Dynamics of a fractional derivative type of a viscoelastic rod with random excitation

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Abstract

The axial vibrations of a viscoelastic rod with a body attached to its end are investigated. The problem is modeled by the constitutive equations with fractional derivatives as well as with the perturbations involving a bounded noise and a white noise process. Weak solutions for the equations given below in two cases of constitutive equations as well as their stochastic moments are determined.

1 Introduction

In many engineering application one is faced with axially loaded viscoelastic rods. Often, the load that is applied at the one end of the rod has two components: the deterministic one and the stochastic one that we call noise. The deterministic part may come from the own weight of the system (for example the weight of the bridge that is supported by a rod) while the stochastic part may come form additional load (wind, or traffic over the bridge, for example). The stochastic component coming for the traffic is bounded, since no matter for how long we wait, there will be no load of arbitrary large intensity. In this paper we want to model axially loaded viscoelastic rod subjected to both unbounded and bounded stochastic noise.

Thus consider a viscoelastic rod, fixed at one end and loaded by a force of intensity F at its free end. Suppose that the density of the rod is small so that the inertial forces are negligible with respect to the force F. The equations describing the motion of the rod free end, in dimensionless form, reads (see [7] for details and for the case when the inertial forces are not neglected)

$$\int_0^1 \phi_\sigma(\gamma)\,{}_0\mathrm{D}_t^\gamma \sigma(t)\,\mathrm{d}\gamma = \int_0^1 \phi_\varepsilon(\gamma)\,{}_0\mathrm{D}_t^\gamma y(t)\,\mathrm{d}\gamma, \tag{1}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)+\sigma(t)=F(t),\ t>0, \tag{2}$$

$$\sigma(0)=0,\ y(0)=y_0,\ \frac{\mathrm{d}}{\mathrm{d}t}y(0)=v_0. \tag{3}$$

The equations above are considered in a weak (distributional) sense; F contains both deterministic and stochastic perturbations, while the constitutive equation (1), representing mechanical model, is considered in the following two cases:

Case 1. $\phi_\sigma(\gamma)=\delta(\gamma),\ \phi_\varepsilon(\gamma)=\omega^2\delta(\gamma)+\lambda b^\gamma,\ \gamma\in[0,1],\ b>0,\ \lambda>0,$
Case 2. $\phi_\sigma(\gamma)=a^\gamma$, $\phi_\varepsilon(\gamma)=\lambda b^\gamma$, $\gamma\in[0,1]$, $b\geq a>0$, $\lambda>0$.

The forcing term F has the form $F(t)=f_0(t)+\xi(t)+B\dot{W}(t),\ t\geq 0$. The function f_0 is a deterministic part which is a locally integrable function on $\mathbb{R}$, such that $f_0(t)=0$, $t<0$, and it is of exponential growth for positive argument: $|f_0(t)|\le Ce^{kt}$, $t>0$, for some $k>0$, $C>0$. Further, ξ represents so-called bounded noise (namely, the cosine function of the Wiener process), B is a constant and $\dot{W}$ is the white noise process. Note that $\dot{W}$ involves shock-type perturbations, while ξ involves bounded perturbations. We assume that the bounded noise and the white noise are independent stochastic processes.

As in [10], we consider a bounded noise $\xi(t)$ in the form

$$\xi(t)=AZ(t)=A\cos\left(\omega t+\eta\tilde{W}(t)+\theta\right),\ t\ge 0, \quad (4)$$

originated from the system

$$\begin{aligned}\xi(t)&=A\cos Z(t),\\ dZ(t)&=\omega dt+\eta d\tilde{W}(t),\ Z(0)=\theta,\end{aligned}$$

where A is the amplitude of the noise, constant η is the noise intensity, constant ω is the frequency, $\tilde{W}$ is the standard Wiener process and the phase angle θ is a random variable, independent of the history of the Wiener process $\tilde{W}$, uniformly distributed in the interval $[0,2\pi]$. The bounded noise ξ is a stationary process (see [10]). Therefore, it has some convenient properties concerning stochastic moments.

Our intention in this paper is to investigate axial random forces in the right hand side of (2). In the special case when the rod is elastic, i.e. $\phi_\sigma=\phi_\varepsilon$, system (1)-(3) reduces to a problem of forced vibrations of a light elastic rod with the body attached to the free end.

Initial conditions (3) specify that the system rod-body is in the rest in the initial time instant. Actually, in this paper we will consider the initial conditions $y(0)=0$ and $\frac{d}{dt}y(0)=0$.

We will find the solution to (1)-(3) so that (2) is satisfied in a weak sense, that is, in the sense of distributions. Let us explain this. Following Schwartz (see [16], [17] or any other book with the theory of distributions), we denote by $\mathcal{D}=\mathcal{D}(\mathbb{R})$ the space of smooth functions φ on $\mathbb{R}$ so that supp φ (the closure of the set where $\varphi\ne 0$) is a compact set. We supply $\mathcal{D}$ with the usual convergence structure and denote by $\mathcal{D}'=\mathcal{D}'(\mathbb{R})$ its dual, the space of continuous linear functionals on $\mathcal{D}$. The solutions to (1)-(3) which will be determined, are continuous functions and their first derivatives are supported by $[0,\infty)$, i.e. their supports are contained in the interval $[0,\infty)$. Their second derivatives satisfy (2) in the weak sense. More precisely, y and σ are weak solutions to (2) if, for every $\phi\in\mathcal{D}$,

$$\begin{aligned}&\left\langle \frac{d^2}{dt^2}y(t)+\sigma(t),\phi(t)\right\rangle\\ &=\int_0^t (y(t)+\sigma(t))\phi''(t)dt=\langle F(t),\phi(t)\rangle.\end{aligned}$$

So, in the sequel equation (2) will be always considered in the weak sense and functions are considered as distributions equal zero in $(-\infty,0)$.

