NUMERICAL VERIFICATION OF DELTA SHOCK WAVES FOR PRESSURELESS GAS DYNAMICS

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1. Introduction

Consider an arbitrary Riemann problem for a pressureless gas dynamic model given by the system

$$\begin{aligned} u_t + (uv)_x &= 0 \\ (uv)_t + (uv^2)_x &= 0, \end{aligned} \tag{1}$$ orig.probl.

and the initial data

$$u(x,0) = \begin{cases} u_l, x < 0 \\ u_r, x > 0 \end{cases}, \; v(x,0) = \begin{cases} v_l, x < 0 \\ v_r, x > 0 \end{cases}, \tag{2}$$ id0

where u and v present a density and a velocity, respectively.

The two eigenvalues of the system are the same $\lambda_1 = \lambda_2 = v$ and the system is weakly hyperbolic. The above problem has a bounded weak entropy solution consisting of combinations of contact discontinuities and vacuum states ($u \equiv 0$) when $v_l \leq v_r$, and a delta shock wave solution when $v_l > v_r$, see [1] or [15]. The subject of the present paper is numerical verification of delta shock wave existence for (1), so we consider only the case $v_l > v_r$. Then a solutions does not contain the vacuum state and we transform it into the evolutionary form

$$\begin{aligned} u_t + w_t &= 0 \\ w_t + (w^2/u)_x &= 0, \end{aligned} \tag{3}$$ 1'

by the substitution $w = uv$. The initial data is now given by

$$u(x,0) = \begin{cases} u_l, x < 0 \\ u_r, x > 0 \end{cases}, w(x,0) = \begin{cases} w_l = u_l v_l, x < 0 \\ w_r = u_r v_r, x > 0 \end{cases}.$$

The measure theoretic solution to (1,refid0) constructed in a number of papers (see [4] or [15], for example) has a distributional limit given by

$$\begin{aligned} U(x,t) &\approx \begin{cases} u_l, x < ct \\ u_r, \; x > ct \end{cases} + (v_l - v_r)\sqrt{u_l u_r} t \delta(x - ct), \\ V(x,t) &\approx \begin{cases} v_l, x < ct \\ v_r, x > ct \end{cases}. \end{aligned}$$

Another possibility, which will be presented here, is to give a solution using generalized function space obtained from nets of smooth functions. Spaces of that type are already successfully used in numerics for PDE's. One can look in the book [2] for some other examples. The particular version of Colombeau generalized functions, $\mathcal{G}_g(\mathbb{R}^2_+)$, used in the present paper is defined in [11].

The admissibility condition for delta shock waves used in the paper is given by the condition

$$\lambda_2(u_l, v_l) \geq \lambda_1(u_l, v_l) \geq c \geq \lambda_2(u_r, v_r) \geq \lambda_1(u_r, v_r).$$

The waves which satisfy the above condition are said to be overcompressive.

First, we solve Riemann problem for system (3) in the case $w_l/u_l > w_r/u_r$ using the above mentioned space of generalized functions $\mathcal{G}_g(\mathbb{R}^2_+)$. The obtained solution can be interpreted as a net of smooth functions possessing the distributional limit which contains the delta function.

Second, we present a numerical solution in a large time interval which gives a reasonable verification of the above solution both of system (1) and its perturbation

$$\begin{aligned} u_t + w_t &= 0 \\ w_t + (w^2/u + \mu u^\gamma)_x &= 0, \end{aligned} \tag{4}$$

1''

where $1 < \gamma < 3$, $w_l/u_l > w_r/u_r$. System (4) is called isentropic gas dynamics model. We take γ to be constant or coupled with μ: $\gamma = \gamma(\mu) \to 1$ as $\mu \to 0$. Contrary to a viscosity approximation when a perturbed system is parabolic or mixed hyperbolic-parabolic, system (4) is hyperbolic so its Riemann problem can be solved by a combination of the usual elementary wave solutions.

In all three cases the obtained results are mutually consistent and they are consistent with generalized solution. Another interpretation of such a results is that the numerical procedure used in this paper is also robust enough for weakly hyperbolic problems.

There is a large class of numerical methods dealing with conservation laws. Roughly speaking, one can consider methods on fixed or moving meshes. As discontinuities propagate in time, the solution at a spatial point can change very rapidly and therefore fixed spatial mesh requires extremely small time step. On the other hand there is no justification for small time steps in smooth regions. That is why a nonuniform mesh with reasonably large spatial step in smooth regions and small step in discontinuity regions should be more efficient for this type of problems. As shocks travel in time, mesh should also be able to adjust in time so that points remain concentrated near discontinuities, thus maintaining a balance between computational costs and accuracy. Time adaptation can be done by static regridding technique, or it can be based on dynamic refinement in which the mesh equation is explicitly derived. Based on the equidistribution principle, which attempts to distribute some measure of solution error over the spatial domain, dynamic refinement naturally generates concentration of mesh points in the regions of discontinuity. This technique leads to the coupled problem consisting of mesh equation based on monitor function and physical PDE, see [3] or [13].

High resolution finite volume methods are employed to solve the physical PDE. One of them is the wave propagation method introduced by LeVeque in [7] and implemented in the software package CLAWPACK [6]. The method is based on Godunov's scheme and Roe's solvers with addition of high resolution terms. One of the implementations of this method, coupled with dynamic refinement of mesh with fixed number of spatial points is presented in [13]. That algorithm, with the necessary adjustment to the specific problem we consider here, will serve as a base for our experiments.

Delta shock waves can be obtained using the following procedure. The first step is the smoothing of initial data (2) over some finite interval where a small

parameter $\varepsilon > 0$ denotes the smoothing width. The second step is to find a smooth weak solution depending on the given perturbation term to the Riemann problem, the so called generalized solution. Interpretation of the solution can be given in a framework of Colombeau generalized functions algebras, like in [9]: Solutions are considered as nets of smooth functions depending on a parameter ε with equality substituted by the distributional convergence as ε tends to zero.

Due to the specific nature of the delta shock waves (they contain δ-functions) it is not possible to follow the solution to (1) numerically in some large time interval. Therefore we will follow the solution only until a time point T where the delta shock is clearly formed.

The situation is different for the perturbed system (4) even for a small value of a perturbation coefficient μ. The solution is a combination of two shock waves in the case $w_l/u_l > w_r/u_r$, and we can follow the numerical solution for a quite long time. There is no need for a generalized solution.

The basic numerical algorithm will be the one presented in [13], with some adaptation to the specific problem we consider. First of all we apply the smoothing technique to initial data in order to avoid non-physical oscillations. The original problem (1) is modified by introducing the perturbation term as will be explained later. The monitor function used to distribute the mesh points is based on the arclength function with a parameter that prevents too many points in the shock regions but allows enough points in these regions. Furthermore the mesh is moving in spatial domain with time in order to follow the waves. These parameters (smoothing, perturbation, mesh parameter and spatial movement of the mesh) have a great influence on performance of the method and therefore need careful adjustment. Several properties of delta shock waves are exploited in order to check the relevance of obtained numerical solution.

