Nonlinear stochastic wave equation with Colombeau stochastic generalized processes

Marko Nedeljkov and Danijela Rajter
Institute of Mathematics, University of Novi Sad,
Trg D. Obradovića 4, 21 000 Novi Sad, Yugoslavia

1 Introduction

A large class of stochastic processes which appear in applications can not be defined in a classical way. One of the best known examples of a such process is certainly the white noise process which happened to be a good model of fluctuating phenomena frequently appearing in dynamic systems. White noise was first correctly defined in connection with the theory of generalized functions (distributions) and the concept of white noise as generalized stochastic process has proved to be a very useful mathematical idealization. The generalized stochastic processes have been at first introduced in [5], [16]. But working with generalized stochastic processes involves distribution spaces which are not suitable for multiplication and thus for dealing with nonlinear stochastic partial differential equations. One of the possible approaches in solving stochastic differential equations uses the Wick product as it is done in [6]. Another one, as in paper [13], uses the weighted L^2-spaces.

In order to overcome the multiplication problem, in this paper we use the theory of Colombeau-type generalized functions spaces (see [2], [4]). This is also done in papers [11], [12], [14] and in similar way in paper [1]. More precisely, we use Colombeau-type algebras constructed in [3] and the energy inequality for wave equation (see [8] and references in it).

The first part of the paper is devoted to one-dimensional nonlinear stochastic wave equations of the form

$$\begin{aligned}&(\partial_t^2-\partial_x^2)U+F(U)\cdot S=0,\\ &U|_{t=0}=A,\ \partial_t U|_{t=0}=B,\end{aligned}$$

where A, B and S are certain Colombeau generalized stochastic processes on $\mathbb{R}$ and $\mathbb{R}^2$, respectively, and the function F is smooth, polynomially bounded together with all its derivatives and such that $F(0)=0$.

Oberguggenberger and Russo considered in [11] a one-dimensional nonlinear stochastic wave equation but in the case when the nonlinear part F is a Lipschitz function with an additive generalized stochastic process. A Colombeau solution is constructed and the limiting behavior of the representing net is obtained.

Here, since we are not dealing with F which is Lipschitz, we use a so-called regularization of the function F and instead of the original equation, which we call nonregularized, we consider the corresponding regularized equation obtained by substituting the function F by a family of smooth Lipschitz functions F_ε, for $\varepsilon \in (0,1)$. We prove existence and uniqueness of the solution to the regularized equation. Finally, we are interested in questions under what conditions given on initial data the solution to the regularized equation is also the solution to the nonregularized one.

In the second part of the paper we are interested in 3-dimensional cubic and subcubic stochastic wave and Klein-Gordon equations containing Colombeau generalized stochastic processes. We consider four different cases depending on the growth rates of L^2-norms of the initial data as well as of L^∞-norms of Colombeau generalized stochastic processes which are added to or multiplied with the nonlinear part.

Suppose that A, B, S, S_1 and S_2 are certain Colombeau generalized stochastic processes and that f and g are globally Lipschitz functions, polynomially bounded together with all their derivatives and such that $f(0) = g(0) = 0$.

The first type of equation we are interested in is

$$\begin{aligned}&(\partial_t^2 - \triangle)U + U^3 \cdot S = 0,\\ &U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B.\end{aligned}$$

The second type of equation is

$$\begin{aligned}&(\partial_t^2 - \triangle)U + U \cdot S + U^3 = 0,\\ &U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B.\end{aligned}$$

Then we consider the equation

$$\begin{aligned}&(\partial_t^2 - \triangle)U + U + U^3 + S = 0,\\ &U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B.\end{aligned}$$

Finally, we are interested in equations of the form

$$\begin{aligned}&(\partial_t^2 - \triangle)U + f(U)S_1 + g(U) + S_2 = 0,\\ &U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B.\end{aligned}$$

In all four cases solutions, considered as Colombeau generalized stochastic processes, are obtained and proved to be unique. Conditions under which we have those unique solutions are different for every equation and that is the reason why we consider them separately.

2 Notation and basic definitions

At the beginning we recall some basic facts from classical stochastic analysis.

Let (Ω, Σ, μ) be a probability space. A weakly measurable mapping

$$X : \Omega \ \to \mathcal{D}'(\mathbb{R}^d)$$

is called a *generalized stochastic process* on $\mathbb{R}^d$.

For each fixed function $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the mapping $\Omega \to \mathbb{R}$ defined by

$$\omega \to \langle X(\omega), \varphi\rangle$$

is a random variable.

The space of generalized stochastic processes will be denoted by $\mathcal{D}'_\Omega(\mathbb{R}^d)$. The *characteristic functional* of a process X is

$$C_X(\varphi) = \int e^{i\langle X(\omega),\varphi\rangle} d\mu(\omega),\ \varphi \in \mathcal{D}(\mathbb{R}^d).$$

We take as probability space the space of tempered distributions $\Omega = \mathcal{S}'(\mathbb{R}^d)$ and for Σ the Borel σ-algebra generated by the weak topology. Then there is a unique probability measure μ on (Ω, Σ) such that

$$\int e^{i\langle X(\omega),\varphi\rangle} d\mu(\omega) = e^{-\frac{1}{2}\|\varphi\|^2_{L^2(\mathbb{R}^d)}},\ \varphi \in \mathcal{S}(\mathbb{R}^d).$$

This is a well known result following from the Bochner-Minlos theorem (we refer to [5] or [6]). *White noise process* $\dot{W} : \Omega \to \mathcal{D}'(\mathbb{R}^d)$ is the identity mapping

$$\langle \dot{W}(\omega), \varphi\rangle = \langle \omega, \varphi\rangle,\ \varphi \in \mathcal{D}(\mathbb{R}^d).$$

It is a generalized Gaussian process with mean zero and variance

$$E(\dot{W}(\varphi)^2) = \|\varphi\|^2_{L^2(\mathbb{R}^d)},$$

where E denotes expectation.

Let us now recall the facts from Colombeau generalized functions theory that we need here. A detailed study of these spaces and their properties one can find in [2], [4], [9] and [10].

Let O be an open subset of $\mathbb{R}^n$. We consider the following spaces:

$\mathcal{E}(O)$ is the space of all mappings $G : (0,1) \times O \to \mathbb{C}$ such that

$$G(\varepsilon, \cdot) = G_\varepsilon \in C^\infty(O),\ \varepsilon > 0.$$

$\mathcal{E}_b([0,T) \times \mathbb{R}^n)$ is the space of all $G_\varepsilon \in \mathcal{E}([0,T) \times \mathbb{R}^n)$ with the property that for all $T > 0$ and $\alpha \in \mathbb{N}_0^n$ there exists $N \in \mathbb{N}$ such that

$$\|\partial^\alpha G_\varepsilon\|_{L^\infty([0,T)\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{-N}).$$

We say that $\|\partial^\alpha G_\varepsilon\|_{L^\infty}$ is moderate or that it has a moderate bound.

