Semigroups and PDEs with perturbations

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1 Introduction

The classical theory of semigroups (see [8]) has been developed in order to find solutions to evolution equations and systems. But there is a broad class of classically non-solvable equations and systems, particularly ones where nonlinear operations (such as multiplication) occur. One of possible ways is to use a Colombeau-like theory of generalized functions spaces, which is in fact a multiplicative algebra (see [1], [3], [4] or [5]). Up to our knowledge, the first attempt is made in [2]. In [6] the theory of C_0-semigroups is combined with the above spaces of generalized functions. In this paper we are using uniformly continuous semigroups which are more suitable to work with, but have a big disadvantage: differentiation is not a bounded operator. Thus operators involving differentiation can not determine uniformly continuous semigroups.

On the other hand, there exists the notion of a regularized derivative (see [4] and [9]) which substitutes the derivative with a bounded integral operator. This is the main point of our attempt, we will now solve equations with such objects using the well developed theory of semigroups of bounded operators.

One should mention that this paper is just a beginning of investigations leading to applications on some classes of nonlinear PDE's.

2 Basic spaces

Let $(E, \|\cdot\|)$ be a Banach space and $\mathcal{L}(E)$ the space of all linear continuous mappings $E \to E$.

$\mathcal{S}E_M([0,\infty):\mathcal{L}(E))$ is the space of nets

$$S_\varepsilon : [0,\infty) \to \mathcal{L}(E),\ \varepsilon \in (0,1)$$

differentiable with respect to $t \in [0,\infty)$, with the property that for every $T > 0$ there exist $N \in \mathbb{N}$, $M > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T)} \left\| \frac{d^\alpha}{dt^\alpha} S_\varepsilon(t) \right\|_{\mathcal{L}(E)} \leq M\varepsilon^{-N},\ \varepsilon < \varepsilon_0,\ \alpha \in \{0,1\}. \tag{1}$$

It is an algebra with respect to composition of operators.

$\mathcal{S}N([0,\infty):\mathcal{L}(E))$ is the space of nets

$$N_\varepsilon : [0,\infty) \to \mathcal{L}(E),\ \varepsilon \in (0,1)$$

differentiable with respect to $t \in [0,\infty)$, with the property that for every $T > 0$ and $a \in \mathbb{R}$ there exist $M > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{t\in[0,T)} \left\| \frac{d^\alpha}{dt^\alpha} N_\varepsilon(t) \right\|_{\mathcal{L}(E)} \le M\varepsilon^a, \ \varepsilon < \varepsilon_0, \ \alpha \in \{0,1\}. \tag{2}$$

One can easily see that it is an ideal of $\mathcal{S}E_M$. Thus, Colombeau-type space can be defined as

$$\mathcal{S}G\left([0,\infty) : \mathcal{L}(E)\right) = \frac{\mathcal{S}E_M\left([0,\infty) : \mathcal{L}(E)\right)}{\mathcal{S}N\left([0,\infty) : \mathcal{L}(E)\right)}.$$

Elements of $\mathcal{S}G\left([0,\infty) : \mathcal{L}(E)\right)$ will be denoted as $S = [S_\varepsilon]$ where S_ε is a representative of the class.

Similarly, one can define following spaces:

$\mathcal{S}E_M(E)$ is the space of nets of linear continuous mappings

$$A_\varepsilon : E \to E, \ \varepsilon \in (0,1)$$

with the property that there exist constants $N \in \mathbb{N}$, $M > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)} \le M\varepsilon^{-N}, \ \varepsilon < \varepsilon_0. \tag{3}$$

$\mathcal{S}N(E)$ is the space of nets of linear continuous mappings

$$A_\varepsilon : E \to E, \ \varepsilon \in (0,1)$$

with the property that for every $a \in \mathbb{R}$ there exist $M > 0$ and $\varepsilon_0 > 0$ such that

$$\|A_\varepsilon\|_{\mathcal{L}(E)} \le M\varepsilon^a, \ \varepsilon < \varepsilon_0. \tag{4}$$

Now,

$$\mathcal{S}G(E) = \frac{\mathcal{S}E_M(E)}{\mathcal{S}N(E)}.$$

Elements of $\mathcal{S}G(E)$ will be denoted as $A = [A_\varepsilon]$ where A_ε is a representative of the class.