2. Generalized solution

We shall briefly repeat some definitions of Colombeau algebra given in [11] and [9]. Denote $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$, $\overline{\mathbb{R}^2_+} := \mathbb{R} \times [0, \infty)$ and let $C_b^\infty(\Omega)$ be the algebra of smooth functions on Ω bounded together with all their derivatives. Let $C_{\overline{b}}^\infty(\mathbb{R}^2_+)$ be a set of all functions $u \in C^\infty(\mathbb{R}^2_+)$ satisfying $u|_{\mathbb{R}\times(0,T)} \in C_b^\infty(\mathbb{R} \times (0, T))$ for every $T > 0$. Let us remark that every element of $C_b^\infty(\mathbb{R}^2_+)$ has a smooth extension up to the line $\{t = 0\}$, i.e. $C_b^\infty(\mathbb{R}^2_+) = C_b^\infty(\overline{\mathbb{R}^2_+})$. This is also true for $C_{\overline{b}}^\infty(\mathbb{R}^2_+)$.

Definition 1. $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$ is the set of all maps $G : (0, 1) \times \mathbb{R}^2_+ \to \mathbb{R}$, $(\varepsilon, x, t) \mapsto G_\varepsilon(x, t)$, where for every $\varepsilon \in (0, 1)$, $G_\varepsilon \in C_{\overline{b}}^\infty(\mathbb{R}^2_+)$ satisfies:
For every $(\alpha, \beta) \in \mathbb{N}_0^2$ and $T > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial_x^\alpha \partial_t^\beta G_\varepsilon(x, t)| = (\varepsilon^{-N}), \text{ as } \varepsilon \to 0.$$

$\mathcal{E}_{M,g}(\mathbb{R}^2_+)$ is an multiplicative differential algebra, i.e. a ring of functions with the usual operations of addition and multiplication, and differentiation which satisfies Leibniz rule.

$\mathcal{N}_g(\mathbb{R}^2_+)$ is the set of all $G \in \mathcal{E}_{M,g}(\mathbb{R}^2_+)$, satisfying:
For every $(\alpha, \beta) \in \mathbb{N}_0^2$, $a \in \mathbb{R}$ and $T > 0$

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |\partial_x^\alpha \partial_t^\beta G_\varepsilon(x, t)| = \mathcal{O}(\varepsilon^a), \text{ as } \varepsilon \to 0.$$

□

Clearly, $\mathcal{N}_g(\mathbb{R}^2_+)$ is an ideal of the multiplicative differential algebra $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$, i.e. if $G_\varepsilon \in \mathcal{N}_g(\mathbb{R}^2_+)$ and $H_\varepsilon \in \mathcal{E}_{M,g}(\mathbb{R}^2_+)$, then $G_\varepsilon H_\varepsilon \in \mathcal{N}_g(\mathbb{R}^2_+)$.

Definition 2. The multiplicative differential algebra $\mathcal{G}_g(\mathbb{R}^2_+)$ of generalized functions is defined by $\mathcal{G}_g(\mathbb{R}^2_+) = \mathcal{E}_{M,g}(\mathbb{R}^2_+)/_g(\mathbb{R}^2_+)$. All operations in $\mathcal{G}_g(\mathbb{R}^2_+)$ are defined by the corresponding ones in $\mathcal{E}_{M,g}(\mathbb{R}^2_+)$. □

If $C_b^\infty(\mathbb{R})$ is used instead of $C_b^\infty(\mathbb{R}^2_+)$ (i.e. drop the dependence on the t variable), then one obtains $\mathcal{E}_{M,g}(\mathbb{R})$, $\mathcal{N}_g(\mathbb{R})$, and consequently, the space of generalized functions on a real line, $\mathcal{G}_g(\mathbb{R})$.

In the sequel, G denotes an element (equivalence class) in $\mathcal{G}_g(\Omega)$ defined by its representative $G_\varepsilon \in \mathcal{E}_{M,g}(\Omega)$.

Since $C_{\overline{b}}^\infty(\mathbb{R}^2_+) = C_{\overline{b}}^\infty(\overline{\mathbb{R}^2_+})$, one can define a restriction of a generalized function to $\{t = 0\}$ in the following way.

For given $G \in \mathcal{G}_g(\mathbb{R}^2_+)$, its restriction $G|_{t=0} \in \mathcal{G}_g(\mathbb{R})$ is the class determined by a function $G_\varepsilon(x, 0) \in \mathcal{E}_{M,g}(\mathbb{R})$. In the same way as above, $G(x - ct) \in \mathcal{G}_g(\mathbb{R})$ is defined by $G_\varepsilon(x - ct) \in \mathcal{E}_{M,g}(\mathbb{R})$.

If $G \in \mathcal{G}_g$ and $f \in C^\infty(\mathbb{R})$ is polynomially bounded together with all its derivatives, then one can easily show that the composition $f(G)$, defined by a representative $f(G_\varepsilon)$, $G \in \mathcal{G}_g$ makes sense. It means that $f(G_\varepsilon) \in \mathcal{E}_{M,g}$ if $G_\varepsilon \in \mathcal{E}_{M,g}$, and $f(G_\varepsilon) - f(H_\varepsilon) \in \mathcal{N}_g$ if $G_\varepsilon - H_\varepsilon \in \mathcal{N}_g$.

The equality in the space of the generalized functions $\mathcal{G}_g$ is to strong for our purpose (see [10] for some nice examples), so we need to define a weaker relation, so called, association.

Definition 3. A generalized function $G \in \mathcal{G}_g(\Omega)$ is said to be *associated with* $u \in \mathcal{D}'(\Omega)$, $G \approx u$, if for some (and hence every) representative G_ε of G, $G_\varepsilon \to u$ in $\mathcal{D}'(\Omega)$ as $\varepsilon \to 0$. Two generalized functions G and H are said to be associated, $G \approx H$, if $G - H \approx 0$. The rate of convergence in $\mathcal{D}'$ with respect to ε is called the order of association. □

A generalized function G is said to be *of a bounded type* if

$$\sup_{(x,t)\in\mathbb{R}\times(0,T)} |G_\varepsilon(x,t)| = \mathcal{O}(1) \text{ as } \varepsilon \to 0,$$

for every $T > 0$.

$G \in \mathcal{G}_g$ is a positive generalized function if there exists its representative G_ε and a real $a > 0$ such that $G_\varepsilon(x,t) \geq a$, for every $(x,t) \in \mathbb{R}^2_+$. This condition on a representative also means that $G \geq a$.

Let $u \in \mathcal{D}'_{L^\infty}(\mathbb{R})$. Let $\mathcal{A}_0$ be the set of all functions $\phi \in C_0^\infty(\mathbb{R})$ satisfying $\phi(x) \geq 0$, $x \in \mathbb{R}$, $\int \phi(x)dx = 1$ and $\operatorname{supp}\phi \subset [-1, 1]$, i.e.

$$\mathcal{A}_0 = \{\phi \in C_0^\infty : \ (\forall x \in \mathbb{R})\phi(x) \geq 0, \ \int \phi(x)dx = 1, \ \operatorname{supp}\phi \subset [-1,1]\}.$$

Let $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, $x \in \mathbb{R}$. Then

$$\iota_\phi : u \mapsto u * \phi_\varepsilon/\mathcal{N}_g,$$

where $u * \phi_\varepsilon/\mathcal{N}_g$ denotes the equivalence class with respect to the ideal $\mathcal{N}_g$, defines a mapping of $\mathcal{D}'_{L^\infty}(\mathbb{R})$ into $\mathcal{G}_g(\mathbb{R})$, where $*$ denotes the usual convolution in $\mathcal{D}'$. It

is clear that ι_ϕ commutes with the derivation, i.e.

$$\partial_x \iota_\phi(u) = \iota_\phi(\partial_x u).$$

Definition 4. (a) $G \in \mathcal{G}_g(\mathbb{R})$ is said to be *a generalized step function* with value (y_0, y_1) if it is of bounded type and

$$G_\varepsilon(y) = \begin{cases} y_0, & y < -\varepsilon \\ y_1, & y > \varepsilon \end{cases}$$

Denote $[G] := y_1 - y_0$.