$\mathcal{N}_b([0,T) \times \mathbb{R}^n)$ is the space of all $G_\varepsilon \in \mathcal{E}([0,T) \times \mathbb{R}^n)$ with the property that for all $T > 0$, $\alpha \in \mathbb{N}_0^n$ and $a \in \mathbb{R}$

$$\|\partial^\alpha G_\varepsilon\|_{L^\infty([0,T)\times\mathbb{R}^n)} = \mathcal{O}(\varepsilon^{a}).$$

We say that $\|\partial^\alpha G_\varepsilon\|_{L^\infty}$ is negligible.

Spaces $\mathcal{E}_b([0,T) \times \mathbb{R}^n)$ and $\mathcal{N}_b([0,T) \times \mathbb{R}^n)$ are algebras and $\mathcal{N}_b([0,T) \times \mathbb{R}^n)$ is an ideal of $\mathcal{E}_b([0,T) \times \mathbb{R}^n)$.

The factor algebra

$$\mathcal{G}_b([0,T)\times\mathbb{R}^n)=\mathcal{E}_b([0,T)\times\mathbb{R}^n)/\mathcal{N}_b([0,T)\times\mathbb{R}^n)$$

is called the algebra of Colombeau generalized functions of bounded type.

Similarly we define the spaces $\mathcal{E}_b(\mathbb{R}^n)$, $\mathcal{N}_b(\mathbb{R}^n)$ and $\mathcal{G}_b(\mathbb{R}^n)$.

Let us remark that $f(\varepsilon)=\mathcal{O}(\varepsilon^b)$ means that $|f(\varepsilon)|\le \mathrm{const}\,\varepsilon^b$ and $f(\varepsilon)=o(\varepsilon^b)$ means $\lim\limits_{\varepsilon\to 0} f(\varepsilon)\varepsilon^{-b}=0$.

In [3] the following construction is given.

$\mathcal{E}_{2,2}([0,T)\times\mathbb{R}^n)$ is the algebra of all $G_\varepsilon\in\mathcal{E}([0,T)\times\mathbb{R}^n)$ with the property that for all $T>0$ and $\alpha\in\mathbb{N}_0^n$ there exists $N\in\mathbb{N}$ such that

$$\|\partial^\alpha G_\varepsilon\|_{L^2([0,T)\times\mathbb{R}^n)}=\mathcal{O}(\varepsilon^{-N}).$$

Again, we say that $\|\partial^\alpha G_\varepsilon\|_{L^2}$ is moderate or that it has a moderate bound.

$\mathcal{N}_{2,2}([0,T)\times\mathbb{R}^n)$ is the algebra of all $G_\varepsilon\in\mathcal{E}([0,T)\times\mathbb{R}^n)$ with the property that for all $T>0$, $\alpha\in\mathbb{N}_0^n$ and $a\in\mathbb{R}$

$$\|\partial^\alpha G_\varepsilon\|_{L^2([0,T)\times\mathbb{R}^n)}=\mathcal{O}(\varepsilon^{a}).$$

We say that $\|\partial^\alpha G_\varepsilon\|_{L^2}$ is negligible.

As above, we define

$$\mathcal{G}_{2,2}([0,T)\times\mathbb{R}^n)=\mathcal{E}_{2,2}([0,T)\times\mathbb{R}^n)/\mathcal{N}_{2,2}([0,T)\times\mathbb{R}^n).$$

One can similarly define spaces $\mathcal{E}_{2,2}(\mathbb{R}^n)$, $\mathcal{N}_{2,2}(\mathbb{R}^n)$ and $\mathcal{G}_{2,2}(\mathbb{R}^n)$.

Let Q denote $[0,T)\times O$ or O. The proof that $\mathcal{N}_{2,2}(Q)$ is an ideal of $\mathcal{E}_{2,2}(Q)$ is given in paper [3]. Sobolev embedding theorems give that $\mathcal{E}_{2,2}(Q)\subset\mathcal{E}_b(Q)$ and $\mathcal{N}_{2,2}(Q)\subset\mathcal{N}_b(Q)$. Thus there exists a canonical mapping $\mathcal{G}_{2,2}(Q)\to\mathcal{G}_b(Q)$. Also, this means that in $\mathcal{G}_{2,2}(Q)$ instead of the L^2-norm on the strip $[0,T)\times\mathbb{R}^n$ one can use the L^∞-norm on $[0,T)$ and the L^2-norm on $\mathbb{R}^n$ and vice versa.

Definition 1 *A $\mathcal{G}_b$-Colombeau generalized stochastic process on a probability space (Ω,Σ,μ) is a mapping $U:\Omega\to\mathcal{G}_b(Q)$ such that there exists a function $U:(0,1)\times Q\times\Omega\to\mathbb{R}$ with the following properties:*

1) For fixed $\varepsilon\in(0,1)$, $(x,\omega)\to U(\varepsilon,x,\omega)$ is jointly measurable in $Q\times\Omega$.

2) $\varepsilon\to U(\varepsilon,\cdot,\omega)$ belongs to $\mathcal{E}_b(Q)$ almost surely in $\omega\in\Omega$, and it is a representative of $U(\omega)$.

By $\mathcal{G}_b^\Omega(Q)$ we denote the algebra of $\mathcal{G}_b$-Colombeau generalized stochastic processes on Ω.

Definition 2 *A $\mathcal{G}_{2,2}$-Colombeau generalized stochastic process on a probability space (Ω,Σ,μ) is a mapping $U:\Omega\to\mathcal{G}_{2,2}(Q)$ such that there exists a function $U:(0,1)\times Q\times\Omega\to\mathbb{R}$ with the following properties:*

1) For fixed $\varepsilon\in(0,1)$, $(x,\omega)\to U(\varepsilon,x,\omega)$ is jointly measurable in $Q\times\Omega$.

2) $\varepsilon\to U(\varepsilon,\cdot,\omega)$ belongs to $\mathcal{E}_{2,2}(Q)$ almost surely in $\omega\in\Omega$, and it is a representative of $U(\omega)$.

By $\mathcal{G}_{2,2}^\Omega(Q)$ we denote the algebra of $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes on Ω.

In the sequel the variable ε will be written as a subindex and U_ε will always denote a representative of U.

3 One-dimensional nonlinear stochastic wave equation

3.1 Preliminary constructions

Consider the problem

$$(\partial_t^2 - \partial_x^2)U + F(U)\cdot S = 0, \tag{1}$$

$$U|_{t=0} = A,\ \partial_t U|_{t=0} = B, \tag{2}$$

where A and B are $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes on $\mathbb{R}$, that is, $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R})$, and $S \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$ is a $\mathcal{G}_{2,2}$-Colombeau generalized stochastic process on $\mathbb{R}^2$ with compact support. We suppose that the function F is smooth, polynomially bounded together with all its derivatives and that $F(0)=0$. We look for a solution $U \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$.

We substitute F by a family of smooth functions F_ε, $\varepsilon \in (0,1)$, which is called the regularization of F. This is done in the following way.

We choose the smooth function F_ε with the property that there exists a net a_ε such that for every $\alpha \in \mathbb{N}_0$ there exist $\varepsilon_0 \in (0,1)$ and $m^\alpha \in \mathbb{N}$ such that

$$F_\varepsilon(y) = F(y),\ \text{for } |y| \le a_\varepsilon,\ \varepsilon < \varepsilon_0$$

$$\|D^\alpha F_\varepsilon(y)\|_{L^\infty} = \mathcal{O}(a_\varepsilon^{m_\alpha}).$$

In the sequel we shall denote $m = \sup_{|\alpha|\le 1} m^\alpha$.