In the last section of the paper we will need the following spaces:

$\mathcal{E}_M\left([0,\infty) : L^p(\mathbb{R}^n)\right)$ is the space of nets of functions

$$G_\varepsilon : [0,\infty) \times \mathbb{R}^n \to \mathbb{C}, \ G_\varepsilon(t,\cdot) \in L^p(\mathbb{R}^n), \text{ for every } t \in [0,\infty),$$

with the property that for every $T > 0$ there exist $C > 0$, $N \in \mathbb{N}$ and $\eta > 0$ such that

$$\sup_{t\in[0,T)} \|\partial_t^\alpha G_\varepsilon(t,\cdot)\|_{L^p} \le C\varepsilon^{-N}, \ \alpha \in \{0,1\}, \ \varepsilon < \eta.$$

$\mathcal{N}\left([0,\infty) : L^p(\mathbb{R}^n)\right)$ is the space of nets of functions $G_\varepsilon \in \mathcal{E}_M\left([0,\infty) : L^p(\mathbb{R}^n)\right)$ with the property that for every $T > 0$ and $a \in \mathbb{R}$ there exist $C > 0$ and $\eta > 0$ such that

$$\sup_{t\in[0,T)} \|\partial_t^\alpha G_\varepsilon(t,\cdot)\|_{L^p} \le C\varepsilon^a, \ \alpha \in \{0,1\}, \ \varepsilon < \eta.$$

Define the quotient space

$$\mathcal{G}\left([0,\infty):L^p(\mathbb{R}^n)\right) = \frac{\mathcal{E}_M\left([0,\infty):L^p(\mathbb{R}^n)\right)}{\mathcal{N}\left([0,\infty):L^p(\mathbb{R}^n)\right)}.$$

In similar way, by omitting t-variable, one can define spaces $\mathcal{E}_M\left(L^p(\mathbb{R}^n)\right)$, $\mathcal{N}\left(L^p(\mathbb{R}^n)\right)$ and $\mathcal{G}\left(L^p(\mathbb{R}^n)\right)$.

Let us note that the above spaces are not algebras with respect to multiplication (which is the case for the original definition of generalized function spaces).

3 Colombeau semigroups

Definition 1 *$S \in \mathcal{S}G\left([0,\infty):\mathcal{L}(E)\right)$ is called a uniformly continuous Colombeau semigroup if it has a representative S_ε which is a uniformly continuous semigroup for every ε small enough*
(i) $S_\varepsilon(0) = I$.
(ii) $S_\varepsilon(t_1 + t_2) = S_\varepsilon(t_1)S_\varepsilon(t_2)$, *for every* $t_1, t_2 \geq 0$.
(iii) $\lim_{t\to 0} \|S_\varepsilon(t) - I\| = 0$.

Proposition 1 *Let S_ε and $\tilde{S}_\varepsilon$ be representatives of a uniformly continuous Colombeau semigroup S, with infinitesimal generators A_ε, and $\tilde{A}_\varepsilon$, respectively, for ε small enough. Then $A_\varepsilon - \tilde{A}_\varepsilon \in \mathcal{S}N(E)$.*

Proof. We have

$$\begin{aligned} A_\varepsilon - \tilde{A}_\varepsilon &= \left.\frac{d^+}{dt}S_\varepsilon(t)\right|_{t=0} - \left.\frac{d^+}{dt}\tilde{S}_\varepsilon(t)\right|_{t=0} \\ &= \left.\frac{d^+}{dt}\Big(S_\varepsilon(t) - \tilde{S}_\varepsilon(t)\Big)\right|_{t=0}. \end{aligned}$$

Since $S_\varepsilon - \tilde{S}_\varepsilon = N_\varepsilon \in \mathcal{S}N\left([0,\infty):\mathcal{L}(E)\right)$ we have that for every $a \in \mathbb{R}$ there exists $M > 0$ such that

$$\left\| \left.\frac{d^+}{dt}\Big(S_\varepsilon(t) - \tilde{S}_\varepsilon(t)\Big)\right|_{t=0} \right\| \leq M\varepsilon^a,$$

for ε small enough.

