\right) = -\tilde{A}_\varepsilon\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x + \tilde{S}_\varepsilon(t-s)A_\varepsilon S_\varepsilon(s)x,\ t \geq s \geq 0.$$

The assumption $A = \tilde{A}$ implies that $A_\varepsilon = \tilde{A}_\varepsilon + R_\varepsilon$, where R_ε is a zero operator. Since $\tilde{A}_\varepsilon$ commutes with $\tilde{S}_\varepsilon$, for every $x \in D(A_\varepsilon)$

$$\frac{d}{ds}\left(\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x\right) = \tilde{S}_\varepsilon(t-s)R_\varepsilon S_\varepsilon(s)x,\ t \geq s \geq 0$$

and this implies

$$\tilde{S}_\varepsilon(t-s)S_\varepsilon(s)x - \tilde{S}_\varepsilon(t)x = \int_0^s \tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)x du,\ t \geq s \geq 0. \tag{12}$$

Putting $s = t$ in (12), we obtain

$$S_\varepsilon(t)x - \tilde{S}_\varepsilon(t)x = \int_0^t \tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)x\, du,\ \ t \geq 0,\ x \in D(A_\varepsilon). \tag{13}$$

Since $D(A_\varepsilon)$ is dense in E, uniform boundedness of S and $\tilde{S}$ on $[0,t]$ implies that 13 holds for every $y \in E$.

Let us prove that $(N_\varepsilon)_\varepsilon = (S_\varepsilon - \tilde{S}_\varepsilon)_\varepsilon \in \mathcal{S}N\left([0,\infty) : \mathcal{L}(E)\right)$.

(13) and Definition 3 imply that for some $C > 0$ and $a, \tilde{a} \in \mathbb{R}$,

$$\begin{aligned}\sup_{t\in[0,T)} \|N_\varepsilon(t)x\| &\leq \sup_{t\in[0,T)} \int_0^t \|\tilde{S}_\varepsilon(t-u)\| \cdot \|R_\varepsilon\| \cdot \|S_\varepsilon(u)\|\|x\|\, du \\ &\leq T\, C\, \varepsilon^{a+\tilde{a}}\, \|R_\varepsilon\|\|x\|, \quad x \in E.\end{aligned}$$

Since $\|R_\varepsilon\| = \mathcal{O}(\varepsilon^b)$, as $\varepsilon \to 0$, for every $b \in \mathbb{R}$, $(N_\varepsilon(t))_\varepsilon$ satisfies condition (4) in Definition 3. Condition (5) follows from the boundedness of $(\tilde{S}_\varepsilon)_\varepsilon$, $(S_\varepsilon)_\varepsilon$ on bounded domain $[0,t)$, the properties of $(R_\varepsilon)_\varepsilon$ and the following expression:

$$\begin{aligned}\left\|\frac{N_\varepsilon(t)}{t}\right\| &= \left\|\frac{1}{t}\int_0^t \tilde{S}_\varepsilon(t-u)R_\varepsilon S_\varepsilon(u)x\, du\right\| \\ &\leq \|\tilde{S}_\varepsilon(t))\| \cdot \|R_\varepsilon\| \cdot \|S_\varepsilon(t))\| \leq \text{const},\ x \in E, t \leq t_0,\end{aligned}$$

for some $t_0 > 0$. Also,

$$\lim_{t\to 0}\frac{N_\varepsilon(t)}{t} = \lim_{t\to 0}\frac{\tilde{S}_\varepsilon(t)x - x}{t} - \lim_{t\to 0}\frac{S_\varepsilon(t)x - x}{t} = R_\varepsilon x,\ x \in D(A_\varepsilon).$$

Since it is enough that (6) holds for a dense subset of E (see the remark after Definition 3) this concludes the proof. □

Remark 1. Let the assumptions of Definition 3 hold. Moreover, assume a stronger assumption than (3):

There exist $M > 0$, $a \in \mathbb{R}$ and $\varepsilon_0 \in (0,1)$ such that

$$\|S_\varepsilon(t)\| \leq M\varepsilon^a e^{\alpha_\varepsilon t},\ \varepsilon < \varepsilon_0,\ t \geq 0,$$

where $0 < \alpha_\varepsilon < \alpha$, for some $\alpha > 0$.

Then we obtain the corresponding subalgebra of $\mathcal{S}G\left([0,\infty) : \mathcal{L}(E)\right)$. For this subalgebra we can formulate the Hille-Yosida theorem in a usual way.

For the whole algebra of Colombeau C_0-semigroups $\mathcal{SG}([0,\infty):\mathcal{L}(E))$ the formulation of the Hille-Yosida-type theorem is an open problem.

3. Schrödinger operators with singular potentials

This section deals with applications of Colombeau C_0-semigroups in solving a class of heat equations with singular potentials and singular data. First note that the multiplication of elements $G \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ and $H \in \mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ gives an element in $\mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$. Indeed, if $(G_\varepsilon)_\varepsilon \in \mathcal{E}_{H^{2,\infty}}(\mathbb{R}^n)$ and $(H_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$ then

$$(G_\varepsilon H_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n).$$

Similarly, if $(G_\varepsilon)_\varepsilon \in \mathcal{N}_{H^{2,\infty}}(\mathbb{R}^n)$ or $(H_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T):\mathbb{R}^n)$, then

$$(G_\varepsilon H_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T):\mathbb{R}^n).$$

Thus, multiplication of potential $V \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ and $u \in \mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ which is expected to be a solution to equation

$$\partial_t u = (\Delta - V)u,\ u(0,x) = u_0(x),$$

makes sense.

Definition 6. Let A be represented by a net $(A_\varepsilon)_\varepsilon$, $\varepsilon \in (0,1)$, of linear operators with the common domain $H^2(\mathbb{R}^n)$ and with ranges in $L^2(\mathbb{R}^n)$. A generalized function $G \in \mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$, $T>0$, is said to be a solution to equation $\partial_t G = AG$ if

$$\sup_{t\in[0,T)} \|\partial_t G_\varepsilon(t,\cdot) - A_\varepsilon G_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^a), \quad \text{for every } a \in \mathbb{R}.$$

3.1. General potential.

Theorem 2. *Let $V \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ be of logarithmic type.*

(i) *Differential operators $A_\varepsilon u = (\Delta - V_\varepsilon)u$, $u \in H^2(\mathbb{R}^n)$, $\varepsilon < \varepsilon_0$, are infinitesimal generators of semigroups S_ε, for every $\varepsilon < \varepsilon_0$, and $(S_\varepsilon)_\varepsilon$ is a representative of a Colombeau C_0-semigroup*

$$S \in \mathcal{SG}\left([0,\infty):\mathcal{L}(L^2(\mathbb{R}^n))\right).$$

(ii) *Let $u_0 = [u_{0\varepsilon}] \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ and let $(S_\varepsilon)_\varepsilon$, $\varepsilon < \varepsilon_0$, be as in (i).*

Then, for every $T>0$, $u(t,x) = Su_0 \in \mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ is the solution to equation

$$\partial_t u(t,x) - \Delta u(t,x) + V(x)u(t,x) = 0,\ u(0,x) = u_0(x). \tag{14}$$

The solution is unique in the space $\mathcal{G}_{C^1,L^2}([0,T):\mathbb{R}^n)$, i.e. if v is also a solution to (14), then $\iota_{L^2}(u) = \iota_{L^2}(v)$.