The case with the general initial data y_0 and v_0 can be always written in the form with the zero initial data. It can be done by the change of variables $y(t)=y_0H(t)+v_0t_++Y(t)$, where H is the Heaviside's function and $t_+=tH$, $t\in\mathbb{R}$. Indeed, substituting $y''=y_0\delta'+v_0\delta+Y'$ and ${}_0D_t^\gamma y={}_0D_t^\gamma Y+{}_0D_t^\gamma H+{}_0D_t^\gamma t_+$ into (1) and (2), and calculating ${}_0D_t^\gamma H$ and ${}_0D_t^\gamma t_+$ one obtains

$$\begin{aligned}&\int_0^1 \phi_\sigma(\gamma)\,{}_0\mathrm{D}_t^\gamma\sigma(t)\,\mathrm{d}\gamma\\ &=\int_0^1 \phi_\varepsilon(\gamma)\,{}_0\mathrm{D}_t^\gamma Y(t)\,\mathrm{d}\gamma,\\ &\frac{\mathrm{d}^2}{\mathrm{d}t^2}Y(t)+\sigma(t)=\tilde{F}(t),\ t>0,\\ &\sigma(0)=0,\ Y(0)=0,\ \frac{\mathrm{d}}{\mathrm{d}t}Y(0)=0,\end{aligned}$$

where $\tilde{F}=-y_0\delta'-v_0\delta-c_1t^{1-\gamma}-c_2t^{2-\gamma}+F$, for some constants c_1 and c_2. Thus, one can take the zero initial data without loosing on generality.

Stochastic vibration of a similar system has been studied in [12], where both F and material density ρ, are assumed to have deterministic and stochastic parts. We shall treat a viscoelastic rod instead of elastic one and we shall assume that $\rho=0$.

Due to the fact that F involves the white noise process, we find the solutions as elements of $\mathcal{S}'_+\left(\mathbb{R},L^2(dW)\right)$. That means the following. If we denote by dW the white noise measure over $(\Omega,\mathcal{B}_\Omega),\Omega=\mathcal{S}'(\mathbb{R})$, $\mathcal{B}_\Omega$ is the Borel algebra generated by the weak topology of the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R})$, $L^2(dW)$ is the white noise space of L^2-functions over Ω with respect to the measure dW and $\mathcal{S}'_+\left(\mathbb{R},L^2(dW)\right)$ denotes the space of continuous liner functionals T, $\mathcal{S}(\mathbb{R})\ni\theta\mapsto T(\theta)\in L^2(dW)$ supported by $[0,\infty)$, that is of vector-valued tempered distributions T with by supp $T\subset[0,\infty)$. The action of $T\in\mathcal{S}'_+\left(\mathbb{R},L^2(dW)\right)$ is given by

$$\langle T,\varphi\rangle=T(\varphi)\in L^2(dW),\ \varphi\in\mathcal{S}(\mathbb{R}),$$

so that $T(\varphi)=0$, for φ equals zero on $[0,\infty)$.

The expectation of the solution to (1)-(3) is the classical solution of the stochastically unperturbed system ($A=B=0$) under the assumption $f_0\in L^1_{loc}(\mathbb{R})$, supp$f\subset[0,\infty)$.

The paper is organized as follows. Our model is described in the continuation of this introduction. Section 2 deals with the fundamental solution to (1)-(3), that is with F equals Dirac's delta distribution δ. We separately analyze two cases of the constitutive equations. Section 3 is devoted to the solution of the quoted system with the stochastic forcing term F, while the stochastic moments of solutions are given in Section 4. The concrete numerical example is presented in Section 5. The proofs of the main results are provided in the appendix.

1.1 Model

Consider a viscoelastic rod of length L in the natural (undeformed) state. Suppose that one end of the rod is fixed. Let m be the mass of a body attached to the free end of the rod. At the initial time moment $t=0$, as well as during the motion, the $\bar{x}$ axis coincides with the axis of the rod. Let $x\in[0,L]$ denote a position of a material point of a rod at the initial time $t=0$. The position of this point at the time $t>0$ is $x+u(x,t)$, where u denotes displacement of the point (see [8]). We consider the motion of attached body, in the case of light rod (the mass of the rod is much smaller than m). The equations that describe this motion are (1)-(3).

Recall that σ denotes the (dimensionless) stress in the rod, $y(t)=u(L,t)$, $t>0$, is the displacement of the free end of the rod, i.e., the displacement of the attached body, F is the force applied to the body and ϕ_σ and ϕ_ε are constitutive functions, or distributions describing the properties of a rod. For the derivation of (1)-(3) see [2]. Note that we use ${}_0\mathrm{D}_t^\gamma$ to denote the left Riemann-Liouville fractional derivative operator of order $\gamma\in(0,1)$:

$${}_0\mathrm{D}_t^\gamma y(t):=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^{-\gamma}}{\Gamma(1-\gamma)}*y(t)\right),\ t>0.$$

Γ is the Euler gamma function and $*$ is the convolution. In the case of $f,g\in L^1_{loc}$, supp$f,g\subset[0,\infty)$, it is defined by

$$(f*g)(t):=\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau,\ t>0,$$

We refer to [11, 13, 15] for some basic definitions and properties of the fractional calculus.