(b) $D \in \mathcal{G}_g(\mathbb{R})$ is said to be *generalized delta function* (*δ-function*, for short) if its representatives are nonnegative functions supported in $[-1, 1]$ such that $\int D_\varepsilon(y)dy = 1$. □

Suppose that the initial data are given by

$$u|_{t=T} = \begin{cases} u_0, & x < X \\ u_1, & x > X \end{cases} \quad v|_{t=T} = \begin{cases} v_0, & x < X \\ v_1, & x > X. \end{cases}$$

Definition 5. *Delta shock wave* is an associated solution to (3) of the form

$$\begin{aligned} u(x,t) &= G(x-ct) + s_1(t)D(x-ct) \\ w(x,t) &= H(x-ct) + s_2(t)D(x-ct), \end{aligned} \tag{5}$$ dsw-i

where

(i) $c \in \mathbb{R}$ is the speed of the wave,
(ii) $s_i(t)$, $t \geq 0$ are smooth functions, $s_i(0) = 0$, $i = 1, 2$.
(iii) G and H are generalized step functions with values (u_0, u_1) and (v_0, v_1) respectively, and D is a generalized delta function. □

Remark 1. The standard choice for a generalized delta function is $D_\varepsilon = \phi_\varepsilon$, $\phi \in \mathcal{A}_0$, i.e. $D = \iota_\phi(\delta)$, where δ is the delta distribution. Also, the standard choice for a representative of a step function is $G = \iota_\phi(g) = g * \phi_\varepsilon / \mathcal{N}_g$, where $g = \begin{cases} y_0, & x < 0 \\ y_1, & x > 0 \end{cases} \in L^\infty$. The above definition does not provide a unique way to interpret the product of generalized step and delta function (as in [9], where the representatives are chosen in a special way), but this fact has not importance in the case of system (3) as one will see later.

We shall use the following three lemmas.

⟨lm1⟩ **Lemma 1.** *Let $A \in \mathcal{G}_g(\mathbb{R}^2_+)$ be of a bounded type, $B \geq \tau > 0$, $\tau \in \mathbb{R}$ be a generalized function in $\mathcal{G}_g(\mathbb{R}^2_+)$ and $D \in \mathcal{G}_g(\mathbb{R})$ be a generalized delta function. Then*

$$\frac{A(x,t)}{B(x,t) + s(t)D(x-ct)} \approx \frac{A(x,t)}{B(x,t)},$$

for any smooth function $s : \mathbb{R}_+ \to \mathbb{R}_+$.

Proof. Take a representatives $B_\varepsilon \geq \tau$ and $D_\varepsilon \geq 0$, $\operatorname{supp} D_\varepsilon \subset [-\varepsilon, \varepsilon]$ of B and D, respectively. Then

$$\begin{aligned} I &= \Big| \iint_{\mathbb{R}^2_+} \Big(\frac{A_\varepsilon(x,t)}{B_\varepsilon(x,t) + s(t)D_\varepsilon(x-ct)} - \frac{A_\varepsilon(x,t)}{B_\varepsilon(x,t)} \Big) \phi(x,t)dxdt \Big| \\ &\leq \iint_{\operatorname{supp}\phi \cap \{(x,t):\, |x-ct| < \varepsilon\}} \Big| \frac{A_\varepsilon(x,t)}{B_\varepsilon(x,t)} \Big| |\phi(x,t)| dxdt. \end{aligned}$$

Since $|A_\varepsilon(x,t)| \le C_1 < \infty$, the integrand of the last integral is bounded. The fact that $\mathrm{mes}(\mathrm{supp}\,\phi \cap \{(x,t):\ |x-ct|<\varepsilon\}) \le \mathrm{const}\cdot\varepsilon$ proves that $I \to 0$ as $\varepsilon \to 0$. □

Lemma 2. *Let A, B and D be as above. Let $s_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i = 1, 2$, be smooth functions. Then*

$$\frac{A(x,t)s_1(t)D(x-ct)}{B(x,t)+s_2(t)D(x-ct)} \approx 0. \tag{6}$$ el2

Proof. Immediately one can see that

$$\Big\| \frac{A_\varepsilon(x,t)s_1(t)D_\varepsilon(x-ct)}{B_\varepsilon(x,t)+s_2(t)D_\varepsilon(x-ct)} \Big\|_{L^\infty(\mathbb{R}^2_+)} = C_\varepsilon < \infty,$$

and

$$\mathrm{mes}\Big(\mathrm{supp}\,\big(\frac{A_\varepsilon(x,t)s_1(t)D_\varepsilon(x-ct)}{B_\varepsilon(x,t)+s_2(t)D_\varepsilon(x-ct)}\big) \cap \mathrm{supp}\,\phi\Big) = \mathcal{O}(\varepsilon),\ \varepsilon \to 0,$$

for every $\phi \in C_0^\infty(\mathbb{R}^\infty_+)$. Thus

$$\iint_{\mathbb{R}^2_+} \Big(\frac{A_\varepsilon(x,t)s_1(t)D_\varepsilon(x-ct)}{B_\varepsilon(x,t)+s_2(t)D_\varepsilon(x-ct)}\Big)\phi(x,t)dxdt \to 0,\ \varepsilon \to 0. \square$$

Remark 2. Let us note that it generalized delta functions above have different representatives, the relation (6) need not to be true. For example, if they have representatives with disjoint supports, then the obtained result would be

$$(A(x,t)/B(x,t))s_1(t)\delta(x-ct)$$

instead of zero.

⟨lm3⟩ **Lemma 3.** *Let A, D and s_i be as above. Suppose that B is of the bounded type. Then*

$$\frac{A(x,t)s_1(t)D^2(x-ct)}{B(x,t)+s_2(t)D(x-ct)} \approx A(x,t)\frac{s_1(t)}{s_2(t)}D(x-ct),$$

provided that $s_1(t)/s_2(t)$ can be continuously prolonged to the point $t = 0$.

Proof. Using the fact that $\mathcal{G}_g(\mathbb{R}^2_+)$ is a multiplicative algebra one gets

$$\begin{aligned}
&\frac{A(x,t)s_1(t)D^2(x-ct)}{B(x,t)+s_2(t)D(x-ct)}\\
=&\frac{A(x,t)s_1(t)D^2(x-ct)+A(x,t)\frac{s_1(t)}{s_2(t)}(t)B(x,t)D(x-ct)}{B(x,t)+s_2(t)D(x-ct)}\\
&-\frac{A(x,t)\frac{s_1(t)}{s_2(t)}(t)B(x,t)D(x-ct)}{B(x,t)+s_2(t)D(x-ct)}\\
=&\frac{A(x,t)\frac{s_1(t)}{s_2(t)}(t)D(x-ct)\big(s_2(t)D(x-ct)+B(x,t)\big)}{B(x,t)+s_2(t)D(x-ct)}\\
&-\frac{A(x,t)\frac{s_1(t)}{s_2(t)}(t)B(x,t)D(x-ct)}{B(x,t)+s_2(t)D(x-ct)}\\
\approx&A(x,t)\frac{s_1(t)}{s_2(t)}(t)D(x-ct).
\end{aligned}$$

In the last association process we have used relation (6). □

Now we are in the position to state the following theorem.