Proof. (i) Let $\varepsilon < \varepsilon_0$ be fixed. The operator A_ε is the infinitesimal generator of the corresponding semigroup $S_\varepsilon : [0,\infty) \to \mathcal{L}(L^2(\mathbb{R}^n))$ defined by the Feynman-Kac formula

$$S_\varepsilon(t)\psi(x) = \int_\Omega \exp\left(-\int_0^t V_\varepsilon(\omega(s))\,ds\right)\psi(\omega(t))d\mu_x(\omega),\ t \geq 0,\ x \in \mathbb{R}^n, \tag{15}$$

for $\psi \in L^2(\mathbb{R}^n)$, $\varepsilon < \varepsilon_0$, where $\Omega = \prod_{t\in[0,\infty)} \overline{\mathbb{R}^n}$ and μ_x is the Wiener measure concentrated at $x \in \mathbb{R}^n$ (cf. [27] or [26]).

Since $V = [V_\varepsilon]$ is of logarithmic type, there exist $C > 0$ and $\eta \in (0,1)$ such that

$$
\begin{aligned}
|S_\varepsilon(t)\psi(x)| &\leq \exp\left(t \sup_{s\in\mathbb{R}^n} |V_\varepsilon(s)|\right) \int_\Omega |\psi(\omega(t))| d\mu_x(\omega) \\
&= \varepsilon^{Ct}(4\pi t)^{-n/2} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) |\psi(y)| dy,
\end{aligned}
$$

for every $t > 0$, $x \in \mathbb{R}^n$ and $\varepsilon < \eta$.

Therefore, there exist $C_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$
\sup_{t\in[0,T)} \|S_\varepsilon(t)\psi\|_{L^2} \leq C_0 \varepsilon^{CT} \|\psi\|_{L^2},\ \varepsilon < \varepsilon_0,
$$

i.e. $(S_\varepsilon(t))_\varepsilon$, $t \geq 0$, satisfies relation (3) and $S = [S_\varepsilon] \in \mathcal{SG}\left([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n))\right)$.

(ii) Let $\varepsilon < \varepsilon_0$. The solution to equation

$$
\partial_t u_\varepsilon(t,x) - \Delta u_\varepsilon(t,x) + V_\varepsilon(x) u_\varepsilon(t,x) = 0,\ u_\varepsilon(0,x) = u_{0\varepsilon}(x) \tag{16}
$$

is given by

$$
u_\varepsilon(t,x) = S_\varepsilon(t) u_{0\varepsilon}(x),\ t \in [0,T),\ x \in \mathbb{R}^n,
$$

and $u_\varepsilon \in C^1\left([0,T) : L^2(\mathbb{R}^n)\right)$. Let us show that $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T) : \mathbb{R}^n)$. Recall, the heat kernel is given by

$$
E_n(t,x) = \begin{cases} \dfrac{1}{(4\pi t)^{n/2}} \exp\left(-\dfrac{x^2}{4t}\right), & t > 0, \\ 0, & t = 0,\ x \in \mathbb{R}^n \end{cases} \tag{17}
$$

and its $L^1(\mathbb{R}^n)$-norm equals 1, for every $t > 0$. Let $\varepsilon < \varepsilon_0$. By the Duhamel principle, solution $u_\varepsilon(t,x)$ to equation (16) satisfies

$$
\begin{aligned}
u_\varepsilon(t,x) &= \int_{\mathbb{R}^n} E_n(t, x-y) u_{0\varepsilon}(y)\, dy \\
&\quad + \int_0^t \int_{\mathbb{R}^n} E_n(t-s, x-y) V_\varepsilon(y) u_\varepsilon(s,y)\, dy\, ds,
\end{aligned} \tag{18}
$$

for $t \in [0,T)$ and $x \in \mathbb{R}^n$. Since $E_n(t,x) \to \delta(x)$ as $t \to 0$, we obtain that $u_\varepsilon(t,x) \to u_{0\varepsilon}(x)$, as $t \to 0$.

Young's inequality implies

$$
\begin{aligned}
\|u_\varepsilon(t,\cdot)\|_{L^2} &\leq \|E_n(t,\cdot)\|_{L^1} \|u_{0\varepsilon}\|_{L^2} + \int_0^t \|E_n(t-s,\cdot)\|_{L^1} \|V_\varepsilon(\cdot)\|_{L^\infty} \|u_\varepsilon(s,\cdot)\|_{L^2}\, ds \\
&= \|u_{0\varepsilon}\|_{L^2} + \int_0^t \|V_\varepsilon(\cdot)\|_{L^\infty} \|u_\varepsilon(s,\cdot)\|_{L^2}\, ds,\ t \in [0,T),\ \varepsilon < \varepsilon_0.
\end{aligned}
$$

Gronwall's inequality gives

$$
\|u_\varepsilon(t,\cdot)\|_{L^2} \leq \|u_{0\varepsilon}\|_{L^2}\ \exp\left(\int_0^t \|V_\varepsilon(\cdot)\|_{L^\infty}\, ds\right),\ t \in [0,T),\ \varepsilon < \varepsilon_0.
$$