The choice of the constitutive functions or Schwartz distributions ϕ_σ and ϕ_ε is not completely arbitrary. Namely, it must be checked whether ϕ_σ and ϕ_ε satisfy the restrictions following from the Second Law of Thermodynamics. The restrictions are obtained in [3, 4, 6]. We refer to [6] for a systematic review of restrictions on parameters if ϕ_σ and ϕ_ε.

2 Fundamental solution of the system (1)-(3)

We investigate the system (1)-(3) in both cases, Case 1 and Case 2, given in the Introduction.

2.1 Case 1

Here we consider the system (1)-(3) with the constitutive functions $\phi_\sigma(\gamma)=\delta(\gamma)$, $\phi_\varepsilon(\gamma)=\omega^2\delta(\gamma)+\lambda b^\gamma$, $\gamma\in[0,1]$, $b>0$, $\lambda>0$, and with the initial data $y_0=0$, $v_0=0$. One obtains

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)+\omega^2 y(t)+\lambda\int_0^1 b^\gamma {}_0D_t^\gamma y(t)d\gamma=F(t) \quad (5)$$
$$y(0)=0,\ \frac{\mathrm{d}}{\mathrm{d}t}y(0)=0.$$

Theorem 1 *Under the assumptions listed above there exists the fundamental solution P to (5), i.e. the solution to (5) with $F=\delta$. For $t>0$, P is given by*

$$\begin{aligned}P(t)=&\\ &2e^{-\alpha t}\cos(\beta t)Re\left(\frac{1}{2s_1+\frac{b}{\ln(bs_1)}-\frac{bs_1-1}{s_1\ln^2(bs_1)}}\right)\\ &+2e^{-\alpha t}\cos(\beta t)Im\left(\frac{1}{2s_1+\frac{b}{\ln(bs_1)}-\frac{bs_1-1}{s_1\ln^2(bs_1)}}\right)\\ &+\int_0^\infty\frac{\lambda(b\xi+1)e^{-\xi t}\ d\xi}{\big((\xi^2+\omega^2)\ln(b\xi)-\lambda(b\xi+1)\big)^2+(\xi^2+\omega^2)^2\pi^2},\end{aligned} \quad (6)$$

and $P(t)=0$ for $t\leq 0$. Here $s_{1,2}=-\alpha\pm\beta i$ are the only zeros of the function $g_1(s):=s^2+\omega^2+\lambda\frac{bs-1}{\ln(bs)}$.

Remark 1 *Note that P is a continuous function in $[0,\infty)$ as well as its first derivative. But, the second (classical) derivative of P does not exist at $t=0$.*

2.2 Case 2

Now we investigate the relevant system (1)-(3) with a different choice of constitutive functions $\phi_\sigma(\gamma)=a^\gamma$, $\phi_\varepsilon(\gamma)=\lambda b^\gamma$, $\gamma\in[0,1]$, $b\geq a>0$, $\lambda>0$. Again, we consider the case with zero initial data. Thus, we consider

$$\begin{aligned}&\int_0^1 a^\gamma {}_0D_t^\gamma\sigma(t)\ d\gamma\\ &=\lambda\int_0^1 b^\gamma {}_0D_t^\gamma y(t)\ d\gamma,\\ &\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)+\sigma(t)=F(t),\ t>0,\\ &\sigma(0)=0,\ y(0)=0,\ \frac{\mathrm{d}}{\mathrm{d}t}y(0)=0.\end{aligned} \quad (7)$$

Theorem 2 *Let P and Q be given as follows. First, $P(t)=0$, for $t<0$, and*

$$\begin{aligned}P(t)=&2e^{-\alpha t}\big(\cos(\beta t)Re(A_1)+\cos(\beta t)Im(A_1)\big)\\ &-\lambda\beta\ln(a/b)\cdot\int_0^\infty\frac{(b\xi+1)(a\xi+1)e^{-\xi t}}{A_2}d\xi,\end{aligned} \quad (8)$$

for $t\geq 0$, where

$$\begin{aligned}A_1:=&\frac{1}{2s_1+\lambda\beta\left(\frac{\ln(b/a)}{\ln^2(bs_1)}\frac{bs_1-1}{as_1-1}-\frac{\ln(bs_1)}{\ln(as_1)}\frac{a-b}{(as_1-1)^2}\right)}\\ A_2:=&\big((\xi^2+\omega^2)(a\xi+1)\ln(b\xi)+\lambda(b\xi+1)\ln(a\xi)\big)^2\\ &+\big((\xi^2+\omega^2)(a\xi+1)+\lambda\beta(b\xi+1)\big)^2\pi^2\end{aligned}$$

Now, $s_{1,2}=-\alpha\pm\beta i$ are the only zeroes of the function $g_1(s)=s^2+\omega^2+\lambda\beta\frac{\ln(as)}{\ln(bs)}\cdot\frac{bs-1}{as-1}$.

Next, $Q(t)=0$, for $t<0$, and

$$\begin{aligned}&Q(t)=(\phi_1*\phi_2)(t),\ \text{for } t\geq 0,\\ &\phi_1(t)=\delta(t)+\ln\frac{a}{b}\int_0^\infty\frac{t^{u-1}b^{-u}a}{\Gamma(u)}du,\\ &\phi_2(t)=\left(\frac{a}{b}\right)^2\delta(t)+\frac{b^2-a^2}{a^4}\sinh\frac{t}{a}.\end{aligned} \quad (9)$$

Then P and Q determine the fundamental solution to (7):

$$\sigma=\lambda P*Q,\ \frac{d^2}{dt^2}P+\lambda P*Q=\delta,$$

P and Q determine the fundamental solution to (7).