⟨th1⟩ **Theorem 1.** *There exists an overcompressive delta shock wave solution to (3,2) if* $u_l, u_r > 0$, $w_l/u_l > w_r/u_r$.

Proof. Let

$$\begin{aligned} u(x,t) &= G(x-ct) + s_1(t)D(x-ct) \\ w(x,t) &= H(x-ct) + s_2(t)D(x-ct) \end{aligned} \tag{7}$$ res

where G and H are generalized step functions with values (u_l, u_r) and (w_l, w_r), resp., $s_i : \mathbb{R}_+ \to \mathbb{R}_+$, $s_i(0) = 0$, $i = 1, 2$, are smooth functions, and D is a generalized delta function. In the sequel we shall omit the argument $x - ct$ where it appears. We have

$$\begin{aligned} \frac{w^2}{u} &= \frac{(H + s_2(t)D)^2}{G + s_1(t)D} = \frac{H^2 + 2Hs_2(t)D + s_2^2(t)D^2}{G + s_1(t)D} \\ &= \frac{H^2}{G + s_1(t)D} + \frac{2Hs_2(t)D}{G + s_1(t)D} + \frac{s_2^2(t)D^2}{G + s_1(t)D} \approx \frac{H^2}{G} + 0 + \frac{s_2^2(t)}{s_1(t)}D, \end{aligned} \tag{8}$$ del

where we have used Lemmas 1-3.

Substituting (7) into the first equation in (3) one gets

$$\begin{aligned} u_t + w_x &\approx -c[G]\delta + s_1'(t)\delta - cs_1(t)\delta' + s_2(t)\delta' + [H]\delta \\ &= (s_1'(t) - c[G] + [H])\delta + (s_2(t) - cs_1(t))\delta' \approx 0. \end{aligned}$$

Thus, $s_1(t) = \sigma t$, $s_2(t) = c\sigma t$ and

$$\sigma = c[G] - [H]. \tag{9}$$ j1

Substitution of (7) into the second equation in (3) and use of (8) yields

$$\begin{aligned} w_t + \left(\frac{w^2}{u}\right)_x &\approx -c[H]\delta + s_2'(t)\delta - cs_2(t)\delta' + \left[\frac{H^2}{G}\right]\delta + \frac{s_2^2(t)}{s_1(t)}\delta' \\ &= \left(c\sigma - c[H] + \left[\frac{H^2}{G}\right]\right)\delta + (c^2\sigma - c^2\sigma)\delta' \\ &= \left(c\sigma - c[H] + \left[\frac{H^2}{G}\right]\right)\delta = 0, \end{aligned}$$

i.e.

$$c(\sigma - [H]) + \left[\frac{H^2}{G}\right] = 0. \tag{10}$$ j2

Solving (9) and (10) gives

$$c = \frac{w_r - w_l \pm |w_r/u_r - w_l/u_l|\sqrt{u_l u_r}}{u_r - u_l}.$$

Taking into the account the overcompressiveness condition

$$w_l/u_l \geq c \geq w_r/u_r,$$

one gets the following final result for the speed of the delta shock wave

$$c = \frac{w_r - w_l + (w_l/u_l - w_r/u_r)\sqrt{u_l u_r}}{u_r - u_l},$$

if $[G] \neq 0$, and otherwise

$$c = \frac{w_l + w_r}{2u_r}.$$

In the both cases

$$\sigma = (w_l/u_l - w_r/u_r)\sqrt{u_l u_r}. \tag{11}$$ sigma

This proves the theorem. □

Remark 3. (a) Let us note that the solution obtained in the theorem is associated to the distributions

$$\begin{aligned} U(x,t) &\approx \begin{cases} u_l, x < ct \\ u_r, x > ct \end{cases} + \left(\frac{w_l}{u_l} - \frac{w_r}{u_r}\right)\sqrt{u_l u_r}\, t\delta(x-ct), \\ W(x,t) &\approx \begin{cases} w_l, x < ct \\ w_r, x > ct \end{cases} + \left(\frac{w_l}{u_l} - \frac{w_r}{u_r}\right)\sqrt{u_l u_r}\, ct\delta(x-ct), \end{aligned} \tag{12}$$ newresenje

where

$$c = \frac{[GH] - [H]\sqrt{u_l u_r}}{[G]} \text{ or } c = \frac{w_l + w_r}{2u_r} \text{ if } [G] = 0.$$

(b) The same limit is obtained in [9] for (1) if one takes $w = uv$ with using singular shock wave solution. But comparing with that one, our solution does not have non-zero correction factors as that one.
(c) Since the value of v on the line $x = ct$ is determined to be c in [1], [4] or [15] the measure-theoretic product uv gives the same solution (12).

3. The numerical algorithm

The algorithm we use here is a modification of the algorithm introduced in [14]. Therefore, we will explain it briefly with a detailed explanation of the changes we made in order to get more efficiency and better resolution.

The solution procedure is based on two independent parts: a mesh redistribution algorithm and a solution algorithm. We shall first explain the solution algorithm.

Let $\{t_n\}$ denote the sequence of time steps with $\Delta t_n = t_{n+1} - t_n$. Assume that a fixed uniform mesh on the computational domain $[a, b]$ is given by

$$x = x(\xi),\ \xi_j = j/(J+1),\ 0 \le j \le J+1,$$

where $\xi \in [0, 1]$, and

$$x(0) = a \text{ and } x(1) = b.$$

The Godunov scheme (see [7]) assumes that the solution is piecewise constant on each subinterval $[x_j, x_{j+1}]$ and the discrete solution is taken as an average value of the actual solution along the lower cell boundary,

$$U_j^n = \frac{1}{\Delta x_j^n} \int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t)dx,$$

where $\Delta x_j^n = x_{j+1/2}^n - x_{j-1/2}^n$ presents the local spatial step. The method requires the solution of Riemann problems at every cell boundary in each time step. Doing so in practice can be very expensive, especially for nonlinear problems, as is the case with problem (1). Therefore, it is advisable to introduce the approximate Riemann solver [12], which is based on the linearized system

$$u_t + \widehat{A} \cdot u_x = 0, \tag{13}$$ ro

where $\widehat{A}$ is an $m \times m$ matrix with the following properties

1. $\widehat{A}(u_l, u_r)(u_r - u_l) = f(u_r) - f(u_l)$
2. $\widehat{A}(u_l, u_r)$ is diagonizable with real eigenvalues

3. $\widehat{A}(u_l, u_r) \longrightarrow f'(\overline{u})$ when $u_l, u_r \longrightarrow \overline{u}$.

Since $\widehat{A}$ is diagonizable with real eigenvalues, we can decompose

$$\widehat{A} = R\Lambda R^{-1},$$

where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_m)$ is a diagonal matrix of eigenvalues and $R = [r_1 \mid r_2 \mid .... \mid r_m]$ is the matrix of the appropriate eigenvectors. Let us introduce the following notation:

$$\begin{aligned} \lambda_p^+ &= \max(\lambda_p, 0),\ \Lambda^+ = \mathrm{diag}(\lambda_1^+, ..., \lambda_m^+), \\ \lambda_p^- &= \min(\lambda_p, 0),\ \Lambda^- = \mathrm{diag}(\lambda_1^-, ..., \lambda_m^-), \\ \widehat{A}^+ &= R\Lambda^+ R^{-1},\ \widehat{A}^- = R\Lambda^- R^{-1}. \end{aligned}$$

Now, for the linearized system (13) Godunov's method takes the form

$$U_j^{n+1} = U_j^n - \frac{\Delta t_n}{\Delta x_j}\left[\widehat{A}^-(U_{j+1}^n - U_j^n) + \widehat{A}^+(U_j^n - U_{j-1}^n)\right]. \tag{14}$$ godunov

Kakve gornja formula ima veze sa "A"?