Since $V = [V_\varepsilon] \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ is of logarithmic type and $(u_{0\varepsilon})_\varepsilon \in \mathcal{E}_{H^2}(\mathbb{R}^n)$, it follows that $\sup_{t\in[0,T)} \|u_\varepsilon(t,\cdot)\|_{L^2}$ has a moderate bound. Differentiation of equation (18) with respect to some spatial variable x_i gives

$$\begin{aligned}
\partial_{x_i} u_\varepsilon(t,x) &= \int_{\mathbb{R}^n} E_n(t,y)\partial_{x_i} u_{0\varepsilon}(x-y)\,dy \\
&+ \int_0^t \int_{\mathbb{R}^n} E_n(t-s,y)\partial_{x_i}(V_\varepsilon(x-y)u_\varepsilon(s,x-y))\,dy\,ds, \\
&= \int_{\mathbb{R}^n} E_n(t,y)\partial_{x_i} u_{0\varepsilon}(x-y)\,dy \\
&+ \int_0^t \int_{\mathbb{R}^n} E_n(t-s,y)(\partial_{x_i} V_\varepsilon(x-y)u_\varepsilon(s,x-y) \\
&+ V_\varepsilon(x-y)\partial_{x_i} u_\varepsilon(s,x-y))\,dy\,ds,
\end{aligned}$$

for $t \in [0,T)$, $x \in \mathbb{R}^n$ and $\varepsilon < \varepsilon_0$. Then, for $t \in [0,T)$ and $\varepsilon < \varepsilon_0$,

$$\begin{aligned}
\|\partial_{x_i} u_\varepsilon(t,\cdot)\|_{L^2} &\leq \|\partial_{x_i} u_{0\varepsilon}\|_{L^2} \\
&+ \int_0^t \Big(\|\partial_{x_i} V_\varepsilon(\cdot)\|_{L^\infty}\|u_\varepsilon(s,\cdot)\|_{L^2} + \|V_\varepsilon(\cdot)\|_{L^\infty}\|\partial_{x_i} u_\varepsilon(s,\cdot)\|_{L^2}\Big)ds.
\end{aligned} \tag{19}$$

Since $\sup_{t\in(0,T)} \|u_\varepsilon(t,\cdot)\|_{L^2}$ and the first two derivatives of V_ε are moderate, Gronwall's inequality implies that $\sup_{t\in[0,T)} \|\partial_{x_i} u_\varepsilon(t,\cdot)\|_{L^2}$ is moderate, too. One can similarly estimate all the second order space derivatives of u_ε. The moderateness of $\sup_{t\in[T_1,T)} \|\partial_t u(t,\cdot)\|_{L^2}$, $T_1 < T$, simply follows from equation (14). Therefore, $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T) : \mathbb{R}^n)$.

It remains to show that the solution $[u_\varepsilon(t,x)]$ is unique in the sense described in the statement of the theorem.

Suppose that $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T) : \mathbb{R}^n)$ are representatives of two solutions to the given Cauchy problem. Then $G_\varepsilon := u_\varepsilon - v_\varepsilon$ satisfies

$$\begin{aligned}
&\partial_t G_\varepsilon(t,x) - (\triangle - V_\varepsilon)G_\varepsilon(t,x) = N_\varepsilon(t,x) \\
&G_\varepsilon(0,x) = N_{0\varepsilon}(x),
\end{aligned}$$

where $(N_\varepsilon)_\varepsilon \in \mathcal{N}_{C^1,L^2}([0,T) : \mathbb{R}^n)$ and $(N_{0\varepsilon})_\varepsilon \in \mathcal{N}_{H^2}([0,T) : \mathbb{R}^n)$.

Therefore, for $x \in \mathbb{R}^n$, $t \in [0,T)$ and $\varepsilon < \varepsilon_0$,

$$\begin{aligned}
G_\varepsilon(t,x) = &\int E_n(t,x-y)N_{0\varepsilon}(y)\,dy \\
&+ \int_0^t \int E_n(t-s,x-y)V_\varepsilon(y)G_\varepsilon(s,y)\,dy\,ds \\
&+ \int_0^t \int E_n(t-s,x-y)N_\varepsilon(s,y)\,dy\,ds,
\end{aligned} \tag{20}$$

where $(N_{0\varepsilon})_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^n)$ and, for every $a \in \mathbb{R}$, $\sup_{t\in[0,T)} \|N_\varepsilon(t,\cdot)\|_{L^2} = \mathcal{O}(\varepsilon^a)$, as $\varepsilon \to 0$. Then Young's and Gronwall's inequalities imply

$$\|G_\varepsilon(t,\cdot)\|_{L^2} \leq \|N_{0\varepsilon}\|_{L^2} + \int_0^t \|V_\varepsilon(\cdot)\|_{L^\infty}\|G_\varepsilon(s,\cdot)\|_{L^2}\,ds + \int_0^t \|N_\varepsilon(s,\cdot)\|_{L^2}\,ds,$$

for $t \in [0,T)$, i.e.,

$$\sup_{t\in[0,T)} \|G_\varepsilon(t,\cdot)\|_{L^2} = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon \to 0,$$

for every $a \in \mathbb{R}$. $\square$

3.2. **Powers of the delta function as a potential.** Let $(\phi_\varepsilon)_\varepsilon$ be a net of mollifiers of the form

$$\phi_\varepsilon = \varepsilon^{-n}\phi(\cdot/\varepsilon),\ \varepsilon \in (0,1), \tag{21}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$, $\int \phi(x)dx = 1$ and $\phi(x) \geq 0$, $x \in \mathbb{R}^n$. It represents the generalized delta function in $\mathcal{G}(\mathbb{R}^n)$.

Powers of the delta function are defined by the representatives

$$\delta^\alpha = \left[\frac{1}{\varepsilon^{n\alpha}}\phi^\alpha\left(\frac{\cdot}{\varepsilon}\right)\right],\ \varepsilon \in (0,1), \tag{22}$$

where $\alpha > 0$.

Let $A_\varepsilon u = (\Delta - \phi_\varepsilon)u$, $u \in H^2(\mathbb{R}^n)$, $\varepsilon < 1$. The operator A_ε is the infinitesimal generator of the semigroup $S_\varepsilon : [0,\infty) \to \mathcal{L}(L^2(\mathbb{R}^n))$, denoted by $S_\varepsilon(t) = e^{(\Delta-\phi_\varepsilon)t}$, $t \geq 0$, $\varepsilon \in (0,1)$ (cf. (15)). It is representative of a Colombeau C_0-semigroup $S \in \mathcal{SG}\big([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n))\big)$.

Feynman-Kac formula gives

$$S_\varepsilon(t)\psi(x) = \int_\Omega \exp\left(-\int_0^t \phi_\varepsilon(\omega(s))\,ds\right)\psi(\omega(t))d\mu_x(\omega),\ \psi \in L^2(\mathbb{R}^n), \tag{23}$$

for every $x \in \mathbb{R}^n$, $t \geq 0$ and $\varepsilon \in (0,1)$.

Since the delta generalized function is represented by a net of non-negative functions, we have that $\{S_\varepsilon,\ \varepsilon \in (0,1),\ t \geq 0\}$ is bounded in $\mathcal{L}(L^2(\mathbb{R}^n))$ (not only moderate) and thus satisfies relation (3).