Denote by $\tilde{F} = [F_\varepsilon]$, where $F_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$ has the properties as above. Then, instead of the nonregularized equation (1)-(2), we consider the regularized one

$$(\partial_t^2 - \partial_x^2)U + \tilde{F}(U)\cdot S = 0, \tag{3}$$

$$U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B, \tag{4}$$

where $S = [S_\varepsilon] \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$ and $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R})$.

Note that for $U_\varepsilon, V_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$ such that $U_\varepsilon - V_\varepsilon \in \mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$, we have that $\tilde{F}(U_\varepsilon) - \tilde{F}(V_\varepsilon) \in \mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$.

3.2 Regularized wave equation

Before giving the main result of this section we prove the following lemma which we shall often use in the sequel. The function space norms with subscripts like L^2, H^1 are always meant to signify $L^2(\mathbb{R}^n)$, $H^1(\mathbb{R}^n)$ and so as in the sequal.

Lemma 1 *([8]) Let $u \in C^1([0,T)) \times H^2(\mathbb{R}^n)$ be a solution to equation*

$$u_{tt} - u_{xx} = f.$$

Then

$$\|\partial_t u(t)\|_{L^2} \le \|(\partial_t u(0), \nabla u(0))\|_{L^2} + \int_0^t \|f(s)\|_{L^2} ds, \tag{5}$$

and

$$\|u(t)\|_{H^1} \leq \max(1,t)\left(\|\partial_t u(0)\|_{L^2} + \|u(0)\|_{H^1} + \int_0^t \|f(s)\|_{L^2}ds\right). \tag{6}$$

Proof. ¿From [15]

$$\|(\partial_t u(t), \nabla u(t))\|_{L^2} \leq \|(\partial_t u(0), \nabla u(0))\|_{L^2} + \int_0^t \|f(s)\|_{L^2}ds,$$

and (5) immediately follows. But

$$u(t) = u(0) + \int_0^t \partial_t u(s)ds.$$

Thus

$$\begin{aligned}
\|u(t)\|_{L^2} &\leq \|u(0)\|_{L^2} + \int_0^t \|\partial_t u(s)\|_{L^2}ds \\
&\leq \|u(0)\|_{L^2} + t\|(\partial_t u(0), \nabla u(0))\|_{L^2} + \int_0^t \int_0^s \|f(r)\|_{L^2}drds \\
&\leq \|u(0)\|_{L^2} + t\|(\partial_t u(0), \nabla u(0))\|_{L^2} + t\int_0^t \|f(s)\|_{L^2}ds.
\end{aligned}$$

Therefore (6) follows and the proof is completed.

In sequel we denote $\gamma = \max(1,T)$.

Theorem 1 *Assume that the net a_ε used in the regularization of the function F, has the property*

$$a_\varepsilon = o\left(\left(\log \varepsilon^{-1}\right)^{\frac{1}{2m}}\right). \tag{7}$$

Then, for every $T > 0$, a solution to problem (3)-(4) almost surely exists in $\mathcal{G}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$. Additionally, if the stochastic process $S \in \mathcal{G}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$ satisfies

$$\|S_\varepsilon\|_{L^2} = o\left(\left(\log \varepsilon^{-1}\right)^{\frac{1}{2}}\right), \tag{8}$$

then the obtained solution is almost surely unique in $\mathcal{G}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$.

Proof. As it is usual in the Colombeau framework, we consider problem (3)-(4) given by the representatives

$$(\partial_t^2 - \partial_x^2)U_\varepsilon + F_\varepsilon(U_\varepsilon) \cdot S_\varepsilon = 0, \tag{9}$$

$$U_\varepsilon|_{\{t=0\}} = A_\varepsilon,\ \partial_t U_\varepsilon|_{\{t=0\}} = B_\varepsilon, \tag{10}$$

where $S_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}([0,T) \times \mathbb{R})$ and $A_\varepsilon, B_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}(\mathbb{R})$.

In the sequel $\omega \in \Omega$ and $\varepsilon \in (0,1)$ will be fixed. Note that F_ε is globally Lipschitz for each fixed ε. Thus Cauchy problem (9)-(10) has a unique solution U_ε. The mapping $(x,t,\omega) \to U_\varepsilon(x,t,\omega)$ is jointly measurable in (x,t) and ω for every fixed ε. This conclusion follows by applying successive approximations method and using the fact that a continuous mapping of a measurable function is also measurable.

By Lemma 1 we have that

$$\begin{aligned} & \max\left(\|\partial_t U_\varepsilon(t)\|_{L^2}, \|U_\varepsilon(t)\|_{H^1}\right) \\ \leq\ & \gamma\Big(\|\partial_t U_\varepsilon(0)\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + \int_0^T \|F_\varepsilon(U_\varepsilon(s))S_\varepsilon(s)\|_{L^2} ds\Big) \\ \leq\ & \gamma\Big(\|\partial_t U_\varepsilon(0)\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + \int_0^T \|F_\varepsilon(U_\varepsilon(s))\|_{L^\infty}\|S_\varepsilon(s)\|_{L^2} ds\Big) \\ \leq\ & \gamma\Big(\|\partial_t U_\varepsilon(0)\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + T\int_0^T a_\varepsilon^m \|S_\varepsilon(s)\|_{L^2} ds\Big),\ t < T. \end{aligned}$$

We immediately obtain the moderate bound for $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1}$.

Note that, for fixed $\omega \in \Omega$, $S_\varepsilon \in \mathcal{E}_{2,2}([0,T)\times\mathbb{R}) \subset \mathcal{E}_b([0,T)\times\mathbb{R})$; that gives us the $\mathcal{E}_b$-property of the regularized stochastic process S_ε which will be used in the considerations that follow.

In order to obtain moderate bounds for higher order derivatives of U_ε, we differentiate equation (9) with respect to the spatial variable x and obtain

$$(\partial_t^2 - \partial_x^2)\partial_x U_\varepsilon + F'_\varepsilon(U_\varepsilon)\partial_x U_\varepsilon S_\varepsilon + F_\varepsilon(U_\varepsilon)\partial_x S_\varepsilon = 0. \tag{11}$$

Energy inequality gives

$$\begin{aligned} & \|(\partial_{tx}U_\varepsilon, \partial_{xx}U_\varepsilon)(t)\|_{L^2} \leq \|(\partial_{tx}U_\varepsilon, \partial_{xx}U_\varepsilon)(0)\|_{L^2} \\ +\ & \int_0^T \|F'_\varepsilon(U_\varepsilon(s))\partial_x U_\varepsilon(s) S_\varepsilon(s)\|_{L^2} ds + \int_0^T \|F_\varepsilon(U_\varepsilon(s))\partial_x S_\varepsilon(s)\|_{L^2} ds \\ \leq\ & \|(\partial_{tx}U_\varepsilon, \partial_{xx}U_\varepsilon)(0)\|_{L^2} + \int_0^T \|F'_\varepsilon(U_\varepsilon(s))\|_{L^\infty}\|\partial_x U_\varepsilon(s)\|_{L^2}\|S_\varepsilon(s)\|_{L^\infty} ds \\ +\ & \int_0^T \|F_\varepsilon(U_\varepsilon(s))\|_{L^\infty}\|\partial_x S_\varepsilon(s)\|_{L^2} ds \\ \leq\ & \|(\partial_{tx}U_\varepsilon, \partial_{xx}U_\varepsilon)(0)\|_{L^2} + \int_0^T a_\varepsilon^m \|\partial_x U_\varepsilon(s)\|_{L^2}\|S_\varepsilon(s)\|_{L^\infty} ds \\ +\ & \int_0^T a_\varepsilon^m \|\partial_x S_\varepsilon(s)\|_{L^2} ds. \end{aligned}$$

Since $S_\varepsilon \in \mathcal{E}_b([0,T)\times\mathbb{R})$ and $\sup_{t\in[0,T)} \|\partial_x U_\varepsilon(t)\|_{L^2}$ is moderate, we obtain that $\sup_{t\in[0,T)} \|\partial_{xx} U_\varepsilon(t)\|_{L^2}$ is moderate, too.