Remark 2 *Note that P and Q have continuous second derivatives in $[0,\infty)$.*

3 Equations with forcing term F

Now we solve (1)-(3) with $F = f_0 + \xi + B\,\dot{W}$, where f_0 is the deterministic part of the force F, ξ is the bounded noise satisfying (4), B is a constant and $\dot{W}$ is the white noise process. If $B=0$ then rare but arbitrarily large perturbations are excluded and only small perturbations, represented by the bounded noise ξ, are considered. As mentioned in the introduction, the bounded noise process and the white noise process are assumed to be independent. Also, f_0 has the form as in the introduction.

3.1 Case 1

Equation (5) now reads

$$\begin{aligned}
&\frac{d^2}{dt^2}y(t)+\omega^2 y(t)+\lambda\int_0^1 b^\gamma{}_0D_t^\gamma y(t)d\gamma\\
&=f_0(t)+A\cos Z(t)+B\,\dot{W}(t),\\
&\frac{dZ}{dt}=\omega+\eta\dot{\tilde{W}}(t),\ t>0,\\
&y(0)=0,\ \frac{dy}{dt}(0)=0,\ Z(0)=\theta,
\end{aligned}\tag{10}$$

where random variable θ is uniformly distributed in the interval $[0,2\pi]$ and it is independent of the Wiener process $\tilde{W}$. Recall, the amplitude of the bounded noise is denoted by A, η denotes the noise intensity and ω is the frequency around which the bounded noise appears. As we already mention, Z can be immediately written in an explicit form (4), $Z(t)=\cos(\omega t+\eta\tilde{W}(t)+\theta)$.

Theorem 3 *Under the assumptions given above, there exists the weak solution to the system (10) and it is given by $y(t)=0$, for $t<0$, and*

$$\begin{aligned}
y(t)&=(f_0+A\cos Z(t)+B\,\dot{W}(t))*P(t)\\
&=\int_0^t P(t-s)f_0(s)ds\\
&\quad+A\int_0^t P(t-s)\cos Z(s)ds\\
&\quad+B\int_0^t P(t-s)dW(t),\ t\geq 0,
\end{aligned}\tag{11}$$

with P is given by (6).

Remark 3 *The simpler form of problem (1)-(3) with the same bounded noise in perturbation of F and with no white noise perturbation ($B=0$) was studied in [10] but in the case when $\phi_\sigma=\delta(\gamma)$, $\phi_\varepsilon=\omega[\delta(\gamma)+\varepsilon\delta(\gamma-\mu)]+2\varepsilon\beta\delta(1-\gamma)$, $\gamma\in[0,1]$, $\varepsilon=$const, $\beta=$const. Another special case was treated in [9] where $\phi_\sigma=\delta(\gamma)$, $\phi_\varepsilon=\omega^2\delta(\gamma)+\varepsilon\delta(\gamma-\mu)$ and $A=0$, $B\neq 0$.*

3.2 Case 2

Equation (7) now reads

$$\begin{aligned}
&\int_0^1 a^\gamma{}_0D_t^\gamma\sigma(t)\,d\gamma=\lambda\int_0^1 b^\gamma{}_0D_t^\gamma y(t)\,d\gamma,\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}y(t)+\sigma(t)=f_0(t)+A\cos Z(t)+B\,\dot{W}(t),\\
&\frac{dZ}{dt}=\omega+\eta\dot{\tilde{W}}(t),\ t>0,\\
&\sigma(0)=0,\ y(0)=0,\ \frac{\mathrm{d}}{\mathrm{d}t}y(0)=0,\ Z(0)=\theta.
\end{aligned}\tag{12}$$

Here we have used the same notation as in the previous case and all the assumptions listed in the previous case are still valid.

Theorem 4 *Under the assumptions given above, there exists a weak solution to the system (12) and it is given by, for $t\geq 0$,*

$$\begin{aligned}
y(t)&=\Big(f_0+A\cos Z(t)+B\,\dot{W}(t)\Big)*P(t),\\
\sigma(t)&=\lambda y(t)*Q(t)=\lambda(f_0*P*Q)(t)\\
&\quad+\lambda(A\cos Z*P*Q)(t)\\
&\quad+\lambda(B\,\dot{W}*P*Q)(t),
\end{aligned}$$

where P and Q are given by (8) and (9), respectively, and $y(t){=}\sigma(t){=}0$, for $t{<}0$.

Note that the functions y,y',σ and σ' are continuous, for $t{\geq}0$.

4 Stochastic moments of the solution

We consider the first and the second (stochastic) moments of the solutions.

4.1 Case 1

The solution to the problem in consideration y is given by (11) and therefore its expectation is

$$\begin{aligned}E(y(t)){=}&(f_0{*}P)(t)\\&+E((\xi{*}P)(t)){+}E((B\dot{W}{*}P)(t))\\={}&(f_0{*}P)(t){+}E\left(\int_0^t \xi(s)P(t{-}s)ds\right)\\&+BE\left(\int_0^t P(t{-}s)dW(s)\right)\\={}&(f_0{*}P)(t){+}\int_0^t P(t{-}s)E(\xi(s))ds,\end{aligned}$$

where P given by (6).

In the calculation above we have used the well known fact that the expectation of the Itô integral $\int_0^t P(t{-}s)dW(s)$ equals zero.

As mentioned in the introduction, $\xi(t)$ is a stationary stochastic process and therefore its mean $E(\xi(t))$ is a constant and its auto-covariance function $K_\xi(t,s){=}E(\xi(t)\xi(s)){-}E(\xi(t))E(\xi(s))$ is a function of the difference of the arguments, i.e., a function of $t{-}s$.