Proposition 4. *Let $\delta^\alpha \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$, $0 < \alpha < 1$, be defined by (22) and let $u_0 \in L^2(\mathbb{R}^n)$ be a continuous and bounded function. Then, by*

$$u_\varepsilon(t,x) = \int_\Omega \exp\Big(-\frac{1}{\varepsilon^{n\alpha}}\int_0^t \phi^\alpha\left(\frac{\omega(s)}{\varepsilon}\right)ds\Big)u_0(\omega(t))d\mu_x(\omega)), \tag{24}$$

where $t \in [0,T)$ and $x \in \mathbb{R}^n$, is defined a representative of a unique solution $u(t,x) \in \mathcal{G}_{C^1,L^2}([0,T) \times \mathbb{R}^n)$ to equation

$$\partial_t u(t,x) - \Delta u(t,x) + \delta^\alpha(x)u(t,x) = 0,\ u(0,x) = u_0(x). \tag{25}$$

The above representative of the solution has a subsequence $u_{\varepsilon_\nu}(t,x)$, $\nu \in \mathbb{N}$, converging in $\mathcal{D}'$ to $u(t,x) = e^{-\Delta t}u_0(x)$, the solution to equation

$$\partial_t u(t,x) - \Delta u(t,x) = 0,\ u(0,x) = u_0(x). \tag{26}$$

Proof. The representative of the solution to equation (25) equals

$$u_\varepsilon(t,x) = S_\varepsilon(t)u_0(x),\ t \in [0,T),\ x \in \mathbb{R}^n,\ \varepsilon < \varepsilon_0,$$

where $S_\varepsilon(t)$ is semigroup generated by operator $A_\varepsilon = \Delta - \dfrac{1}{\varepsilon^{n\alpha}}\phi^\alpha\left(\dfrac{\cdot}{\varepsilon}\right)$, $\varepsilon < \varepsilon_0$. The Feynman-Kac formula gives (24). Similarly as in the proof of Theorem 2 it follows that $S_\varepsilon(t)u_0(\cdot) \in L^2\left([0,T) \times \mathbb{R}^n\right)$, for every $T > 0$ and $\varepsilon < \varepsilon_0$. Using this and the fact that $\{S_\varepsilon(\cdot)u_0(\cdot);\ \varepsilon < \varepsilon_0\}$ is bounded in $L^2\left([0,T) \times \mathbb{R}^n\right)$ and hence relatively compact with respect to the weak topology, we obtain that there exists a sequence $\{\varepsilon_\nu\}_{\nu\in\mathbb{N}}$ such that

$$S_{\varepsilon_\nu}(t)u_0(x) \to u(t,x),\ \varepsilon_\nu \to 0,$$

in the sense of weak topology in $L^2\left([0,T)\times\mathbb{R}^n\right)$. Let $x\in\mathbb{R}^n$, $t\in[0,T)$ and $\varepsilon<\varepsilon_0$. Using Duhamel's principle we have

$$\begin{aligned} u_\varepsilon(t,x) &= \int_{\mathbb{R}^n} E_n(t,x-y)u_0(y)\,dy \\ &+ \int_0^t\int_{\mathbb{R}^n} E_n(t-s,x-y)\frac{1}{\varepsilon^{n\alpha}}\phi^\alpha\left(\frac{y}{\varepsilon}\right)\ u_\varepsilon(s,y)\,dy\,ds, \\ &= \int_{\mathbb{R}^n} E_n(t,x-y)u_{0\varepsilon}(y)\,dy \\ &+ \int_0^t\int_{\mathbb{R}^n} E_n(t-s,x-y\varepsilon)\varepsilon^{n(1-\alpha)}\phi(y)\ u_\varepsilon(s,y\varepsilon)\,dy\,ds. \end{aligned}$$

(24) and the assumption on the initial data and ϕ imply $\|u_\varepsilon(t,\cdot)\|_{L^\infty}<\infty$.
Let $\psi\in\mathcal{D}([0,T)\times\mathbb{R}^n)$ and

$$\begin{aligned} J_{\varepsilon_\nu} =& \int_0^T\int_{\mathbb{R}^n}\int_0^t\int_{\mathbb{R}^n} E_n(t-s,x-y\varepsilon_\nu)\varepsilon_\nu^{n(1-\alpha)}\phi(y)u_{\varepsilon_\nu}(s,y\varepsilon_\nu)\ dy\ ds\ \psi(t,x)\,dx\,dt \\ =&\varepsilon_\nu^{n(1-\alpha)}\int_0^T\int_{\mathbb{R}^n}\int_0^t\int_{\mathbb{R}^n}\frac{1}{(4\pi(t-s))^{n/2}}\exp\left(-\frac{(x-\varepsilon_\nu y)^2}{4(t-s)}\right)\phi(y) \\ & u_{\varepsilon_\nu}(s,\varepsilon_\nu y)\ dy\ ds\ \psi(t,x)dx\,dt,\ \varepsilon<\varepsilon_0,\ \nu\in\mathbb{N}. \end{aligned}$$

After a suitable change of variables, using the non-negativity of sub-integral function and Fubini-Tonelli theorem one obtains

$$\begin{aligned} J_{\varepsilon_\nu} =&\varepsilon^{n(1-\alpha)}\int_0^T\int_{\mathbb{R}^n}\Big(\int_0^t\int_{\mathbb{R}^n}\frac{1}{(4\pi s)^{n/2}}\exp\left(-\frac{(x-\varepsilon_\nu y)^2}{4s}\right) \\ & \phi(y)\psi(t,x)u_{\varepsilon_\nu}(t-s,\varepsilon_\nu y)\,dx\,ds\Big)\,dy\,dt \\ =&\int_0^T\int_{\mathbb{R}^n}\varepsilon^{n(1-\alpha)}H_{\varepsilon_\nu}(y,t)\,dy\,dt,\ \varepsilon<\varepsilon_0,\ \nu\in\mathbb{N}. \end{aligned}$$