Besides that, the scheme requires the time step to satisfy the Courant-Friedrichs-Levy stability condition ([8])

$$\nu = \max_{j,p}\left|\frac{\Delta t_n}{\Delta x_j}\lambda_p(U_j^n)\right| \leq 1. \tag{15}$$ cfl

Although, in practice a more restrictive condition $\nu \leq 0.9$ is used. It is also useful to mention that the Godunov scheme is implemented in the software package CLAWPACK ([6]) used by us.

We use the Roe solver for problem (3) to get a linearized system

$$u_t + \widehat{A}u_x = 0.$$

The matrix $\widehat{A}$ is of the form

$$\widehat{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{w_r w_l}{u_l u_r} & \dfrac{w_r}{u_r} + \dfrac{w_l}{u_l} \end{bmatrix}$$

with eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{w_r}{u_r} + \frac{w_l}{u_l} \pm \left|\frac{w_r}{u_r} - \frac{w_l}{u_l}\right|\right)$$

and the corresponding eigenvectors

$$r_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} \text{ and } r_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.$$

Since the solution to a Riemann problem of a linear hyperbolic system of PDE's is consisting of jumps of the form

$$[U] = \sum_p \alpha_p r_p,$$

(see [7]) we have

$$\begin{aligned}\begin{bmatrix} u_r - u_l \\ w_r - w_l \end{bmatrix} =& \alpha_1 r_1 + \alpha_2 r_2 \\ =& \alpha_1 \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}.\end{aligned} \tag{16}$$

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Since $\lambda_1 \neq \lambda_2$ relation (16) yields

$$\begin{aligned} u_r - u_l =& \alpha_1 + \alpha_2 \\ w_r - w_l =& \alpha_1\lambda_1 + \alpha_2\lambda_2, \end{aligned}$$

"A"

so we have

$$(\alpha_1, \alpha_2) = \left(\frac{w_r - w_l + \lambda_2(u_l - u_r)}{\lambda_1 - \lambda_2}, \frac{w_l - w_r + \lambda_1(u_r - u_l)}{\lambda_1 - \lambda_2} \right).$$

Let us now introduce another way we handled weakly hyperbolic system (3) without using a perturbation. Since in that case we have $\lambda_1 = \lambda_2$, it holds $r_1 = r_2$, and (16) gives

$$\begin{aligned} u_r - u_l =& \alpha_1 + \alpha_2 \\ w_r - w_l =& (\alpha_1 + \alpha_2)\lambda_1. \end{aligned}$$

One of possible solutions of the above system is

$$(\alpha_1, \alpha_2) = (0, u_r - u_l).$$

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Let us now explain the mesh redistribution algorithm.

The equidistribution principle (a detailed explanation can be found in [**?**]) is formulated as $Mx_\xi =$ constant or equivalently

$$(Mx_\xi)_\xi = 0 \tag{17}$$

2.8

for a monitor function $M(x, y) > 0$. Generally speaking, the monitor function is an appropriately chosen measure of numerical solution of the physical PDE. In order to solve the mesh redistribution equation (17), in [14] it is suggested to take an artificial time τ and solve

$$x_\tau = (Mx_\xi)_\xi, \ 0 < \xi < 1 \tag{18}$$

3.1

with boundary conditions $x(0, \tau) = a$ and $x(1, \tau) = b$. Discretizeeing (18) we get

$$\widetilde{x}_j = x_j + \frac{\Delta\tau}{\Delta\xi^2}[M_j(x_{j+1} - x_j) - M_{j-1}(x_j - x_{j-1})], \tag{19}$$

3.2

where $\Delta\xi = 1/(J+1)$. Solving (19) with boundary conditions $x_0 = a$ and $x_{J+1} = b$ leads to a new grid.

In [14] it is also suggested to use the following Gauss-Seidel type iteration to solve the mesh moving equation (17):

$$M_j^n(x_{j+1}^n - x_j^{n+1}) - M_{j-1}^n(x_j^{n+1} - x_{j-1}^{n+1}) = 0. \tag{20}$$

3.31

In the above mentioned paper it is demonstrated that the new mesh $\{x^{n+1}\}$ generated by (20) keeps the monotonic order of $\{x^n\}$.

In this paper, we will introduce an alternative approach. We will use a Newton-type iteration to solve (17):

$$M_j(x_{j+1}^{n+1} - x_j^{n+1}) - M_{j-1}(x_j^{n+1} - x_{j-1}^{n+1}) = 0. \tag{21}$$

4.22

Let us demonstrate that the new mesh $\{x^{n+1}\}$ generated by (20) keeps the monotonic order of $\{x^n\}$.

Lemma 4. *Assume $x_j^n > x_{j-1}^n$, for $1 \le j \le J$. If the new mesh $\{x^{n+1}\}$ is obtained by using Newton's iterative scheme (21), then $x_j^{n+1} > x_{j-1}^{n+1}$, for $1 \le j \le J$.*

Proof. From (21) we have

$$M_j x_{j+1}^{n+1} - (M_j + M_{j-1}) x_j^{n+1} + M_{j-1} x_{j-1}^{n+1} = 0,$$

which gives

$$-\alpha_j x_{j+1}^{n+1} + x_j^{n+1} - \beta_j x_{j-1}^{n+1} = 0, \tag{22}$$ 4.24

after dividing by $-(M_j + M_{j-1})$. Here

$$\alpha_j = \frac{M_j}{M_j + M_{j-1}} \text{ and } \beta_j = \frac{M_{j-1}}{M_j + M_{j-1}}.$$

Obviously, $\alpha_j, \beta_j > 0$. Since $\alpha_j + \beta_j = 1$, equation (22) yields

$$(\beta_j - 1) x_{j+1}^{n+1} + x_j^{n+1} \pm \beta_j x_j^{n+1} - \beta_j x_{j-1}^{n+1} = 0,$$

which implies

$$(x_j^{n+1} - x_{j+1}^{n+1}) - \beta_j (x_{j-1}^{n+1} - x_j^{n+1}) = \beta_j (x_j^{n+1} - x_{j+1}^{n+1}),$$

i.e.

$$(x_j^{n+1} - x_{j+1}^{n+1}) - \beta_j (x_{j-1}^{n+1} - x_j^{n+1}) = (1 - \alpha_j) \beta_j (x_j^{n+1} - x_{j+1}^{n+1}),$$

which gives

$$(x_j^{n+1} - x_{j+1}^{n+1}) - (1 - \alpha_j)(x_j^{n+1} - x_{j+1}^{n+1}) = \beta_j (x_{j-1}^{n+1} - x_j^{n+1}),$$

i.e.

$$\alpha_j (x_j^{n+1} - x_{j+1}^{n+1}) = \beta_j (x_{j-1}^{n+1} - x_j^{n+1}). \tag{23}$$ 4.25