It implies that for every $a \in \mathbb{R}$ there exists $M > 0$ such that

$$\|A_\varepsilon - \tilde{A}_\varepsilon\| \leq M\varepsilon^a$$

for ε small enough. Thus, $A_\varepsilon - \tilde{A}_\varepsilon \in \mathcal{S}N(E)$ and the proof is completed. □

Definition 2 *$A \in \mathcal{S}G(E)$ is called the infinitesimal generator of a uniformly continuous Colombeau semigroup $S \in \mathcal{S}G\left([0,\infty):\mathcal{L}(E)\right)$ if A_ε is the infinitesimal generator of the representative S_ε, for every $\varepsilon < \varepsilon_0$, $\varepsilon_0 \in (0,1)$.*

Definition 3 *Let h_ε be a positive net satisfying $h_\varepsilon < \varepsilon^{-1}$. It is said that $A \in \mathcal{S}G(E)$ is of h_ε-type if there exist its representative A_ε such that*

$$\|A_\varepsilon\|_{\mathcal{L}(E)} = \mathcal{O}(h_\varepsilon),\ \varepsilon \to 0. \tag{5}$$

A $\mathcal{G}\left([0,\infty) : L^p(\mathbb{R}^n)\right)$ is said to be of h_ε-type if there exists its representative G_ε such that

$$\|G_\varepsilon\|_{L^p} = \mathcal{O}(h_\varepsilon),\ \varepsilon \to 0.$$

In the classical theory of semigroups of bounded linear operators the following theorem holds.

Theorem 1 *([8], Theorem 1.2) A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

In our case the following lemma holds.

Lemma 1 *Every $A \in \mathcal{S}G(E)$ of h_ε-type, where $h_\varepsilon \le C \log \frac{1}{\varepsilon}$, is the infinitesimal generator of some $T \in \mathcal{S}G\left([0,\infty) : \mathcal{L}(E)\right)$.*

Proof. According to Theorem 1 every bounded operator A_ε is the infinitesimal generator of the uniformly continuous semigroup

$$T_\varepsilon(t) = e^{tA_\varepsilon} = \sum_{n=0}^{\infty} \frac{(tA_\varepsilon)^n}{n!}.$$

Let us show that $(T_\varepsilon)_\varepsilon \in \mathcal{S}E_M\left([0,\infty) : \mathcal{L}(E)\right)$. We have that

$$\|T_\varepsilon(t)\| \le \sum_{n=0}^{\infty} \frac{\|tA_\varepsilon\|^n}{n!} \le M \sum_{n=0}^{\infty} \frac{1}{n!}(th_\varepsilon)^n = Me^{th_\varepsilon}.$$

Since $h_\varepsilon \le C \log \dfrac{1}{\varepsilon}$ we have that

$$\sup_{t \in [0,T)} \|T_\varepsilon(t)\| \le M\varepsilon^{-TC},$$

for ε small enough. Since

$$\frac{d}{dt} T_\varepsilon(t) = A_\varepsilon,$$

for every ε small enough, we have

$$\left\| \frac{d}{dt} T_\varepsilon(t) \right\| = \|A_\varepsilon\| \le C \log \frac{1}{\varepsilon} \le C\varepsilon^{-1},$$

for every such ε, i.e., $(T_\varepsilon)_\varepsilon \in \mathcal{S}E_M\left([0,\infty) : \mathcal{L}(E)\right)$. Thus, the proof is completed. $\square$

Proposition 2 *Let A be the infinitesimal generator of a uniformly continuous Colombeau semigroup S, and B be the infinitesimal generator of a uniformly continuous Colombeau semigroup T. If $A = B$, then $S = T$.*

Proof. Let $N_\varepsilon = A_\varepsilon - B_\varepsilon \in \mathcal{SN}(E)$. We have

$$\frac{d}{dt}(S_\varepsilon(t) - T_\varepsilon(t))x = A_\varepsilon(S_\varepsilon(t) - T_\varepsilon(t))x + N_\varepsilon T_\varepsilon(t)x.$$