Now, using the fact that the expectation of the bounded noise ξ is a constant, one has

$$\begin{aligned}E(\xi(t)){=}E(\xi(0)){=}&E\left[A\cos\left(\eta\tilde{W}(0){+}\theta\right)\right]\\={}&E(A\ \cos\theta){=}A\ E(\cos\theta)\\={}&A\int_0^{2\pi}\frac{1}{2\pi}\cos x dx{=}0,\end{aligned}$$

where we have used that $\tilde{W}(0){=}0$ and that θ is uniformly distributed over the interval $[0,2\pi]$.

Thus,

$$E(y(t)){=}f_0(t){*}P(t).$$

We see that the solution is unbiased, i.e., the expectation of the solution equals to the solution of the corresponding deterministic equations obtained by replacing stochastic elements by theirs expectations.

The second moment of the solution, for $t{\geq}0$, is

$$\begin{aligned}&E(y^2(t)){=}E\left(((f_0{*}P)(t))^2\right)\\&\quad+E\left(((\xi{*}P)(t))^2\right){+}E\left(\left((B\dot{W}{*}P)(t)\right)^2\right)\\&={(f_0(t){*}P(t))}^2{+}E\left(\left(\int_0^t P(t{-}s)\xi(s)ds\right)^2\right)\\&\quad+B^2E\left(\left(\int_0^t P(t{-}s)dW(s)\right)^2\right)\\&\quad+2(f_0(t){*}P(t))E\left(\int_0^t P(t{-}s)\xi(s)ds\right)\\&\quad+2(f_0(t){*}P(t))E\left(\int_0^t P(t{-}s)dW(s)\right)\\&\quad+2E(\xi(t){*}P(t))E\left(\int_0^t P(t{-}s)dW(s)\right)\\&={(f_0(t){*}P(t))}^2{+}E\left(\left(\int_0^t P(t{-}s)\xi(s)ds\right)^2\right)\\&\quad+B^2\int_0^t P^2(t{-}s)\ ds,\end{aligned}$$

where we have used that the bounded noise ξ and the white noise $\dot{W}$ are independent and that, for $t{\geq}0$,

$$\begin{aligned}E\left(\int_0^t P(t{-}s)dW(s)\right)&{=}0,\\\left(\int_0^t P(t{-}s)\xi(s)\right)&{=}0,\\E\left(\left(\int_0^t P(t{-}s)dW(s)\right)^2\right)&{=}\int_0^t P^2(t{-}s)ds.\end{aligned}$$

In order to calculate the second moment of the solution $y(t)$ it remains to investigate the second moment of the convolution $\xi(t)*P(t)$:

$$E\left(\left(\int_0^t P(t-s)\xi(s)ds\right)^2\right)$$
$$=E\left(\int_0^t P(t-s)\xi(s)ds\cdot\int_0^t P(t-u)\xi(u)du\right)$$
$$=\int_0^t\int_0^t P(t-s)P(t-u)E(\xi(s)\xi(u))dsdu.$$

For the auto-correlation function of the bounded noise ξ we refer to [10]. It is given by

$$E(\xi(s)\xi(u))=\frac{1}{2}A^2\cos(\omega(s-u))e^{-\frac{\eta^2}{2}|s-u|}, \quad (13)$$

for $s>0,\ u>0$. Note that, since the bounded noise ξ is a mean zero process, the auto-correlation function and the auto-covariance function coincide. Further, both of them are the functions only of the difference of the argument which is, as mentioned, the well known property of stationary processes.

Now, by using (13) one obtains, for $t\geq 0$,

$$E\left(\left(\int_0^t P(t-s)\xi(s)ds\right)^2\right)$$
$$=\int_0^t\int_0^t P(t-s)P(t-u)\frac{1}{2}A^2\cos(\omega(s-u))$$
$$\cdot e^{-\frac{\eta^2}{2}|s-u|}dsdu:=I_1(P), \quad (14)$$

which is well defined integral.

Finally, the second moment of the solution is

$$E(y(t))=(f_0*P(t))^2 +B^2\int_0^t P^2(t-s)ds+I_1(P).$$

Thus, the dispersion of the solution is finite, i.e.,

$$D(y(t))=E(y^2(t))-E^2(y(t)) =B^2\int_0^t P^2(t-s)ds+I_1(P).$$

4.2 Case 2

The solution to the problem in consideration is given in Theorem 4.

The expectation of the solution is

$$\begin{aligned}E(y(t))&=(f_0*P)(t)+E((\xi*P)(t)) +E\left((B\dot{W}*P)(t)\right)\\&=(f_0*P)(t),\ t\geq 0,\\E(\sigma(t))&=\lambda E((f_0*P*Q)(t)) +\lambda E((\xi*P*Q)(t)) +\lambda E\left((B\dot{W}*P*Q)(t)\right)\\&=\lambda(f_0*P*Q)(t),\end{aligned}$$

where P and Q are given by (8) and (9), respectively. Thus, the solution is again unbiased.

The second moment of the solution is, similarly as in the Case 1,

$$E(y^2(t))=(f_0*P(t))^2+B^2\int_0^t P^2(t-s)ds+I_1(P),$$

and

$$E(\sigma^2(t))=\lambda^2((f_0*P*Q)(t))^2 +\lambda^2\int_0^t (P*Q)^2(t-s)ds +\lambda^2 I_1(P*Q),$$

where I_1 is defined in (14).