Suppose

$$x_{j-1}^{n+1} > x_j^{n+1}, \text{ i.e. } x_{j-1}^{n+1} - x_j^{n+1} > 0 \tag{24}$$ 4.26

for some j, $1 < j < J$. Relations (23), (24) and positivity of α_j and β_j yields

$$x_j^{n+1} - x_{j+1}^{n+1} > 0, \text{ i.e. } x_j^{n+1} > x_{j+1}^{n+1}.$$

Continuing in such a way we get

$$a = x_0^{n+1} > ... > x_{j-1}^{n+1} > x_j^{n+1} > x_{j+1}^{n+1} > ... > x_J^{n+1} = b,$$

which is impossible. Therefore, $x_j^{n+1} < x_{j+1}^{n+1}$ for all j, $1 \le j \le J$. □

Let us make some discussion about the monitor function M now. If M is the arc-length function, i.e.

$$M = \sqrt{1 + |u_x|^2},$$

then the corresponding centered finite difference approximation is given by

$$M_j = \sqrt{1 + \left| \frac{\overline{U}_{j+1} + \overline{U}_j}{x_{j+1} - x_j} \right|},$$

where

$$\overline{U}_j = (U_{j+1} \Delta x_j + U_j \Delta x_{j+1}) / (\Delta x_{j+1} + \Delta x_j).$$

As M is largest where the solution changes most rapidly, the spatial points concentrate in regions with large gradient changes. In order to avoid local oscillation of non-smoothness due to large gradient changes, it is useful to replace the mesh

function with a regularized version $\widetilde{M}_i$. The regularized function we use in this paper is suggested in [14] and is given by

$$\widetilde{M}_j \approx \frac{1}{4}(M_{j+1} + 2M_j + M_{j-1}). \tag{25}$$ mtilda

Using (21) and (25) we get

$$\widetilde{M}_j^{n,m} x_{j+1}^{n,m+1} - (\widetilde{M}_j^{n,m} + \widetilde{M}_{j-1}^{n,m}) x_j^{n,m+1} + \widetilde{M}_{j-1}^{n,m} x_{j-1}^{n,m+1} = 0. \tag{26}$$ jednac_mreze

To balance the number of points inside a steep internal layer, we use a regularizing factor α in the following manner:

$$M = \sqrt{1 + \frac{1}{\alpha}|u_x|^2},$$

where $\alpha > 1$. The factor α allows us to reduce the magnitude of the monitor function in situations where $|u_x|$ is very large, thereby avoiding over-resolution of steep layers, while also ensuring that M still retains a significant peak near these discontinuities. Different approaches in scaling α, based on the maximum solution value, maximum derivative value or the average value of the derivative over the spatial domain, suggested in [3], [7] and [13] respectively, have been successful with linearized mesh equation, but have no sense in a nonlinear case. Therefore, in [14] the regularizing factor is suggested to be taken for free. However, in the region where the monitor function has high magnitude, there is a significant number of points, so Δx_j goes to zero. Thus, in some time step, while moving the mesh from $\{x_j^{n,m}\}$ to $\{x_j^{n,m+1}\}$ the CFL number (15) can go out of range (i.e. $\nu > 0.9$). So one has to interrupt the moving mesh procedure by taking the previous mesh $\{x_j^{n,m}\}$, although $\|x_j^{n,m} - x_j^{n,m-1}\| > \varepsilon$. In order to avoid such interrupting of the numerical procedure in the case $\nu > 0.9$, we suggest increasing the regularizing factor with some fixed amount and performing the current time step again.

Since the shock travel within spatial domain with time it is necessary to generate mesh that is also moving within spatial domain. Otherwise we would not be able to follow the solution for longer time intervals. This mesh adjustment is done using the following procedure: The current spatial domain is divided into two parts according to the position of the maximum of the numerical solution. If the interval on the left side of the maximum is longer than the right one, the first point from the left interval is cut and added to the end of the other interval. The procedure is to be repeated until the two intervals are of equal length.

Using the algorithm proposed in [14] with the modifications we explained above we get the following numerical procedure.

Algorithm.

Step 1: Given an initial solution U^0 at time $t = t^0$, equidistribute the mesh exactly using a discretization of the exact equidistribution principle $(Mx)_\xi = 0$. Given an initial value α^*, set $\alpha = \alpha^*$.

$$\Downarrow$$

Step 2: Increase the time level to $t = t^{n+1}$ and take a guess at the new mesh positions using $\left\{x_j^{n+1,0}\right\} = \{x_j^n\}$ and move grid from $\left\{x_j^{n+1,m}\right\}$ to $\left\{x_j^{n+1,m+1}\right\}$ using (26) and compute $\left\{U_j^{n+1,m+1}\right\}$on the new grid based on the Godunov scheme

(14) with $\nu \leq 0.9$. If $\nu > 0.9$, go set $\alpha := \alpha + 10$ and go to the beginning of Step 2. Repeat the updating procedure until $\left\| x^{n+1,m+1} - x^{n+1,m} \right\| \leq \varepsilon$.

$$\Downarrow$$

Step 3: Compute $\left\{U_j^{n+1}\right\}$ on the new mesh $\left\{x_j^{n+1}\right\}$obtained in the pervious step to get the solution approximations at time level t_{n+1}.

$$\Downarrow$$

Step 4: If $t_{n+1} \leq T$, go to step 2.

4. Application to the solutions with singular shock

4.1. **Pressureless system.** Denote with u_s and w_s the singular parts of the delta shock wave (5), i.e.

$$\begin{aligned} u_s(x,t) =& s_1(t)D(x-ct) \\ w_s(x,t) =& s_2(t)D(x-ct), \end{aligned}$$

and set

$$Q(t) := \int U_s(x,t)dx \text{ and } P(t) := \int W_s(x,t)dx, \ t > 0.$$

Clearly, Q and P represent the surfaces above the non-constant parts of the solution components. The definition of delta function imply $\int D dx \approx 1$, so $Q \approx s_1(t)$. By using (11) and (12) one gets

$$Q \approx \sigma t \approx \left(\frac{w_l}{u_l} - \frac{w_r}{u_r}\right)\sqrt{u_l u_r}t \tag{27}$$ 5.39

$$P \approx c\sigma t \approx c\left(\frac{w_l}{u_l} - \frac{w_r}{u_r}\right)\sqrt{u_l u_r}t. \tag{28}$$ 5.40

From (27) and (28) it follows that both P and Q are linearly time dependent, so their ratio is constant, i.e. $P/Q = c$.

4.2. **Perturbation by a hyperbolic system.** Consider now the isentropic (p-system) gas dynamics system

$$\begin{aligned} u_t + (uv)_x &= 0 \\ (uv)_t + (uv^2 + \mu p(u))_x &= 0 \end{aligned}$$

with the initial data

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}, v(x,0) = \begin{cases} v_l, & x < 0 \\ v_r, & x > 0 \end{cases},$$

where $p(u) = \mu u^\gamma$, $\gamma \in (1,3)$. After a renormalization, one can take $\mu = (\gamma - 1)^2/(4\gamma)$ and when $\mu \to 0$, then $\gamma \to 1$. In numerical tests we shell do both cases: when γ is a constant and when it tends to 1.

Since we are doing the case when the vacuum state does not appear, it is possible to look at the system after a change of variables $uv \mapsto w$,

$$\begin{aligned} u_t + (w)_x &= 0 \\ w_t + (w^2/u + \mu p(u))_x &= 0 \end{aligned}$$

and the initial data are

$$u(x,0) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}, w(x,0) = \begin{cases} w_l = u_l v_l, x < 0 \\ w_r = u_r v_r, x > 0 \end{cases},$$

where $w_l/u_l > w_r/u_r$.