Differentiating equation (11) we obtain

$$\begin{aligned} & (\partial_t^2 - \partial_x^2)\partial_{xx} U_\varepsilon + F''_\varepsilon(U_\varepsilon)(\partial_x U_\varepsilon)^2 S_\varepsilon + F'_\varepsilon(U_\varepsilon)\partial_{xx} U_\varepsilon S_\varepsilon \\ & + 2F'_\varepsilon(U_\varepsilon)\partial_x U_\varepsilon \partial_x S_\varepsilon + F_\varepsilon(U_\varepsilon)\partial_{xx} S_\varepsilon = 0. \end{aligned}$$

Similarly as above, by using Sobolev embedding theorems, we get

$$\begin{aligned} & \|(\partial_{txx}U_\varepsilon, \partial_{xxx}U_\varepsilon)(t)\|_{L^2} \leq \|(\partial_{txx}U_\varepsilon, \partial_{xxx}U_\varepsilon)(0)\|_{L^2} \\ +\ & \int_0^T \|F''_\varepsilon(U_\varepsilon(s))(\partial_x U_\varepsilon(s))^2 S_\varepsilon(s)\|_{L^2} ds + \int_0^T \|F'_\varepsilon(U_\varepsilon(s))\partial_{xx} U_\varepsilon(s) S_\varepsilon(s)\|_{L^2} ds \end{aligned}$$

$$
\begin{aligned}
&+ \; 2\int_0^T \|F_\varepsilon'(U_\varepsilon(s))\partial_x U_\varepsilon(s)\partial_x S_\varepsilon(s)\|_{L^2} ds + \int_0^T \|F_\varepsilon(U_\varepsilon(s))\partial_{xx} S_\varepsilon(s)\|_{L^2} ds \\
&\le \; \|(\partial_{txx}U_\varepsilon, \partial_{xxx}U_\varepsilon)(0)\|_{L^2} + \int_0^T \|F_\varepsilon''(U_\varepsilon(s))\|_{L^\infty}\|\partial_x U_\varepsilon(s)\|_{L^4}^2 \|S_\varepsilon(s)\|_{L^\infty} ds \\
&+ \; \int_0^T \|F_\varepsilon'(U_\varepsilon(s))\|_{L^\infty}\|S_\varepsilon(s)\|_{L^\infty}\|\partial_{xx}U_\varepsilon(s)\|_{L^2} ds \\
&+ \; 2\int_0^T \|F_\varepsilon'(U_\varepsilon(s))\|_{L^\infty}\|\partial_x U_\varepsilon(s)\|_{L^2}\|\partial_x S_\varepsilon(s)\|_{L^\infty} ds \\
&+ \; \int_0^T \|F_\varepsilon(U_\varepsilon(s))\|_{L^\infty}\|\partial_{xx} S_\varepsilon(s)\|_{L^2} ds \\
&\le \; \|(\partial_{txx}U_\varepsilon, \partial_{xxx}U_\varepsilon)(0)\|_{L^2} + C\int_0^T a_\varepsilon^{m_2}\|\partial_x U_\varepsilon(s)\|_{H_1}^2 \|S_\varepsilon(s)\|_{L^\infty} ds \\
&+ \; \int_0^T a_\varepsilon^{m}\|S_\varepsilon(s)\|_{L^\infty}\|\partial_{xx}U_\varepsilon(s)\|_{L^2} ds \\
&+ \; 2\int_0^T a_\varepsilon^{m}\|\partial_x S_\varepsilon(s)\|_{L^\infty}\|\partial_x U_\varepsilon(s)\|_{L^2} ds + \int_0^T a_\varepsilon^{m}\|\partial_{xx} S_\varepsilon(s)\|_{L^2} ds.
\end{aligned}
$$

Using the same argument as above we obtain that $\sup_{t\in[0,T)}\|\partial_{xxx}U_\varepsilon(t)\|_{L^2}$ is moderate.

In order to obtain moderate bounds for the L^2-norm of the m-th order derivative of U_ε, $\partial_x^m U_\varepsilon$, we only have to give bounds of the term that contains the highest order derivative of U_ε because in all other terms derivatives of order at most $m-2$ appear. Their L^∞-norms are bounded by L^2- norms of derivatives of order at most $m-1$ which are moderate from the previous step.

The term that contains the derivative of order $m-1$ (highest order derivative) is of the form

$$
\int_0^T \|F_\varepsilon'(U_\varepsilon(s))\partial_x^{(m-1)}U_\varepsilon(s)\|_{L^2} ds.
$$

Now we have

$$
\begin{aligned}
\int_0^T \|F_\varepsilon'(U_\varepsilon(s))\partial_x^{(m-1)}U_\varepsilon(s)\|_{L^2} ds &\le \int_0^T \|F_\varepsilon'(U_\varepsilon)\|_{L^\infty}\|\partial_x^{(m-1)}U_\varepsilon(s)\|_{L^2} ds \\
&\le \int_0^T a_\varepsilon^m \|\partial_x^{(m-1)}U_\varepsilon(s)\|_{L^2} ds.
\end{aligned}
$$

Since we have from the previous step that $\sup_{t\in[0,T)}\|\partial_x^{(m-1)}U_\varepsilon(t)\|_{L^2}$ has a moderate bound, the moderate bound for the L^2-norm of an arbitrary order derivative follows.

Derivatives of U_ε with respect to the time variable t can be estimated by derivatives of U_ε with respect to the spatial variable x by using the equation which we solve and by differentiating it. This argument will be also used in the uniqueness proof without particular mentioning. Thus, we proved that $U_\varepsilon \in \mathcal{E}_{2,2}^{\Omega}([0,T)\times\mathbb{R})$, i.e., $U = [U_\varepsilon] \in \mathcal{G}_{2,2}^{\Omega}([0,T)\times\mathbb{R})$ is solution to problem (3)-(4).

Let us show that this solution is unique in $\mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$, i.e. that for given two solutions to equation (9), $U_{1\varepsilon}, U_{2\varepsilon} \in \mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$, their difference $\bar{U}_\varepsilon := U_{1\varepsilon} - U_{2\varepsilon}$ belongs to $\mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$.