Duhamel principle and $S_\varepsilon(0) = T_\varepsilon(0) = I$ imply

$$(S_\varepsilon(t) - T_\varepsilon(t))x = \int_0^t S_\varepsilon(t-s)N_\varepsilon T_\varepsilon(s)x ds.$$

One can easily show that $\|S_\varepsilon(t) - T_\varepsilon(t)\| \leq C\varepsilon^a$, for every real a, because $N_\varepsilon \in \mathcal{SN}(E)$. The same bounds for t-derivative of $S_\varepsilon(t) - T_\varepsilon(t)$ can be obtained by a successive differentiation of the above term. □

4 Differential equations with regularized derivatives

Definition 4 *Let $\alpha \in \mathbb{N}_0^n$. Regularized α-th derivative of a generalized function G_ε is defined by the representative*

$$\tilde{\partial}^\alpha_{h_\varepsilon} G_\varepsilon = G_\varepsilon * \partial^\alpha \phi_{h_\varepsilon}, \tag{6}$$

where $\phi_{h_\varepsilon}(x) = h_\varepsilon^n \phi(xh_\varepsilon)$, $\phi(y) = \phi_1(y_1) \cdot \ldots \cdot \phi_1(y_n)$, $\phi_1 \in C_0^\infty(\mathbb{R})$, $\phi_1(\xi) \geq 0$ and $\int \phi_1(\xi)d\xi = 1$.

For definition and some basic properties of regularized derivatives we refer to [9] and [4].

Lemma 2 *Suppose that $f \in L^2(\mathbb{R}^n)$ and let h_ε be a net from Definition 4. Then*

$$\|\tilde{\partial}^\alpha_{h_\varepsilon} f\|_{L^2} \leq C_{\phi,\alpha} h_\varepsilon^{|\alpha|} \|f\|_{L^2}.$$

Proof. For $f \in L^2(\mathbb{R}^n)$ we have

$$\left\|\tilde{\partial}^\alpha_{h_\varepsilon} f\right\|_{L^2} = \|f * \partial^\alpha \phi_{h_\varepsilon}\|_{L^2} \leq \|f\|_{L^2} \cdot \|\partial^\alpha \phi_{h_\varepsilon}\|_{L^1}.$$

Note that

$$\begin{aligned}
\|\partial^\alpha \phi_{h_\varepsilon}\|_{L^1} &= \int_{\mathbb{R}^n} |\partial^\alpha (h_\varepsilon^n \phi(xh_\varepsilon))| \, dx \\
&= h_\varepsilon^{n+|\alpha|} \int_{\mathbb{R}^n} |(\partial^\alpha \phi)(xh_\varepsilon)| \, dx \\
&= h_\varepsilon^{|\alpha|} \int_{\mathbb{R}^n} |\partial^\alpha \phi(y)| \, dy = h_\varepsilon^{|\alpha|} \|\partial^\alpha \phi\|_{L^1} \leq C_{\phi,\alpha} h_\varepsilon^{|\alpha|}.
\end{aligned}$$

This implies the assertion. □

Using the spaces introduced at the end of Section 2 one can prove the following theorem.

Theorem 2 *Let $u_0 \in \mathcal{G}\left(L^2(\mathbb{R}^n)\right)$, $n \leq 3$, and $h_\varepsilon = \left(\log \frac{1}{\varepsilon}\right)^{1/(2m)}$. Let $A \in \mathcal{SG}\left(L^2(\mathbb{R}^n)\right)$ be represented by the nets of operators*

$$A_\varepsilon = \sum_{|\alpha| \leq m} a_{\alpha,\varepsilon}(x)\ \tilde{\partial}^\alpha_{h_\varepsilon},\ a_\alpha \in L^\infty(\mathbb{R}^n), \tag{7}$$

$$A_\varepsilon : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

where $a_\alpha \in \mathcal{G}(L^\infty(\mathbb{R}^n))$ is of $\left(\log \frac{1}{\varepsilon}\right)^{1/2}$-type. The generalized function $u \in \mathcal{G}([0,\infty) : L^2(\mathbb{R}^n))$, represented by $u_\varepsilon(t) = S_\varepsilon(t)u_{0\varepsilon}$, $S(t) \in \mathcal{SG}\left([0,\infty) : L^2(\mathbb{R}^n)\right)$, is the Colombeau semigroup generated by A, which uniquely solves the Cauchy problem

$$\begin{aligned} \frac{d}{dt}u(t) &= Au(t) \\ u(0) &= u_0 \in \mathcal{G}(L^2(\mathbb{R}^n)). \end{aligned} \tag{8}$$