The isentropic system is strictly hyperbolic with both of the fields being genuinely nonlinear. The shock curves are given by

$$S_i:\ w_r - w_l = \frac{w_l}{u_l}(u_r - u_l) + (-1)^i\sqrt{\frac{u_r}{u_l}\frac{\mu u_r^\gamma - \mu u_l^\gamma}{u_r - u_l}}(u_r - u_l)$$
$$(-1)^i(u_r - u_l) < 0,\ u_l, u_r > 0.$$

In [1], the authors proved that for each pair (u_l, w_l), (u_r, w_r) such that $w_l/u_l > w_r/u_r$, solution consists from two shock waves, and this solution tends to a delta shock wave as $\mu \to 0$. The obtained delta shock wave in the limit is the same as the one solving pressureless system (when $\mu = 0$). With the same arguments as in that article, one can prove that this stays true for renormalized γ. These facts are verified numerically here after pressureless system.

5. Numerical Results

Let us now consider the system (3) with the initial data (2). Since the initial conditions are discontinuous, the selection of an appropriate initial mesh is of particular importance. In order to allow mesh points to concentrate on or near initial discontinuities, the data must be smoothed over some finite width. We therefore replace (2) with a smoothed function of the form

$$\widetilde{U}(x) = U_l + \frac{1}{2}(U_r - U_l)\left(1 + \tanh\left(\frac{x}{\varepsilon}\right)\right),$$

where $U_l = (u_l, w_l)$, $U_r = (u_r, w_r)$ and $\varepsilon = 0.005$ as the smoothing width.

The description of parameters used in our examples can be found in Table 1.

Parameter	Description
t	time
$[x_1, x_2]$	spatial domain
J	number of mesh points
α^*	initial value of the regularizing factor
α	final value of the regularizing factor obtained by the program
μ	perturbation coefficient tending to zero
γ	$\in (1, 3)$ is fixed or depending on μ

Table 1. The description of parameters used in our examples

We use the following data for numerical examples.

$$U_l = (1, 0.2),\ U_r = (1.2, 0.2),\ \frac{x_2 - x_1}{J} = \frac{1}{20},\ \alpha^* = 10.$$

We compare the results obtained without and with perturbation by the isentropic system. In the later case we take $\mu \in \{0.01, 0.001, 0.0001\}$, $\gamma = 1 = 2\mu + 2\sqrt{\mu + \mu^2}$ and $\gamma = \frac{5}{3}$. Theorem 1 gives the predicted speed $c = 0.18257$ and mass quotient $P/Q = 0.18257$. Also one could check whether both of P and Q are linearly time dependent or not.

The results are the summarized in the following tables.

$\mu = 0$	P	Q	P/Q
$t_1 = 12.1435$	0.07693	0.42481	0.18110
$t_2 = 24.0287$	0.15087	0.83043	0.18168
$t_3 = 36.0157$	0.22886	1.25012	0.18307
$t_4 = 48.1054$	0.30914	1.68753	0.18319
$t_5 = 60.136$	0.38428	2.10093	0.18291

TABLE 2. System without perturbation

$\mu = 0.01, \gamma = \frac{5}{3}$	P	Q	P/Q	c_l	c_r
$t_1 = 6.1106$	0.01911	0.14452	0.13209	0.04909	0.29457
$t_2 = 12.0250$	0.03751	0.23418	0.16017	0.04158	0.29106
$t_3 = 18.0682$	0.05633	0.31383	0.17948	0.04981	0.30440
$t_4 = 24.1149$	0.07515	0.39349	0.19099	0.05805	0.31101
$t_5 = 30.0423$	0.09361	0.49316	0.18982	0.05325	0.31289

TABLE 3. Fixed γ, $\mu = 0.01$

$\mu = 0.001, \gamma = \frac{5}{3}$	P	Q	P/Q	c_l	c_r
$t_1 = 20.0015$	0.12631	0.71752	0.17603	0.14000	0.2200
$t_2 = 40.0338$	0.25264	1.37278	0.18403	0.14488	0.22231
$t_3 = 60.0916$	0.37906	2.02281	0.18690	0.14145	0.2230
$t_4 = 80.1589$	0.50549	2.71331	0.18630	0.14346	0.22081
$t_5 = 100.0650$	0.63087	3.41852	0.18455	0.14490	0.21986

TABLE 4. Fixed γ, $\mu = 0.001$

$\mu = 0.0001, \gamma = \frac{5}{3}$	P	Q	P/Q	c_l	c_r
$t_1 = 60.1454$	0.39319	2.14957	0.18292	0.17458	0.19785
$t_2 = 120.021$	0.78106	4.24315	0.18408	0.17663	0.19580
$t_3 = 180.134$	1.16688	6.36874	0.18322	0.17598	0.19707
$t_4 = 240.068$	1.55000	8.51507	0.18199	0.17745	0.19619
$t_5 = 300.183$	1.93262	10.6857	0.18086	0.17656	0.19655

TABLE 5. Fixed γ, $\mu = 0.0001$

$\mu = 0.01, \gamma = 1.221$	P	Q	P/Q	c_l	c_r
$t_1 = 6.0973$	0.02554	0.19425	0.13146	0.07380	0.27881
$t_2 = 12.0013$	0.05016	0.31376	0.15987	0.07499	0.27070
$t_3 = 18.0444$	0.07454	0.43330	0.17394	0.07759	0.27709
$t_4 = 24.0924$	0.10059	0.54286	0.18530	0.07471	0.27394
$t_5 = 30.0142$	0.12529	0.65243	0.19203	0.07663	0.27987

TABLE 6. Variable γ, $\mu = 0.01$

$\mu = 0.001, \gamma = 1.06553$	P	Q	P/Q	c_l	c_r
$t_1 = 20.0051$	0.12887	0.73363	0.17567	0.15496	0.20495
$t_2 = 40.0855$	0.25841	1.43772	0.17973	0.15467	0.20955
$t_3 = 60.0263$	0.38695	2.12212	0.18234	0.15493	0.20658
$t_4 = 80.1492$	0.51660	2.80646	0.18407	0.15721	0.20587
$t_5 = 100.103$	0.64511	3.49079	0.18480	0.15884	0.20878

TABLE 7. Variable γ, $\mu = 0.001$

$\mu = 0.0001, \gamma = 1.0202$	P	Q	P/Q	c_l	c_r
$t_1 = 100.148$	0.64534	3.53227	0.18270	0.18073	0.19172
$t_2 = 200.071$	1.25455	6.83558	0.18353	0.18093	0.19143
$t_3 = 300.024$	1.84601	10.0449	0.18378	0.18165	0.19165
$t_4 = 400.021$	2.42775	13.2112	0.18376	0.18174	0.19124
$t_5 = 500.026$	3.00301	16.3527	0.18364	0.18179	0.19099