The following holds:

$$
\begin{aligned}
&(\partial_t^2 - \partial_x^2)\bar{U}_\varepsilon + (F_\varepsilon(U_{1\varepsilon}) - F_\varepsilon(U_{2\varepsilon}))S_\varepsilon + N_\varepsilon = 0,\\
&\bar{U}_\varepsilon|_{t=0} = N_{1\varepsilon},\ \partial_t\bar{U}_\varepsilon|_{t=0} = N_{2\varepsilon},
\end{aligned}
\tag{12}
$$

where $N_{1\varepsilon}, N_{2\varepsilon} \in \mathcal{N}^{\Omega}_{2,2}(\mathbb{R})$ and $N_\varepsilon \in \mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R})$.

Using Lemma 1 we obtain

$$
\begin{aligned}
&\max\left(\|\partial_t\bar{U}_\varepsilon(t)\|_{L^2}, \|\bar{U}_\varepsilon(t)\|_{H^1}\right)\\
\le\ & \gamma\Big(\|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon\|_{L^2}ds + \int_0^T \|(F_\varepsilon(U_{1\varepsilon}) - F_\varepsilon(U_{2\varepsilon}))S_\varepsilon\|_{L^2}ds\Big)\\
\le\ & \gamma\Big(\|N_{2\varepsilon}\|_{L^2}\|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon\|_{L^2}ds + \int_0^T \|F'_\varepsilon(\tilde{U}_\varepsilon)\|_{L^\infty}\|\bar{U}_\varepsilon\|_{L^\infty}\|S_\varepsilon\|_{L^2}ds\Big)\\
\le\ & \gamma\Big(\|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon\|_{L^2}ds + \int_0^T a_\varepsilon^m C\|\bar{U}_\varepsilon\|_{H^1}\|S_\varepsilon\|_{L^2}ds\Big),
\end{aligned}
$$

for some $\tilde{U}_\varepsilon \in (\min(U_{1\varepsilon}, U_{2\varepsilon}), \max(U_{1\varepsilon}, U_{2\varepsilon}))$.

Since S_ε satisfies (8) and the net a_ε satisfies (7), Gronwall's type inequality implies that $\sup_{t\in[0,T)}\|\bar{U}_\varepsilon(t)\|_{H^1}$ is negligible.

Let us consider higher order derivatives of $\bar{U}_\varepsilon$ and show that their L^2-norms are also negligible. For that purpose we differentiate equation (12) with respect to the spatial variable x. We obtain

$$
\begin{aligned}
&(\partial_t^2 - \partial_x^2)\partial_x\bar{U}_\varepsilon + F'_\varepsilon(U_{1\varepsilon})\partial_x U_{1\varepsilon}S_\varepsilon - F'_\varepsilon(U_{2\varepsilon})\partial_x U_{2\varepsilon}S_\varepsilon\\
&+(F_\varepsilon(U_{1\varepsilon}) - F_\varepsilon(U_{2\varepsilon}))\partial_x S_\varepsilon + \partial_x N_\varepsilon = 0.
\end{aligned}
$$

Energy inequality gives

$$
\begin{aligned}
&\|\partial_x\bar{U}_\varepsilon(t)\|_{H^1} \le \|(\partial_x N_{2\varepsilon}, \partial_{xx}N_{1\varepsilon})\|_{L^2} + \int_0^T \|\partial_x N_\varepsilon\|_{L^2}ds\\
+\ & \int_0^T \|F'_\varepsilon(U_{1\varepsilon})\partial_x U_{1\varepsilon} - F'_\varepsilon(U_{1\varepsilon})\partial_x U_{2\varepsilon}\|_{L^\infty}\|S_\varepsilon\|_{L^2}ds\\
+\ & \int_0^T \|F'_\varepsilon(U_{1\varepsilon})\partial_x U_{2\varepsilon} - F'_\varepsilon(U_{2\varepsilon})\partial_x U_{2\varepsilon}\|_{L^\infty}\|S_\varepsilon\|_{L^2}ds\\
+\ & \int_0^T \|F'_\varepsilon(\tilde{U}_\varepsilon)\|_{L^\infty}\|\bar{U}_\varepsilon\|_{L^\infty}\|\partial_x S_\varepsilon\|_{L^2}ds\\
\le\ & \|(\partial_x N_{2\varepsilon}, \partial_{xx}N_{1\varepsilon})\|_{L^2} + \int_0^T \|\partial_x N_\varepsilon\|_{L^2}ds + \int_0^T a_\varepsilon^m\|\partial_x\bar{U}_\varepsilon\|_{L^\infty}\|S_\varepsilon\|_{L^2}ds\\
+\ & \int_0^T a_\varepsilon^m\|\bar{U}_\varepsilon\|_{L^\infty}\|\partial_x U_{2\varepsilon}\|_{L^\infty}\|S_\varepsilon\|_{L^2}ds + C_1\int_0^T a_\varepsilon^{m_2}\|\bar{U}_\varepsilon\|_{L^\infty}\|\partial_x S_\varepsilon\|_{L^2}ds\\
\le\ & \|(\partial_x N_{2\varepsilon}, \partial_{xx}N_{1\varepsilon})|_{L^2} + \int_0^T \|\partial_x N_\varepsilon\|_{L^2}ds + C\Big(\int_0^T a_\varepsilon^m\|\partial_x\bar{U}_\varepsilon\|_{H^1}\|S_\varepsilon\|_{L^2}ds\\
+\ & \int_0^T a_\varepsilon^m\|\partial_x\bar{U}_\varepsilon\|_{L^2}\|\partial_x U_{2\varepsilon}\|_{L^\infty}\|S_\varepsilon\|_{L^2}ds + \int_0^T a_\varepsilon^{m_2}\|\bar{U}_\varepsilon\|_{H^1}\|\partial_x S_\varepsilon\|_{L^2}ds\Big),
\end{aligned}
$$

where $\tilde{U}_\varepsilon \in (\min(U_{1\varepsilon}, U_{2\varepsilon}), \max(U_{1\varepsilon}, U_{2\varepsilon}))$.

¿From the previous step we have that $\sup_{t\in[0,T)} \|\partial_x \bar{U}_\varepsilon(t)\|_{L^2}$ is negligible. Since a_ε and S_ε satisfy relations (7) and (8), respectively, Gronwall's inequality implies that $\sup_{t\in[0,T)} \|\partial_{xx} \bar{U}_\varepsilon(t)\|_{L^2}$ is negligible, too. Similarly, one can show that the L^2- norm of an arbitrary order derivative of $\bar{U}_\varepsilon$ is negligible.

Thus the proof is completed.

Remark 1 *In particular, one could choose the stochastic process S_ε to be smoothed white noise such that*

$$\dot{W}_\varepsilon = \left(\dot{W} * \frac{1}{h_\varepsilon}\phi\left(\frac{\cdot}{h_\varepsilon}\right)\right)\xi_\varepsilon,$$

where

$$h_\varepsilon = \frac{1}{\sqrt[2m]{\log \varepsilon^{-1}}},$$

and ξ_ε is a nonnegative net of smooth, compactly supported cut-off functions converging to the identity. The cut-off procedure is necessary to obtain L^2-moderate properties of the above function $\dot{W}_\varepsilon$.