The dispersion of the solution is finite, i.e., for $t\geq 0$,

$$D(y(t))=\int_0^t P^2(t-s)ds+I_1(P),$$

and

$$D(\sigma(t))=\int_0^t (P*Q)^2(t-s)ds+\lambda^2 I_1(P*Q).$$

5 Numerical example

We consider a special case of a rod described by (10) with the following values of parameters: $\omega^2=1$, $A=0$, $B=0.1$, $f=1,\lambda=1$, $b=0.5$. The solution is given by (11) with P determined by (6). In the next Figure we show the results. The line with $*$ represents the fundamental solution without the noise, while the thin line represents the solution with noise. It is obtained by using Matlab procedure for simulating stochastic integration.

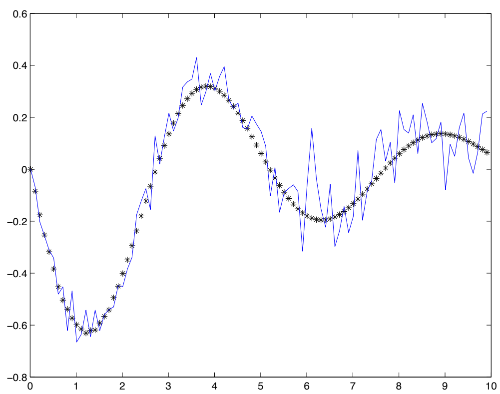

Figure 1: The solution (11) for $\omega^2=1$, $A=0$, $B=0.1$, $f\equiv1$, $\lambda=1$, $b=0.5$

6 Conclusion

We studied systems (10) and (12) that correspond to the axially loaded, distributed order viscoelastic rod with a mass attached at its end. The axial force has deterministic unbounded and bounded stochastic parts. The existence of weak solutions is proved and their properties are examined. We showed that the solutions are unbiased and that their dispersions are finite. Also we presented a numerical example in which we show that the stochastic part of the load produces displacements that are comparable with the fundamental ones.

7 Appendix: Proofs of the theorems

Proof of the Theorem 1

In order to solve (5), we use the Laplace transform method. Recall that the Laplace transform of $y\in L^1_{loc}(\mathbb{R})$, $y\equiv0$ in $(-\infty,0]$ and $|y(t)|\le ce^{kt}$, $t>0$, for some $k>0$, is defined by

$$\widehat{y}(s)=\mathcal{L}[f(t)](s):=\int_0^\infty y(t)e^{-st}\mathrm{d}t,\ \mathrm{Re}(s)>k$$

and analytically continued into the appropriate domain D.

Applying formally the Laplace transform to (5) with $F=\delta$, we obtain

$$(s^2+\omega^2)\hat{y}(s)+\lambda\hat{y}(s)\int_0^1(bs)^\gamma a^\gamma d\gamma=1.$$

Thus, the Laplace transform of the fundamental solution is

$$g(s)=\frac{1}{s^2+\omega^2+\lambda\frac{bs-1}{\ln(bs)}},\ s\in D,$$

where D is the appropriate domain excluding the half-line $(\infty,0]$ and the poles of the function.

Let us now find the locations of poles for g, i.e. zeros of the function $g_1(s)=s^2+\omega^2+\lambda\dfrac{bs-1}{\ln(bs)}$. (Note that it does not have a singularity in the point $s=1/b$.)

Put $C=C_R\cup C_1\cup C_r\cup C_2$, $C_R=\{s\colon s=Re^{i\phi},\ \phi\in[-\pi,\pi]\}$, $C_1=\{s\colon s=-t+0i,\ t\in[r,R]\}$, $C_r=\{s\colon s=re^{i\phi},\ \phi\in[-\pi,\pi]\}$, and $C_2=\{s\colon s=-t-0i,\ t\in[r,R]\}$, with $r>0$ arbitrary small and $R>0$ arbitrary large. The change of the argument of $g_1(s)$ as s is moving along the curve C_R is the same as the change for the function s^2 instead of g_1 (since R is large) and that is 4π. The change of the argument along C_r is obviously zero, since $g_1(s)\approx\omega^2$ for small r. Also, g_1 moves close to the positive side of real axis as s lies at C_1 and C_2, so the change of the argument along these two curves is zero. Thus, $g_1(s)$ has only two zeros in the complex plane that are conjugate to each other, in addition. Let us denote them by $s_{1,2}$.

In order to find out where the zeros lie, let us calculate the change of argument over the contour $\tilde{C}=C_R\cup C_1\cup C_r\cup C_2$, $C_R=\{s\colon s=Re^{i\phi},\ \phi\in[-\pi/2,\pi/2]\}$, $C_1=\{s\colon s=ti,\ t\in[r,R]\}$, $C_r=\{s\colon s=re^{i\phi},\ \phi\in[-\pi/2,\pi/2]\}$, and $C_2=\{s\colon s=-ti,\ t\in[r,R]\}$, with $r>0$ arbitrary small and $R>0$ arbitrary large. As above, the change of the argument over C_R is 2π and zero over C_r. Substitution of g_1 into C_1 and C_2 and direct calculation show that the change of the argument over each of these contours is $-\pi$, and the change of the argument for g_1 over the complete contour C is thus zero. That means that a real part of these zeros is negative.

Thus the function $g(s)$ is holomorphic for $\mathrm{Re}(s)>0$ as well as $sg(s)$ and $s^2g'(s)$ being bounded in the same half-plane. So, by Corollary 2.5.2 in [1], there exists a continuous function P equals zero in $(-\infty,0)$ having the function g as its Laplace transform. That function is a fundamental solution of (5).

One can use the Bromwich contour in order to get an inverse Laplace transform of g. Denote by C_γ the right-handed part of the contour C intersected by the line $\mathrm{Im}(s)=\gamma>0$. Keeping the same notation as there, we have the contour $C_\gamma=I_\gamma\cup C_R\cup C_1\cup C_r\cup C_2$, where C_R now denotes the intersected part of old C_R and I_γ is the intersected part of the line $\mathrm{Im}(s)=\gamma$.