Now, for $\varepsilon<\varepsilon_0$, $\nu\in\mathbb{N}$, $y\in\mathbb{R}^n$ and $t\in[0,T)$

$$\begin{aligned} |H_{\varepsilon_\nu}(y,t)| \le& \int_0^t\int_{\mathbb{R}^n}\frac{1}{(4\pi s)^{n/2}}\exp\left(-\frac{(x-\varepsilon_\nu y)^2}{4s}\right) \\ & |\phi(y)|\cdot|\psi(t,x)|\cdot|u_{\varepsilon_\nu}(t-s,\varepsilon_\nu y)|\,dx\,ds \\ \le& C\int_0^t\int_{\mathbb{R}^n}\frac{1}{(4\pi s)^{n/2}}\exp\left(-\frac{(x-\varepsilon_\nu y)^2}{4s}\right)\,dx\,ds = Ct, \end{aligned}$$

since we have proved that $\{u_\varepsilon,\ \varepsilon<\varepsilon_0\}$ is bounded.
Therefore, using Lebesgue's dominated convergence theorem gives

$$\lim_{\varepsilon\to 0} J_{\varepsilon_\nu} = \int_0^T\int_{\mathbb{R}^n}\lim_{\varepsilon\to 0}\varepsilon^{n(1-\alpha)}H_{\varepsilon_\nu}(y,t)\,dy\,dt = 0.$$

□

In a similar way as above we have

Proposition 5. *Let $\delta \in \mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ be defined by (21) and u_0 be a continuous bounded and $L^2(\mathbb{R}^n)$-function. Let u be the solution to equation (14), with $V = \delta$, is constructed in Theorem 2 and $S_\varepsilon(t)$ be the semigroup generated by operator $A_\varepsilon = \Delta - \phi_\varepsilon$, $\varepsilon < \varepsilon_0$. Then, there is a decreasing sequence $\{\varepsilon_\nu\}_{\nu\in\mathbb{N}}$ converging to zero such that*

$$u_{\varepsilon_\nu}(t,x) = S_{\varepsilon_\nu}(t)u_0(x) \to u(t,x),\ \varepsilon_\nu \to 0,\ (t,x) \in (0,T)\times\mathbb{R}^n,$$

in the sense of weak topology in $L^2\left((0,T)\times\mathbb{R}^n\right)$, $T > 0$.

Remark 2. We can treat the above equation when the δ-function is substituted with a positive linear combination of its powers $\sum_{i=1}^{M} \alpha_i \delta^i(x)$, $M \in \mathbb{N}$, $\alpha_i > 0$, $i = 1,\dots M$. Then

$$\exp\left(-\int_0^1 \left(\phi_\varepsilon(\omega(s))\right)^i\ ds\right) \le 1,\ i \le M,$$

i.e., $\left(S_\varepsilon(t)\psi(x)\right)_\varepsilon$, $\varepsilon < \varepsilon_0$, defined by (15) satisfies

$$\sup_{t\in[0,T)} \|S_\varepsilon(t)\psi\|_{L^2} = \mathcal{O}(1).$$

Thus, $\left(S_\varepsilon(t)u_{0\varepsilon}\right)_\varepsilon$ is a representative of the solution to equation

$$\partial_t u(t,x) = \left(\Delta - \sum_{i=1}^{M} \alpha_i \delta^i(x)\right) u(t,x),\ t \in [0,T),\ x \in \mathbb{R}^n.$$

Remark 3. The first mathematically rigorous investigation of a singular perturbation was carried over in [2], where a self-adjoint realization of $-\Delta + \alpha\delta$ is obtained via the Krein theory of self-adjoint extensions of $-\Delta$ over $C_0^\infty(\mathbb{R}^3\backslash\{0\})$.

By Theorem 1.3.1 in [1], in case $n = 1$, $-\Delta + \delta$ is a self-adjoint operator with the domain $D = \{\psi;\ \psi \in H^1(\mathbb{R}),\ (-\Delta+\delta)\psi \in L^2(\mathbb{R})\}$. In this case, with any delta net $(\delta_\varepsilon)_\varepsilon$ instead of δ, the strong resolvent convergence leads to a solution of Schrödinger equation

$$-i\ \partial_t u = (\Delta - \delta)u,\ \ u|_{t=0} = u_0.$$

In fact, the net of semigroups $(S_\varepsilon)_\varepsilon$ determined by $i(\Delta - \delta_\varepsilon)_\varepsilon$, converges to the semigroup $e^{it(\Delta-\delta)}$ and $S_\varepsilon u_0, u_0 \in D$, converges to the solution $e^{it(\Delta-\delta)}u_0$ of the corresponding Cauchy problem in the sense of weak convergence.

If $n = 2$ or $n = 3$, $\delta \in H^{-2}$ and the rank one form unbounded perturbations are performed in [1] (cf. Sections 1.4 and 1.5). In these cases our approximate solutions, has to be done by a suitable delta net, for example, the one given in Lemma 1.5.3 in [1]. With $(\log|\log\varepsilon|)^{-1}$ instead of ε the convergence to the solution semigroup for the Schrödinger equation with potential δ can be also obtained. This analysis will be given in a separate paper.

Remark 4. We will prove the existance of an L^2 function satisfying

$$\partial_t u(t,x) - \Delta u(t,x) + V(x)u(t,x) = 0,\ u(0,x) = \delta(x), \tag{27}$$

in the sense of distriibutins, where V is a locally bounded function on $\mathbf{R}^n$ so that the Schrödinger operator is the self-adjoint one (for instance of Stummel class, [26]).

With $(\phi_\varepsilon)_\varepsilon$ as in (21) and Feynman-Kac formula we have a net of approximated solutions

$$u_\varepsilon(x,t) = \int_\Omega \exp\left(-\int_0^t V(\omega(s))\,ds\right)\phi_\varepsilon(\omega(t))d\mu_x(\omega),\ t \geq 0,\ x \in \mathbb{R}^n. \tag{28}$$

There exist $C > 0$ and $\eta \in (0,1)$ such that

$$|u_\varepsilon(x,t)| \leq \int_\Omega |\phi_\varepsilon(\omega(t))|d\mu_x(\omega) = C(4\pi t)^{-n/2}\int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right)\phi_\varepsilon(y)dy,$$

for every $t \geq 0$, $x \in \mathbb{R}^n$ and $\varepsilon < \eta$. This implies that there exists a net $(\varepsilon_\nu)_\nu$ decreasing to zero such that $(u_{\varepsilon_\nu})_\nu$ converges to $u \in L^2([0,T] \times \mathbb{R}^n)$ weakly in $L^2([0,T] \times \mathbb{R}^n)$. Then,

$$\begin{aligned} u_\varepsilon(t,x) = &\int_{\mathbb{R}^n} E_n(t,x-y)\phi_{\varepsilon_\nu}(y)\,dy \\ &+ \int_0^t \int_{\mathbb{R}^n} E_n(t-s,x-y)V(y)u_{\varepsilon_\nu}(s,y)\,dy\,ds, \end{aligned} \tag{29}$$

leads to the equality in the sense of distributions:

$$u(t,x) = E_n(t,x) + \int_0^t \int_{\mathbb{R}^n} E_n(t-s,x-y)V(y)u(s,y)\,dy\,ds. \tag{30}$$

Thus, we obtain a distributional solution to (27).