Proof. Using Lemma 2,

$$\begin{aligned} \|A_\varepsilon u_\varepsilon(t)\|_{L^2} &\leq \sum_{|\alpha|\leq m} \|a_{\alpha,\varepsilon}\|_{L^\infty} \|\tilde{\partial}^\alpha_{h_\varepsilon} u_\varepsilon(t)\|_{L^2} \\ &\leq C_1 \left(\log \frac{1}{\varepsilon}\right)^{1/2} \sum_{|\alpha|\leq m} C_{\phi,\alpha} \left(\log \frac{1}{\varepsilon}\right)^{|\alpha|/(2m)} \|u_\varepsilon(t)\|_{L^2} \\ &\leq C \log \frac{1}{\varepsilon} \|u_\varepsilon(t)\|_{L^2}. \end{aligned}$$

By Lemmas 1 and 2, $A \in \mathcal{SG}(L^2(\mathbb{R}^n))$ is the infinitesimal generator of some $S \in \mathcal{SG}([0,\infty) : \mathcal{L}(L^2(\mathbb{R}^n)))$. By well known classical results it follows that $u_\varepsilon(t) = S_\varepsilon(t)u_{0\varepsilon}$ solves

$$\frac{d}{dt}u_\varepsilon(t) = A_\varepsilon u_\varepsilon(t),\ u_\varepsilon(0) = u_{0\varepsilon},$$

for every ε small enough.

Let us show that this solution is unique in $\mathcal{G}([0,\infty) : L^2(\mathbb{R}^n))$.

The function $w_\varepsilon := u_\varepsilon - v_\varepsilon$ satisfies

$$\frac{d}{dt}w_\varepsilon(t) = A_\varepsilon w_\varepsilon(t) + N_\varepsilon(t),\ w_\varepsilon(0) = w_{0\varepsilon},$$

where $N_\varepsilon(t) \in \mathcal{N}([0,\infty) : L^2(\mathbb{R}^n))$ and $w_{0\varepsilon} \in \mathcal{N}(L^2(\mathbb{R}^n))$. Then

$$w_\varepsilon(t) = S_\varepsilon(t)w_{0\varepsilon} + \int_0^t S_\varepsilon(t-s)N_\varepsilon(s)\,ds, \tag{9}$$

and

$$\|w_\varepsilon(t)\|_{L^2} \leq \|S_\varepsilon(t)w_{0\varepsilon}\|_{L^2} + \int_0^t \|S_\varepsilon(t-s)N_\varepsilon(s)\|_{L^2} ds\ t \in [0,\infty),$$

from where we obtain the $\mathcal{N}$-bound for $\|w_\varepsilon(t)\|_{L^2}$.

Equation (4) implies

$$\|\frac{d}{dt}w_\varepsilon(t)\|_{L^2} \leq \|A_\varepsilon w_\varepsilon(t)\|_{L^2} + \|N_\varepsilon(t)\|_{L^2}.$$

Since, as we showed in previous step, $\|w_\varepsilon(t)\|_{L^2}$ has the $\mathcal{N}$-bound and $N_\varepsilon(t) \in \mathcal{N}([0,T):L^2(\mathbb{R}^n))$ we obtain that $\|\frac{d}{dt}w_\varepsilon(t)\|_{L^2}$ has the $\mathcal{N}$-bound, too.

Thus, $w_\varepsilon := u_\varepsilon - v_\varepsilon \in \mathcal{N}([0,\infty):L^2(\mathbb{R}^n))$. $\square$

Definition 5 *The solution u_ε of the problem (8) introduced in Theorem 2 is called generalized solution of the equation*

$$\frac{d}{dt}u(t) = \sum_{|\alpha|\leq m} a_\alpha(\cdot)\partial^\alpha u(t)$$

with regularized derivatives.

Appendix

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