The solution P is the limit

$$P(t)=\lim_{R\to\infty,r\to 0}\frac{1}{2\pi i}\int_{I_\gamma}e^{st}g(s)ds,t>0.$$

We have, for $t>0$,

$$\begin{aligned}&\int_{I_\gamma}e^{st}g(s)ds\\ &=2\pi i\Big(\mathrm{Res}\Big(\frac{e^{st}}{f(s)},s_1\Big)+\mathrm{Res}\Big(\frac{e^{st}}{f(s)}s_2\Big)\Big)\\ &\quad-\int_{C_R}e^{st}g(s)ds-\int_{-R}^{-r}e^{s\xi}g(\xi+0i)d\xi\\ &\quad+\int_{C_r}e^{st}g(s)ds+\int_{-R}^{-r}e^{s\xi}g(\xi-0i)d\xi.\end{aligned}$$

The integrals over C_r and C_R vanish in the limit, $s_{1,2}$ are (conjugate) poles of the first order, and, for $t\geq 0$,

$$\begin{aligned}P(t)=&\sum_{j=1}^{2}\frac{e^{s_jt}}{2s_j+\frac{b}{\ln(bs_j)}-\frac{bs_j-1}{s_j\ln^2(bs_j)}}\\ &+\frac{1}{2\pi i}\int_0^\infty\frac{e^{-\xi t}d\xi}{\xi^2+\omega^2-\lambda\frac{b\xi+1}{\ln(b\xi)-\pi i}}.\\ &-\frac{1}{2\pi i}\int_0^\infty\frac{e^{-\xi t}d\xi}{\xi^2+\omega^2-\lambda\frac{b\xi+1}{\ln(b\xi)+\pi i}}.\end{aligned}$$

If we put $s_1=-\alpha+\beta i$, $s_2=-\alpha-\beta i$, $\alpha>0$, we obtain that $P(t)=0$, $t<0$, and that it is given by (6).

Proof of the Theorem 2

System (7) was treated in [5] in a similar manner as we did here investigating the Case 1. Again, we denote by P the fundamental solution and by g its Laplace transform. We have

$$g(s)=\frac{1}{s^2+\omega^2+\lambda\beta\frac{ln(as)}{\ln(bs)}\frac{bs-1}{as-1}},\ s\in D,\qquad (15)$$

where D is the appropriate domain excluding the half-line $(\infty,0]$ and the poles of the function.

This function has two conjugate zeros $s_{1,2}=-\alpha\pm\beta i$, $\alpha>0$. Further, P has been explicitly calculated and it is given by (8). Note that (7) and (15) imply

$$\hat{\sigma}=\lambda g(s)\frac{\ln(as)}{\ln(bs)}\frac{bs-1}{as-1},\ s\in D,$$

so that

$$\sigma(t)=\lambda P(t)*Q(t),\ t\geq 0,\text{and } Q(t)=0,\ t<0,$$

where (see [5]) Q is given by (9).

Proof of the Theorem 3

We will show that (11) is satisfied in the weak sense.

Let $h\in L^1_{loc}$, such that $h=0$ for $t<0$. We know that

$${}_0D_t^\gamma h(t)=\frac{d}{dt}(f_{1-\gamma}*h)(t),\ t>0,$$

where

$$f_{1-\gamma}(t)=\frac{1}{\Gamma(1-\gamma)}\ t_+^{-\gamma},\ t>0.$$

This implies

$$\int_0^1 b^\gamma {}_0D_t^\gamma h(t) d\gamma = \frac{d}{dt}\int_0^1 b^\gamma (f_{1-\gamma} * h)(t) d\gamma,\ t>0.$$

Now, denoting

$$\check{f}_{1-\gamma}(t) = f_{1-\gamma}(-t),$$

one obtains, for $\phi \in \mathcal{D}$,

$$\begin{aligned}&\left\langle \frac{d}{dt}\int_0^1 b^\gamma (f_{1-\gamma}*h)(t)d\gamma, \phi(t)\right\rangle\\ &= -\int_0^1 b^\gamma \langle (f_{1-\gamma}*h)(t), \phi'(t)\rangle d\gamma\\ &= \left\langle h(t), \int_0^1 b^\gamma \left(\check{f}_{1-\gamma}*\phi'\right)(t)d\gamma\right\rangle. \qquad (16)\end{aligned}$$

Let $h = P*B\dot{W}$. By (16) one obtains

$$\begin{aligned}&\Big\langle \frac{d^2}{dt^2}(P*B\dot{W})(t) + \omega^2\cdot(P*B\dot{W})(t)\\ &\quad +\lambda\int_0^1 b^\gamma {}_0D_t^\gamma (P*B\dot{W})(t)d\gamma, \phi(t)\Big\rangle\\ &= \Big\langle (P*B\dot{W})(t), \frac{d^2}{dt^2}\phi(t) + \omega^2\phi(t)\\ &\quad +\lambda\int_0^1 b^\gamma\Big(\check{f}_{1-\gamma}*\phi'\Big)(t)d\gamma\Big\rangle\\ &= \Big\langle B\dot{W}(t), \check{P}*\Big(\frac{d^2}{dt^2}\phi + \omega^2\phi\\ &\quad +\lambda\int_0^1 b^\gamma\Big(\check{f}_{1-\gamma}*\phi'\Big)(t)d\gamma\Big)(t)\Big\rangle\\ &= \Big\langle B\dot{W}(t), \Big\langle\Big(\frac{d^2}{dt^2}\check{P} + \omega^2\check{P}\\ &\quad +\lambda\int_0^1 b^\gamma \frac{d}{dt}\Big(\check{P}*f_{1-\gamma}\Big)d\gamma\Big)(t-u), \phi(u)\Big\rangle\Big\rangle\\ &= \Big\langle B\dot{W}(t), \Big\langle\Big(\frac{d^2}{dt^2}\check{P} + \omega^2\check{P}\\ &\quad +\lambda\int_0^1 b^\gamma {}_0D_t^\gamma \check{P} d\gamma\Big)(t-u), \phi(u)\Big\rangle\Big\rangle\\ &= \Big\langle B\dot{W}(t), (\check{\delta}*\phi)(t)\Big\rangle = \Big\langle B\dot{W}(t), \phi(t)\Big\rangle\\ &= B\int_0^\infty \phi(t) dW(t),\end{aligned}$$