3.3. **Lipschitz nonlinear case.** We will use the well known inequality

$$\| \mid \nabla g|^2\|_{L^2(\mathbb{R}^n)} \leq \| \nabla g \|^2_{L^4(\mathbb{R}^n)} \leq \| \nabla g \|^2_{H^1(\mathbb{R}^n)},\ g \in H^2(\mathbb{R}^n), \tag{31}$$

which holds for $n \leq 4$. But in the sequel we assume that $n \leq 3$ because in this case $\mathcal{E}_{C^1,H^2}([0,T) : \mathbb{R}^n)$ is an algebra.

Lemma 2. *Let $n \leq 3$. Suppose that a function $f : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies $f(t,\cdot) \in C^2(\mathbb{R}^n)$, $f(\cdot,y) \in C^1([0,T))$, $t \in (0,T)$, $T > 0$, $y \in \mathbb{R}^n$ and*

$$|f(t,y)| \leq L_0(t)|y|,\ |\partial_y f(t,y)| \leq L_1(t),\ |\partial_{yy} f(t,y)| \leq L_2(t), \tag{32}$$

for some positive bounded functions $L_i : [0,T) \to \mathbb{R}$, $i = 0,1,2$.

Then, by

$$(u_\varepsilon)_\varepsilon \mapsto (f(t,u_\varepsilon))_\varepsilon$$

are defined mappings

$$\mathcal{E}_{C^1,H^2}([0,T) : \mathbb{R}^n) \to \mathcal{E}_{C^0,H^2}([0,T) : \mathbb{R}^n)$$

and

$$\mathcal{N}_{C^1,H^2}([0,T) : \mathbb{R}^n) \to \mathcal{N}_{C^0,H^2}([0,T) : \mathbb{R}^n)$$

inducing the mapping $u \mapsto [f(t,u_\varepsilon)]$

$$\mathcal{G}_{C^1,H^2}([0,T) : \mathbb{R}^n) \to \mathcal{G}_{C^0,H^2}([0,T) : \mathbb{R}^n).$$

Proof. We will only prove that $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$ implies $(f(t,u_\varepsilon))_\varepsilon \in \mathcal{E}_{C^0,H^2}([0,T):\mathbb{R}^n)$. The other parts of the proof follow in a similar way.

We have to show that there exists $a\in\mathbb{R}^n$ such that

$$\sup_{t\in[0,T)} \|f(t,u_\varepsilon)\|_{H^2(\mathbb{R}^n)} = \mathcal{O}(\varepsilon^a), \quad \text{as } \varepsilon\to 0.$$

Relation (32) implies

$$\|f(t,u_\varepsilon)\|_{L^2(\mathbb{R}^n)} \le L_0(t)\|u_\varepsilon\|_{L^2(\mathbb{R}^n)},\ t\in[0,T),\ \varepsilon<1.$$

Since $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$, we obtain that $\sup_{t\in[0,T)}\|f(t,u_\varepsilon)\|_{L^2(\mathbb{R}^n)}$ has the moderate bound.

After differentiation with respect to some spatial variable x_i we obtain

$$\begin{aligned}\|\partial_y f(t,u_\varepsilon)\partial_{x_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)} &\le \|\partial_y f(t,u_\varepsilon)\|_{L^\infty(\mathbb{R}^n)}\|\partial_{x_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)}\\ &\le L_1(t)\|\partial_{x_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

As above we obtain the moderate bound for $\sup_{t\in[0,T)}\|f_y(t,u_\varepsilon)\partial_{x_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)}$.

After another differentiation with respect to x_i, using (31), the estimate $|L_1(t)|,|L^2(t)|\le C,\ t\in[0,T)$, and the fact that $\mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$ is an algebra for $n\le 3$, we obtain

$$\begin{aligned}&\|\partial_{yy}f(t,u_\varepsilon)(\partial_{x_i}u_\varepsilon)^2+\partial_y f(t,u_\varepsilon)\partial_{x_i,x_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)}\\ &\le \|\partial_{yy}f(t,u_\varepsilon)\|_{L^\infty(\mathbb{R}^n)}\|(\partial_{x_i}u_\varepsilon)^2\|_{L^2(\mathbb{R}^n)}+\|\partial_y f(t,u_\varepsilon)\|_{L^\infty(\mathbb{R}^n)}\|\partial_{x_ix_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)}\\ &\le L_2(t)\|\partial_{x_i}u_\varepsilon\|^2_{L^4(\mathbb{R}^n)}+L_1(t)\|\partial_{x_ix_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)}\\ &\le C\left(\|\partial_{x_ix_i}u_\varepsilon\|^2_{L^2(\mathbb{R}^n)}+\|\partial_{x_ix_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)}\right).\end{aligned}$$

Since, by assumption, $\|\partial_{x_ix_i}u_\varepsilon\|_{L^2(\mathbb{R}^n)}$ has a moderate bound, the assertion follows. □

Theorem 3. *Let $n\le 3$, $T>0$, $V\in\mathcal{G}_{H^{2,\infty}}(\mathbb{R}^n)$ be of logarithmic type and $u_0\in\mathcal{G}_{H^2}(\mathbb{R}^n)$. Suppose that a function $f:[0,T)\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies the conditions of Lemma 2.*

Then, there exists a solution $u(t,x)\in\mathcal{G}_{C^1,H^2}([0,T):\mathbb{R}^n)$ to equation

$$\partial_t u(t,x)=(\Delta-V)u(t,x)+f(t,u(t,x)),\ u(0,x)=u_0(x). \tag{33}$$

Moreover, if potential V is of log-log type, then the solution to equation (33) is unique in $\mathcal{G}_{C^1,L^2}([0,T):\mathbb{R}^n)$.

Proof. Let $\varepsilon<\varepsilon_0$ be fixed. Consider the approximated equation

$$\partial_t u_\varepsilon(t,x)=(\Delta-V_\varepsilon)u_\varepsilon(t,x)+f(t,u_\varepsilon(t,x)),\ u_\varepsilon(0,x)=u_{0\varepsilon}(x). \tag{34}$$

As we have shown in Theorem 2 (i), $(S_\varepsilon)_\varepsilon$ defined by (15) is an element of $\mathcal{SE}_M\big([0,\infty):\mathcal{L}(L^2(\mathbb{R}^n))\big)$.