3.3 The regularized and the nonregularized equation

Theorem 2 *Let G, a primitive function of F, be nonnegative and $G(0) = 0$. Let the Colombeau generalized stochastic process $S \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R})$ be nonnegative and depend only on the variable x, i.e., there exists a representative S_ε of S such that $S_\varepsilon(x) \geq 0$, for all ε small enough and $x \in \mathbb{R}$. Suppose that*

$$\|B_\varepsilon\|_{L^2} + \|A_\varepsilon\|_{H^1} = o(a_\varepsilon), \text{ as } \varepsilon \to 0, \tag{13}$$

where a_ε is the corresponding net used in the regularization of the function F.

Then, for every $T > 0$, the solution to the regularized equation (3)-(4) is also the solution to the nonregularized equation (1)-(2).

Proof. Standard energy estimates procedure gives

$$\frac{1}{2}\partial_t \int \Big((\partial_t U_\varepsilon(x,t))^2 + (\partial_x U_\varepsilon(x,t))^2 + G_\varepsilon(U_\varepsilon(x,t))S_\varepsilon(x)\Big)dx = 0.$$

Since $G_\varepsilon \geq 0$, by using the procedure given in the proof of Lemma 1, one can see that

$$\max\left(\|\partial_t U_\varepsilon(t)\|_{L^2}, \|U_\varepsilon(t)\|_{H^1}\right) \leq \gamma\Big(\|B_\varepsilon\|_{L^2} + \|A_\varepsilon\|_{H^1}\Big).$$

Using (13) and

$$\|U_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq C\|U_\varepsilon(t)\|_{H^1(\mathbb{R})},\ t \in [0,T),$$

for some $C > 0$, we obtain

$$\|U_\varepsilon(t)\|_{L^\infty(\mathbb{R})} \leq a_\varepsilon,\ t \in [0,T).$$

Thus,

$$F_\varepsilon(U_\varepsilon) = F(U_\varepsilon),$$

and the proof is completed.

Remark 2 *In particular, one can take the stochastic process S in Theorem 2 to be positive noise which depends only on the spatial variable x and with the corresponding growth property. Positive noise ([6]) is generalized stochastic process with representative*

$$\exp\left(\dot{W} * \varphi_\varepsilon - \frac{1}{2}\|\varphi_\varepsilon\|^2_{L^2}\right) \in \mathcal{E}(\mathbb{R}),$$

where $\dot{W}$ is the white noise. Again, one uses the cut-off procedure in order to obtain L^2-moderate properties of the above function. The same assertion holds.

4 3-dimensional stochastic wave and Klein-Gordon equations

In this section we consider the 3-dimensional stochastic cubic and subcubic wave and Klein-Gordon equations containing Colombeau generalized stochastic processes. There are four different cases depending on growth rates of the L^2-norms of the initial data as well as on the L^∞-norms of the Colombeau generalized stochastic processes.

4.1 Cubic wave equation with nonnegative stochastic process

We consider the problem

$$(\partial_t^2 - \triangle)U + U^3 \cdot S = 0, \tag{14}$$

$$U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B, \tag{15}$$

where we suppose that $A, B \in \mathcal{G}^\Omega_{2,2}(\mathbb{R}^3)$ are $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes such that

$$\|B_\varepsilon\|_{L^2} + \|A_\varepsilon\|_{H^1} = o\left(\left(\log\varepsilon^{-1}\right)^{1/4}\right), \tag{16}$$

and $S \in \mathcal{G}^\Omega_b(\mathbb{R}^3)$ is nonnegative $\mathcal{G}_b$-Colombeau generalized stochastic process which depends only on the variable x and such that

$$\|S_\varepsilon\|_{L^\infty} = o\left(\left(\log\varepsilon^{-1}\right)^{1/2}\right). \tag{17}$$

Theorem 3 *Let the stochastic processes $A, B \in \mathcal{G}^\Omega_{2,2}(\mathbb{R}^3)$ satisfy condition (16) and $S \in \mathcal{G}^\Omega_b(\mathbb{R}^3)$ be a nonnegative stochastic process that depends only on the variable x and satisfies (17). Then, for every $T > 0$, problem (14)-(15) has a unique solution almost surely in $\mathcal{G}^\Omega_{2,2}([0,T) \times \mathbb{R}^3)$.*

Proof. In the sequel $\omega \in \Omega$ and $\varepsilon \in (0,1)$ will be fixed.

Problem (14,15) given by the representatives reads

$$(\partial_t^2 - \triangle)U_\varepsilon + U_\varepsilon^3 \cdot S_\varepsilon = 0, \tag{18}$$

$$U_\varepsilon|_{\{t=0\}} = A_\varepsilon,\ \partial_t U_\varepsilon|_{\{t=0\}} = B_\varepsilon, \tag{19}$$

where $A_\varepsilon, B_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}(\mathbb{R}^3)$ satisfy (16) and $S_\varepsilon \in \mathcal{E}^{\Omega}_b(\mathbb{R}^3)$ is nonnegative and satisfies (17).

Using the nonnegativity of a primitive of the nonlinear term, similarly as in Lemma 1,

$$\max\left(\|\partial_t U_\varepsilon(t)\|_{L^2}, \|U_\varepsilon(t)\|_{H^1}\right) \le \gamma\Big(\|\partial_t U_\varepsilon(0)\|_{L^2} + \|U_\varepsilon(0)\|_{H^1}\Big)$$

and we immediately obtain the moderate bound for $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1}$. Moreover, $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1} = o\left(\left(\log \varepsilon^{-1}\right)^{1/4}\right)$.

As we remarked in the previous section, derivatives of U_ε with respect to time variable t can be estimated by derivatives of U_ε with respect to the spatial variables by using the equation which we solve and by differentiating it. This is the reason why we shall estimate only the space derivatives.

In order to obtain moderate bounds for L^2-norms of higher order derivatives of U_ε, we differentiate equation (18) with respect to some spatial variable and obtain

$$(\partial_t^2 - \triangle)\nabla U_\varepsilon + 3U_\varepsilon^2 \nabla U_\varepsilon S_\varepsilon + U_\varepsilon^3 \nabla S_\varepsilon = 0. \tag{20}$$

Energy inequality and Sobolev embedding theorems give

$$\begin{aligned}
& \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(t)\|_{L^2} \le \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} \\
+ \quad & 3\int_0^T \|U_\varepsilon^2(s)\nabla U_\varepsilon(s) S_\varepsilon\|_{L^2} ds + \int_0^T \|U_\varepsilon^3(s)\nabla S_\varepsilon\|_{L^2} ds \\
\le \quad & \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} + 3\int_0^T \|U_\varepsilon(s)\|_{L^6}^2 \|\nabla U_\varepsilon(s)\|_{L^6} \|S_\varepsilon\|_{L^\infty} ds \\
+ \quad & \int_0^T \|U_\varepsilon(s)\|_{L^6}^3 \|\nabla S_\varepsilon\|_{L^\infty} ds \\
\le \quad & \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} \\
+ \quad & C\Big(\int_0^T \|U_\varepsilon(s)\|_{H^1}^2 \|\nabla U_\varepsilon(s)\|_{H^1} \|S_\varepsilon\|_{L^\infty} ds + \int_0^T \|U_\varepsilon(s)\|_{H^1}^3 \|\nabla S_\varepsilon\|_{L^\infty} ds\Big).
\end{aligned}$$

Here and in the sequel C will denote some positive real. The first term in the right-hand side has the moderate bound. From the previous step and the fact that $S_\varepsilon \in \mathcal{E}_b(\mathbb{R}^3)$ we obtain that the third term has the moderate bound. Also, we know that $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1} = o\left(\left(\log \varepsilon^{-1}\right)^{1/4}\right)$. Using all those arguments as well as (17), one can apply Gronwall's inequality and obtain the moderate bound for $\sup_{t\in[0,T)} \|\nabla^2 U_\varepsilon(t)\|_{L^2}$.