where we have used that P is a fundamental solution to (5) and that $\check{P}*\check{f}_{1-\gamma}*\phi' = \frac{d}{dt}(\check{P}*\check{f}_{1-\gamma})*\phi = \frac{d}{dt}\check{P}_f*\phi$, with $P_f = P*f_{1-\gamma}$.

All the operations above are justified by the fact that supp ϕ is compact and by the definition and the properties of Itô integral for the white noise measure. Especially, the Fubini's Theorem is used several times in the calculations above.

Similar (even simpler) calculations gives

$$\begin{aligned}&\Big\langle \frac{d^2}{dt^2}(P*f_0)(t) + \omega^2\cdot(P*f_0)(t)\\ &+\lambda\int_0^1 b^\gamma {}_0D_t^\gamma (P*f_0)(t)d\gamma, \phi(t)\Big\rangle = \Big\langle f_0, \phi\Big\rangle,\end{aligned}$$

and, by the same procedure one gets

$$\begin{aligned}&\Big\langle \frac{d^2}{dt^2}(P*A\cos Z)(t) + \omega^2\cdot(P*A\cos Z)(t)\\ &\quad +\lambda\int_0^1 b^\gamma {}_0D_t^\gamma (P*A\cos Z)(t)d\gamma, \phi(t)\Big\rangle\\ &= \Big\langle A\cos Z, \phi\Big\rangle.\end{aligned}$$

Thus, knowing that $\xi(t) = A\cos Z(t),\ t>0$, we proved that

$$\begin{aligned}&\Big\langle \frac{d^2}{dt^2}(P*(f_0+\xi+\dot{W}))(t) + \omega^2(P*(f_0+\xi+\dot{W}))(t)\\ &\quad +\lambda\int_0^1 b^\gamma {}_0D_t^\gamma (P*(f_0+\xi+\dot{W}))(t)d\gamma, \phi(t)\Big\rangle\\ &= \Big\langle f_0+\xi+\dot{W}, \phi\Big\rangle,\end{aligned}$$

i.e. that (11) is a weak solution to (10). Actually, we know more: f_0*P is the classical solution to the system (1)-(3) for $F = f_0$.

Proof of the Theorem 4

By a direct substitution of the above expressions we get

$$\begin{aligned}&\int_0^1 a^\gamma {}_0D_t^\gamma \sigma(t)\ d\gamma\\ &-\lambda\int_0^1 b^\gamma {}_0D_t^\gamma y(t)\ d\gamma = 0,\ t>0,\end{aligned}$$

in the strong sense. By using the same argument and notation as in the stochastic Case 1 one obtains, for $\phi \in \mathcal{D}$,

$$
\begin{aligned}
&\Big\langle \frac{d^2}{dt^2}(B\dot{W}*P)(t)+\lambda(B\dot{W}*P*Q)(t),\phi(t)\Big\rangle \\
=&\Big\langle \frac{d^2}{dt^2}(B\dot{W}*P)(t),\phi(t)\Big\rangle \\
&+\Big\langle \lambda(B\dot{W}*P*Q)(t),\phi(t)\Big\rangle \\
=&\Big\langle (B\dot{W}*P)(t),\frac{d^2}{dt^2}\phi(u)\Big\rangle \\
&+\Big\langle (B\dot{W}*P)(t),\lambda\check{Q}*\phi(t)\Big\rangle \\
=&\Big\langle B\dot{W}(t),\check{P}*\frac{d^2}{dt^2}(u)\phi\Big\rangle \\
&+\Big\langle B\dot{W}(t),\Big\langle \lambda\check{P}*\check{Q}(t-u),\phi(u)\Big\rangle\Big\rangle \\
=&\Big\langle B\dot{W}(t),\Big\langle\Big(\frac{d^2}{dt^2}\check{P}+\lambda\check{P}*\check{Q}\Big)(t-u),\phi(u)\Big\rangle\Big\rangle \\
=&\Big\langle B\dot{W}(t),(\check{\delta}*\phi)(t)\Big\rangle=\Big\langle B\dot{W}(t),\phi(t)\Big\rangle.
\end{aligned}
$$

In the similar manner, one obtains

$$
\begin{aligned}
&\Big\langle \frac{d^2}{dt^2}(f_0*P)(t)+\lambda(f_0*P*Q)(t),\phi(t)\Big\rangle \\
=&\Big\langle f_0(t),\phi(t)\Big\rangle, \text{ and} \\
&\Big\langle \frac{d^2}{dt^2}(A\cos Z*P)(t)+\lambda(A\cos Z*P*Q)(t),\phi(t)\Big\rangle \\
=&\Big\langle A\cos Z(t),\phi(t)\Big\rangle.
\end{aligned}
$$

Thus, the proof is completed.

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