For every fixed $\varepsilon\in(0,1)$ the classical solution to (34) exists and satisfies

$$u_\varepsilon(t,x)=S_\varepsilon(t)u_{0\varepsilon}(x)+\int_0^t S_\varepsilon(t-s)f(s,u_\varepsilon(s,x))\,ds,\ t\in[0,T),\ x\in\mathbb{R}^n. \tag{35}$$

Let us show that $(u_\varepsilon)_\varepsilon\in\mathcal{E}_{C^1,H^2}([0,T):\mathbb{R}^n)$.

First,

$$
\begin{aligned}
u_\varepsilon(t,x) = & \int_{\mathbb{R}^n} E_n(t,x-y)u_{0\varepsilon}(y)dy \\
& + \int_0^t \int_{\mathbb{R}^n} E_n(t-s,x-y)V_\varepsilon(y)u_\varepsilon(s,y)dyds \\
& + \int_0^t \int_{\mathbb{R}^n} E_n(t-s,x-y)f(s,u_\varepsilon(s,y))ds,\ (t,x)\in[0,T)\times\mathbb{R}^n.
\end{aligned} \tag{36}
$$

Therefore,

$$
\begin{aligned}
\|u_\varepsilon(t,\cdot)\|_{L^2} \leq & \ \|E_n(t,\cdot)\|_{L^1}\|u_{0\varepsilon}\|_{L^2} + \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|V_\varepsilon(\cdot)\|_{L^\infty}\|u_\varepsilon(s,\cdot)\|_{L^2}ds \\
+ & \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|f(s,u_\varepsilon(s,\cdot))\|_{L^2}ds \\
\leq & \ \|u_{0\varepsilon}\|_{L^2} + \int_0^t \|V_\varepsilon(\cdot)\|_{L^\infty}\|u_\varepsilon(s,\cdot)\|_{L^2}ds \\
+ & \int_0^t L_0(s)\ \|u_\varepsilon(s,\cdot)\|_{L^2}ds,\ t\in[0,T),
\end{aligned}
$$

i.e.,

$$
\|u_\varepsilon(t,\cdot)\|_{L^2} = \|u_{0\varepsilon}\|_{L^2}\exp\left(\int_0^t (\|V_\varepsilon(\cdot)\|_{L^\infty} + L_0(s))\,ds\right) = \mathcal{O}(\varepsilon^a),\quad \text{as } \varepsilon\to 0, \tag{37}
$$

uniformly for $t\in[0,T)$, for some $a\in\mathbb{R}$, since V is of logarithmic type.

Note, $u_{0\varepsilon}\in H^2(\mathbb{R}^n)$ and inequality (31) imply the moderate bound for $\|u_{0\varepsilon}(\cdot)\|_{L^4}$. Therefore, the same procedure as above, with $\|\cdot\|_{L^4}$ instead of $\|\cdot\|_{L^2}$, gives the moderate bound for $\|u_\varepsilon(t,\cdot)\|_{L^4}$.

Differentiating (36) with respect to some spatial variable, we have

$$
\begin{aligned}
\partial_{x_i}u_\varepsilon(t,x) = & \int_{\mathbb{R}^n} E_n(t,y)\partial_{x_i}u_{0\varepsilon}(x-y)\,dy \\
+ & \int_0^t\int_{\mathbb{R}^n} E_n(t-s,y)\partial_{x_i}(V_\varepsilon(x-y)u_\varepsilon(s,x-y))\,dy\,ds \\
+ & \int_0^t\int_{\mathbb{R}^n} E_n(t-s,y)\partial_{x_i}f(s,u_\varepsilon(s,x-y))dyds, \\
= & \int_{\mathbb{R}^n} E_n(t,y)\partial_{x_i}u_{0\varepsilon}(x-y)\,dy \\
+ & \int_0^t\int_{\mathbb{R}^n} E_n(t-s,y)(\partial_{x_i}V_\varepsilon(x-y)u_\varepsilon(s,x-y) \\
+ & V_\varepsilon(x-y)\partial_{x_i}u_\varepsilon(s,x-y))\,dy\,ds \\
+ & \int_0^t\int_{\mathbb{R}^n} E_n(t-s,y)\partial_u f(s,u_\varepsilon(s,x-y))\partial_{x_i}u_\varepsilon(s,x-y)dyds,
\end{aligned} \tag{38}
$$

for $(t,x)\in[0,T)\times\mathbb{R}^n$. Young's inequality implies

$$\begin{aligned}
\|\partial_{x_i}u_\varepsilon(t,\cdot)\|_{L^2} &\leq \|E_n(t,\cdot)\|_{L^1}\|\partial_{x_i}u_{0\varepsilon}(\cdot)\|_{L^2}\\
&+ \int_0^t \|E_n(t-s,\cdot)\|_{L^1}(\|\partial_{x_i}V_\varepsilon(\cdot)\|_{L^\infty}\|u_\varepsilon(s,\cdot)\|_{L^2}\\
&+ \|V_\varepsilon(\cdot)\|_{L^\infty}\|\partial_{x_i}u_\varepsilon(s,\cdot)\|_{L^2})\,ds\\
&+ \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|\partial_u f(s,u_\varepsilon(s,\cdot))\|_{L^\infty}\|\partial_{x_i}u_\varepsilon(s,\cdot)\|_{L^2}\,ds,\\
&\leq \|\partial_{x_i}u_{0\varepsilon}(\cdot)\|_{L^2}+\int_0^t(\|\partial_{x_i}V_\varepsilon(\cdot)\|_{L^\infty}\|u_\varepsilon(s,\cdot)\|_{L^2}\\
&+ \|V_\varepsilon(\cdot)\|_{L^\infty}\|\partial_{x_i}u_\varepsilon(s,\cdot)\|_{L^2})\,ds\\
&+ \int_0^t L_1(s)\|\partial_{x_i}u_\varepsilon(s,\cdot)\|_{L^2}\,ds,
\end{aligned}$$

uniformly for $t\in[0,T)$. This implies

$$\begin{aligned}
\|\partial_{x_i}u_\varepsilon(t,\cdot)\|_{L^2} \leq &(\|\partial_{x_i}u_{0\varepsilon}(\cdot)\|_{L^2}+\int_0^t \|\partial_{x_i}V_\varepsilon(\cdot)\|_{L^\infty}\|u_\varepsilon(s,\cdot)\|_{L^2})\\
&\cdot\exp(\int_0^t(\|V_\varepsilon(\cdot)\|_{L^\infty}+L_1(s))\,ds,\ t\in[0,T).
\end{aligned}$$

Using the facts that $\sup_{t\in[0,T)}\|u_\varepsilon(t,\cdot)\|_{L^2}$ has the moderate bound and that V is of logarithmic type we obtain the moderate bound for $\sup_{t\in[0,T)}\|\partial_{x_i}u_\varepsilon(t,\cdot)\|_{L^2}$.