By another differentiation with respect to some spatial variable we obtain

$$(\partial_t^2 - \triangle)\nabla^2 U_\varepsilon + 6U_\varepsilon(\nabla U_\varepsilon)^2 S_\varepsilon + 3U_\varepsilon^2 \nabla^2 U_\varepsilon S_\varepsilon + 6U_\varepsilon^2 \nabla U_\varepsilon \nabla S_\varepsilon + U_\varepsilon^3 \nabla^2 S_\varepsilon = 0.$$

Similarly as above one can get

$$\begin{aligned}
& \|(\partial_t \nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(t)\|_{L^2} \le \|(\partial_t \nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(0)\|_{L^2} \\
+ \quad & 6\int_0^T \|U_\varepsilon(s)\|_{L^6} \|\nabla U_\varepsilon(s)\|_{L^6}^2 \|S_\varepsilon\|_{L^\infty} ds
\end{aligned}$$

$$
\begin{aligned}
&+ \quad 3\int_0^T \|U_\varepsilon(s)\|^2_{L^\infty}\|\nabla^2 U_\varepsilon(s)\|_{L^2}\|S_\varepsilon\|_{L^\infty}ds \\
&+ \quad 6\int_0^T \|U_\varepsilon(s)\|^2_{L^6}\|\nabla U_\varepsilon(s)\|_{L^6}\|\nabla S_\varepsilon\|_{L^\infty}ds + \int_0^T \|U_\varepsilon(s)\|^3_{L^6}\|\nabla^2 S_\varepsilon\|_{L^\infty}ds \\
&\leq \quad \|(\partial_t\nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(0)\|_{L^2} + C\Big(\int_0^T \|U_\varepsilon(s)\|_{H^1}\|\nabla U_\varepsilon(s)\|^2_{H^1}\|S_\varepsilon\|_{L^\infty}ds \\
&+ \quad \int_0^T \|U_\varepsilon(s)\|^2_{H^2}\|\nabla^2 U_\varepsilon(s)\|_{L^2}\|S_\varepsilon\|_{L^\infty}ds \\
&+ \quad \int_0^T \|U_\varepsilon(s)\|^2_{H^1}\|\nabla U_\varepsilon(s)\|_{H^1}\|\nabla S_\varepsilon\|_{L^\infty}ds + \int_0^T \|U_\varepsilon(s)\|^3_{H^1}\|\nabla^2 S_\varepsilon\|_{L^\infty}ds\Big).
\end{aligned}
$$

Since $\sup_{t\in[0,T)}\|\nabla U_\varepsilon(t)\|_{L^2}$ and $\sup_{t\in[0,T)}\|\nabla^2 U_\varepsilon(t)\|_{L^2}$ have moderate bounds we obtain the moderate bound for $\sup_{t\in[0,T)}\|\nabla^3 U_\varepsilon(t)\|_{L^2}$.

Similarly, one can obtain moderate bounds for L^2-norms of higher order derivatives of U_ε. Thus, we have just proved that $U_\varepsilon \in \mathcal{E}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$, i.e., $U = [U_\varepsilon] \in \mathcal{G}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$ is a solution to problem (14)-(15).

Let us show that this solution is unique in $\mathcal{G}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$, i.e. that for given two solutions to equation (18), $U_{1\varepsilon}, U_{2\varepsilon} \in \mathcal{E}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$, their difference $\bar{U}_\varepsilon := U_{1\varepsilon} - U_{2\varepsilon}$ belongs to $\mathcal{N}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$. The following holds:

$$
\begin{aligned}
&(\partial_t^2 - \triangle)\bar{U}_\varepsilon + (U^3_{1\varepsilon} - U^3_{2\varepsilon})S_\varepsilon + N_\varepsilon = 0, \\
&\bar{U}_\varepsilon|_{t=0} = N_{1\varepsilon},\ \partial_t\bar{U}_\varepsilon|_{t=0} = N_{2\varepsilon},
\end{aligned} \tag{21}
$$

where $N_{1\varepsilon}, N_{2\varepsilon} \in \mathcal{N}^\Omega_{2,2}(\mathbb{R}^3)$ and $N_\varepsilon \in \mathcal{N}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$.

Using Lemma 1 and Sobolev embedding theorems we obtain

$$
\begin{aligned}
&\max\left(\|\partial_t\bar{U}_\varepsilon(t)\|_{L^2}, \|\bar{U}_\varepsilon(t)\|_{H^1}\right) \leq \gamma\Big(\|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon(s)\|_{L^2}ds \\
&+ \quad \int_0^T \|\left(U^2_{1\varepsilon}(s) + U_{1\varepsilon}(s)U_{2\varepsilon}(s) + U^2_{2\varepsilon}(s)\right)\bar{U}_\varepsilon S_\varepsilon\|_{L^2}ds\Big) \\
&\leq \quad \gamma\Big(\|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon(s)\|_{L^2}ds \\
&+ \quad \int_0^T \left(\|U_{1\varepsilon}(s)\|^2_{L^6} + \|U_{1\varepsilon}(s)\|_{L^6}\|U_{2\varepsilon}(s)\|_{L^6} + \|U_{2\varepsilon}(s)\|^2_{L^6}\right)\|\bar{U}_\varepsilon\|_{L^6}\|S_\varepsilon\|_{L^\infty}ds\Big) \\
&\leq \quad \gamma\Big(\|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon(s)\|_{L^2}ds\Big) \\
&+ \quad C\int_0^T \left(\|U_{1\varepsilon}(s)\|^2_{H^1} + \|U_{1\varepsilon}(s)\|_{H^1}\|U_{2\varepsilon}(s)\|_{H^1} + \|U_{2\varepsilon}(s)\|^2_{H^1}\right)\|\bar{U}_\varepsilon\|_{H^1}\|S_\varepsilon\|_{L^\infty}ds.
\end{aligned}
$$

Since we know that $\sup_{t\in[0,T)}\|U_{i\varepsilon}(t)\|_{H^1} = o\left(\left(\log\varepsilon^{-1}\right)^{1/4}\right)$, $i \in \{1,2\}$, and $\|S_\varepsilon\|_{L^\infty} = o\left(\left(\log\varepsilon^{-1}\right)^{1/2}\right)$, by using Gronwall's inequality we obtain that $\sup_{t\in[0,T)}\|\bar{U}_\varepsilon(t)\|_{H^1}$ is negligible.