TABLE 8. Variable γ, $\mu = 0.0001$

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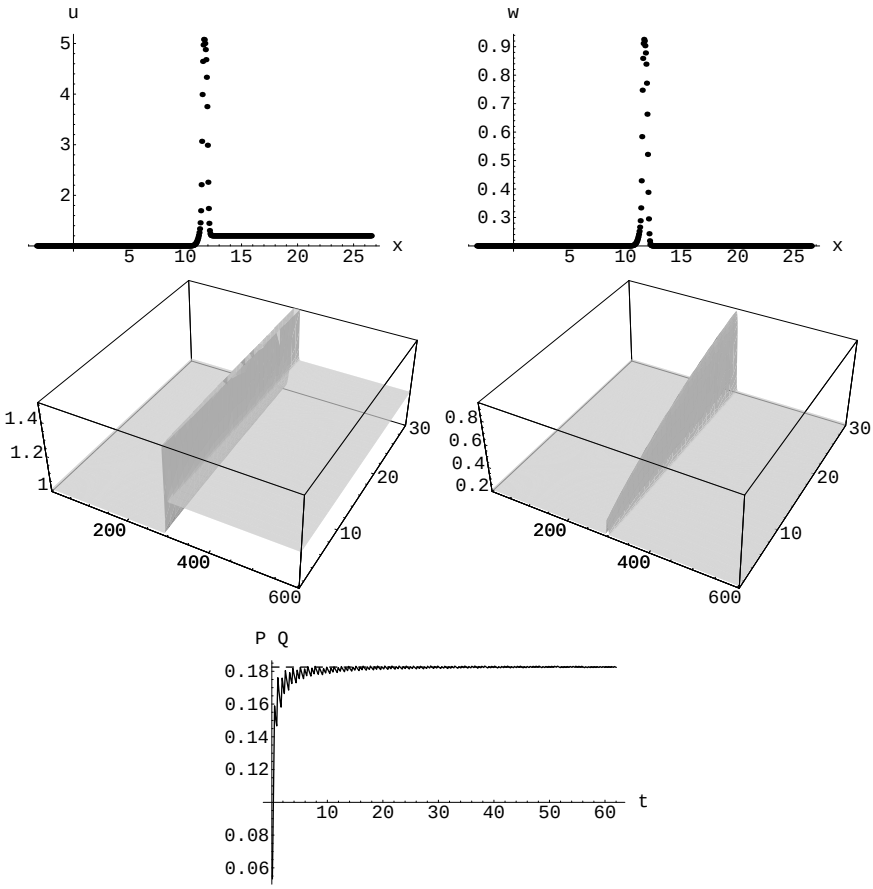

FIGURE 1. The system without perturbation

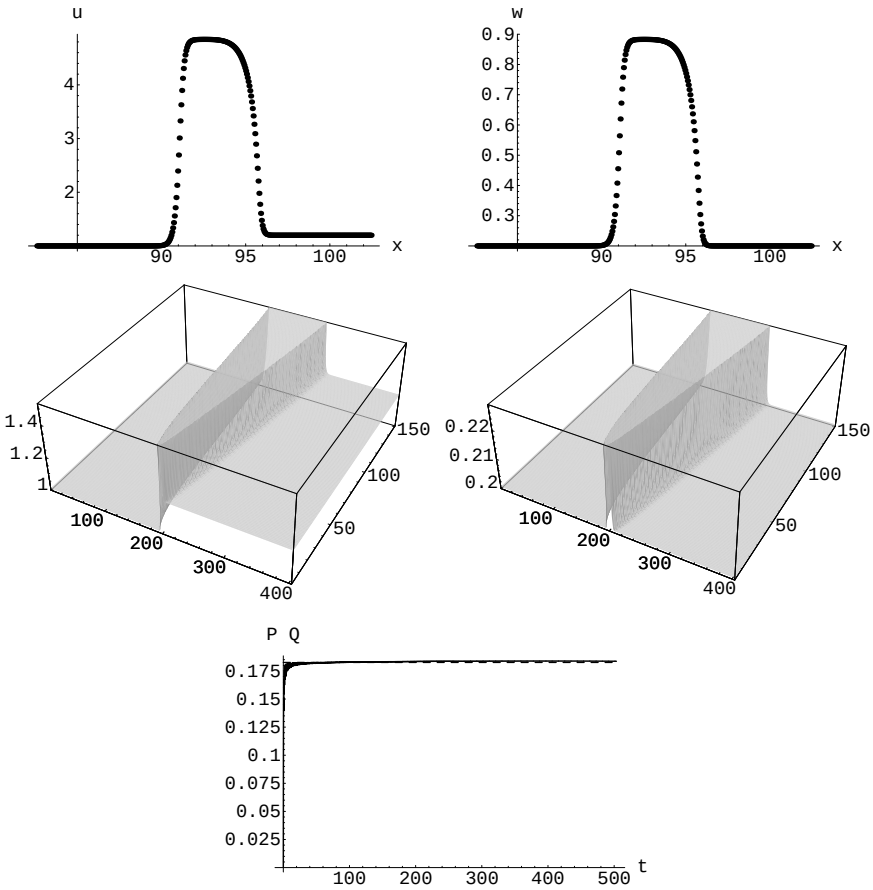

FIGURE 2. Example of a perturbation, variable γ, $\mu = 0.0001$

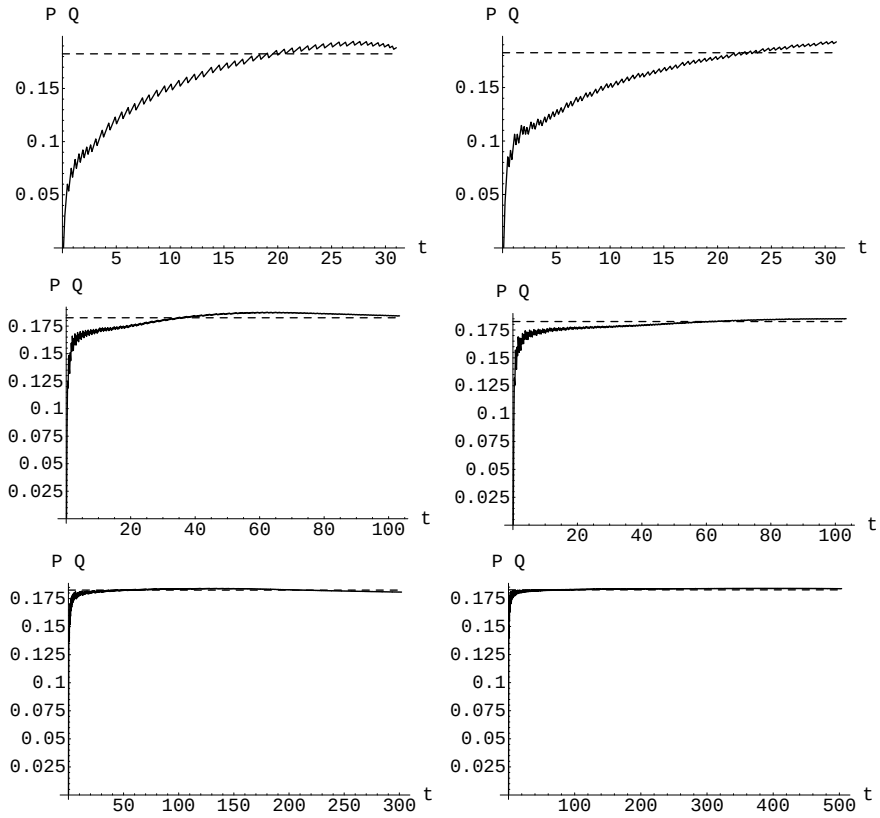

FIGURE 3. Mass quotients: left column – fixed γ, right column – variable γ, mu decreasing from above