Since $\|\partial_{x_i}u_{0\varepsilon}(\cdot)\|_{L^4}$ and $\sup_{t\in[0,T)}\|u_\varepsilon(t,\cdot)\|_{L^4}$ have moderate bounds (the first one by inequality (31) and the second one by the previous step), the same procedure gives us the moderate bound for $\sup_{t\in[0,T)}\|\partial_{x_i}u_\varepsilon(t,\cdot)\|_{L^4}$, too.

After another differentiation, one can obtain the moderateness bounds also. Let us show the uniqueness of the solution.

Note that if potential V is of log-log type (as assumed for the uniqueness) then from (37) follows that solution to equation (33) is of logarithmic type.

Let u_ε and v_ε, be solutions to equation (34). Denote $G_\varepsilon=u_\varepsilon-v_\varepsilon$ and define

$$g_\varepsilon(t,x)=\int_0^1\partial_u f(t,\sigma u_\varepsilon(t,x)+(1-\sigma)v_\varepsilon(t,x))\,d\sigma,\ t\in[0,T),\ x\in\mathbb{R}^n.$$

Then, G_ε is a solution to equation

$$\begin{aligned}
&\partial_t G_\varepsilon(t,x)=(\Delta-V_\varepsilon)G_\varepsilon(t,x)+g_\varepsilon(t,x)G_\varepsilon(t,x)+N_\varepsilon(t,x),\\
&(G_\varepsilon(0,x))_\varepsilon=(N_{0\varepsilon}(x))_\varepsilon\in\mathcal{N}_{H^2}(\mathbb{R}^n),\ \varepsilon<\varepsilon_0,
\end{aligned}$$

where $(N_\varepsilon)_\varepsilon\in\mathcal{N}_{C^1,L^2}([0,T):\mathbb{R}^n)$. From the definition of g_ε, conditions given on the function f and the fact that u and v are both of logarithmic type it follows that

$$\|g_\varepsilon(t,x)\|_{L^2}\leq C\,(\|u_\varepsilon\|_{L^2}+\|v_\varepsilon\|_{L^2})=\mathcal{O}(\log\varepsilon^{-1}),\ t\in[0,T).$$

Let $V_{1\varepsilon} = V_\varepsilon - g_\varepsilon$, $\varepsilon < \varepsilon_0$. Then function V_1 is of logarithmic type. Now,

$$\begin{aligned} G_\varepsilon(t,x) = & \int_{\mathbb{R}^n} E_n(t,x-y)N_{0\varepsilon}(y)dy \\ & + \int_0^t \int_{\mathbb{R}^n} E_n(t-s,x-y)V_{1\varepsilon}(y)G_\varepsilon(s,y)dyds \qquad (39) \\ & + \int_0^t \int_{\mathbb{R}^n} E_n(t-s,x-y)f(s,G_\varepsilon(s,y))ds,\ (t,x)\in[0,T)\times\mathbb{R}^n, \end{aligned}$$

for $\varepsilon < \varepsilon_0$. This implies

$$\begin{aligned} \|G_\varepsilon(t,\cdot)\|_{L^2} \le & \|E_n(t,\cdot)\|_{L^1}\|N_{0\varepsilon}\|_{L^2} + \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|V_{1\varepsilon}(\cdot)\|_{L^\infty}\|G_\varepsilon(s,\cdot)\|_{L^2}ds \\ & + \int_0^t \|E_n(t-s,\cdot)\|_{L^1}\|f(s,G_\varepsilon(s,\cdot))\|_{L^2}ds \\ \le & \|N_{0\varepsilon}\|_{L^2} + \int_0^t \|V_{1\varepsilon}(\cdot)\|_{L^\infty}\|G_\varepsilon(s,\cdot)\|_{L^2}ds \\ & + \int_0^t L_0(s)\ \|G_\varepsilon(s,\cdot)\|_{L^2}ds,\ t\in[0,T), \end{aligned}$$

and

$$\|G_\varepsilon(t,\cdot)\|_{L^2} = \|N_{0\varepsilon}\|_{L^2}\exp\left(\int_0^t (\|V_{1\varepsilon}(\cdot)\|_{L^\infty} + L_0(s))\,ds\right) = \mathcal{O}(\varepsilon^a),$$

uniformly for $t\in[0,T)$, for every $a\in\mathbb{R}$, since $(N_{0\varepsilon}(x))_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^n)$ and V_1 is of logarithmic type. Therefore, the solution is unique in $\mathcal{G}_{C^1,L^2}([0,T):\mathbb{R}^n)$. □

Proposition 6. *Let $n\le 3$, $T>0$, $V\in H^{2,\infty}(\mathbb{R}^n)$, $u_0=\delta$ (thus $u_{0,\varepsilon} = \frac{1}{\varepsilon^n}\phi(\frac{\cdot}{\varepsilon})$, $\varepsilon\in(0,1)$) and a function $f:[0,T)\times\mathbb{R}^n\to\mathbb{R}^n$ satisfy first two conditions of (32).*

Then, there exists a function $u(t,x)\in L^1([0,T):\mathbb{R}^n)$ satisfying equation

$$\partial_t u(t,x) = (\Delta - V)u(t,x) + f(t,u(t,x)),\ u(0,x) = u_0(x).$$

in the sense of distributions.

Proof. As in (36)-(37) but with the L^1-norm instead of L^2-norm (where it appears) we have

$$||u_\varepsilon||_{L^1}, ||\partial_{x_i}u_\varepsilon||_{L^1} < const,\ \varepsilon\in(0,1).$$

By Rellich lemma this implies the existence of an L^1-convergent sequence $(u_{\varepsilon_\nu})_\nu$. Denote by $u\in L^1([0,T]\times\mathbf{R}^n)$ its limit. It has a subsequence converging to u almost everywhere. By (35) we have

$$u_\varepsilon(t,x) = E_n(t,x) + \int_0^t\int_{\mathbb{R}^n} E_n(t-s,x-y)V(y)u(s,y)dyds$$

$$+ \int_0^t\int_{\mathbb{R}^n} E_n(t-s,x-y)f(s,u(s,y))ds,\ (t,x)\in[0,T)\times\mathbb{R}^n.$$

Now it is easy to see that u is a distribution solution to the equation. □

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