Let us consider higher order derivatives of $\bar{U}_\varepsilon$ and show that their L^2-norms are also negligible. For that purpose we differentiate equation (21) with respect to some spatial variable and obtain

$$(\partial_t^2-\triangle)\nabla\bar{U}_\varepsilon+\left(3U_{1\varepsilon}^2\nabla U_{1\varepsilon}-3U_{2\varepsilon}^2\nabla U_{2\varepsilon}\right)S_\varepsilon+\left(U_{1\varepsilon}^3-U_{2\varepsilon}^3\right)\nabla S_\varepsilon+\nabla N_\varepsilon=0.$$

Energy inequality and Sobolev embedding theorems give

$$\begin{aligned}
&\|(\partial_t\nabla\bar{U}_\varepsilon,\nabla^2\bar{U}_\varepsilon)(t)\|_{L^2}\le\|(\nabla N_{2\varepsilon},\nabla^2N_{1\varepsilon})\|_{L^2}+\int_0^T\|\nabla N_\varepsilon(s)\|_{L^2}ds\\
+&\quad 3\int_0^T\|U_{1\varepsilon}^2(s)\nabla U_{1\varepsilon}(s)-U_{2\varepsilon}^2(s)\nabla U_{2\varepsilon}(s)\|_{L^2}\|S_\varepsilon\|_{L^\infty}ds\\
+&\quad \int_0^T\|\left(U_{1\varepsilon}^2(s)+U_{1\varepsilon}(s)U_{2\varepsilon}(s)+U_{2\varepsilon}^2(s)\right)\bar{U}_\varepsilon(s)\|_{L^2}\|\nabla S_\varepsilon\|_{L^\infty}ds\\
\le&\quad \|(\nabla N_{2\varepsilon},\nabla^2N_{1\varepsilon})\|_{L^2}+\int_0^T\|\nabla N_\varepsilon(s)\|_{L^2}ds\\
+&\quad 3\int_0^T\|U_{1\varepsilon}^2(s)\nabla U_{1\varepsilon}(s)-U_{1\varepsilon}^2(s)\nabla U_{2\varepsilon}(s)\|_{L^2}\|S_\varepsilon\|_{L^\infty}ds\\
+&\quad 3\int_0^T\|U_{1\varepsilon}^2(s)\nabla U_{2\varepsilon}(s)-U_{2\varepsilon}^2(s)\nabla U_{2\varepsilon}(s)\|_{L^2}\|S_\varepsilon\|_{L^\infty}ds\\
+&\quad \int_0^T\left(\|U_{1\varepsilon}(s)\|_{L^6}^2+\|U_{1\varepsilon}(s)\|_{L^6}\|U_{2\varepsilon}(s)\|_{L^6}+\|U_{2\varepsilon}(s)\|_{L^6}^2\right)\|\bar{U}_\varepsilon\|_{L^6}\|\nabla S_\varepsilon\|_{L^\infty}ds\\
\le&\quad \|(\nabla N_{2\varepsilon},\nabla^2N_{1\varepsilon})\|_{L^2}+\int_0^T\|\nabla N_\varepsilon(s)\|_{L^2}ds\\
+&\quad 3\int_0^T\|U_{1\varepsilon}(s)\|_{L^6}^2\|\nabla\bar{U}_\varepsilon(s)\|_{L^6}\|S_\varepsilon\|_{L^\infty}ds\\
+&\quad 3\int_0^T\|\left(U_{1\varepsilon}(s)-U_{2\varepsilon}(s)\right)\left(U_{1\varepsilon}(s)+U_{2\varepsilon}(s)\right)\nabla U_{2\varepsilon}(s)\|_{L^2}\|S_\varepsilon\|_{L^\infty}ds\\
+&\quad C_1\int_0^T\left(\|U_{1\varepsilon}(s)\|_{H^1}^2+\|U_{1\varepsilon}(s)\|_{H^1}\|U_{2\varepsilon}(s)\|_{H^1}+\|U_{2\varepsilon}(s)\|_{H^1}^2\right)\|\bar{U}_\varepsilon\|_{H^1}\|\nabla S_\varepsilon\|_{L^\infty}ds\\
\le&\quad \|(\nabla N_{2\varepsilon},\nabla^2N_{1\varepsilon})\|_{L^2}+\int_0^T\|\nabla N_\varepsilon(s)\|_{L^2}ds\\
+&\quad C\Big(\int_0^T\|U_{1\varepsilon}(s)\|_{H^1}^2\|\nabla\bar{U}_\varepsilon(s)\|_{H^1}\|S_\varepsilon\|_{L^\infty}ds\\
+&\quad \int_0^T\|\bar{U}_\varepsilon(s)\|_{H^1}\left(\|U_{1\varepsilon}(s)\|_{H^1}+\|U_{2\varepsilon}(s)\|_{H^1}\right)\|\nabla U_{2\varepsilon}(s)\|_{H^1}\|S_\varepsilon\|_{L^\infty}ds\\
+&\quad \int_0^T\left(\|U_{1\varepsilon}(s)\|_{H^1}^2+\|U_{1\varepsilon}(s)\|_{H^1}\|U_{2\varepsilon}(s)\|_{H^1}+\|U_{2\varepsilon}(s)\|_{H^1}^2\right)\|\bar{U}_\varepsilon\|_{H^1}\|\nabla S_\varepsilon\|_{L^\infty}ds\Big).
\end{aligned}$$

Using similar arguments as above we obtain that $\sup_{t\in[0,T)}\|\nabla^2\bar{U}_\varepsilon(t)\|_{L^2}$ is negligible. Similarly, one can show that the L^2-norms of all derivatives of $\bar{U}_\varepsilon$ are negligible. That concludes the proof.

Remark 3 *Note that any Colombeau stochastic generalized process S can be regularized in such way that estimate (17) holds. This remark could be added after each further assertion when we need estimates on a stochastic term.*

4.2 Cubic wave equation with multiplicative stochastic process

We consider the problem

$$(\partial_t^2 - \triangle)U + U \cdot S + U^3 = 0, \tag{22}$$

$$U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B, \tag{23}$$

where the stochastic processes $A, B \in \mathcal{G}_{2,2}^{\Omega}(\mathbb{R}^3)$ are such that

$$\|B_\varepsilon\|_{L^2} + \|A_\varepsilon\|_{H^1} = o\left(\left(\log \varepsilon^{-1}\right)^{1/2}\right), \tag{24}$$

and $S \in \mathcal{G}_b^{\Omega}([0,T) \times \mathbb{R}^3)$ is such that

$$\|S_\varepsilon\|_{L^\infty} = o\left(\log\left(\log \varepsilon^{-1}\right)^{\frac{1}{2T}}\right). \tag{25}$$

Theorem 4 *Let the $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes $A, B \in \mathcal{G}_{2,2}^{\Omega}(\mathbb{R}^3)$ satisfy condition (24) and the $\mathcal{G}_b$-Colombeau stochastic process $S \in \mathcal{G}_b^{\Omega}([0,T) \times \mathbb{R}^3)$ satisfy (25). Then, for every $T > 0$, problem (22)-(23) has a unique solution almost surely in $\mathcal{G}_{2,2}^{\Omega}([0,T) \times \mathbb{R}^3)$.*

Proof. In the sequel $\omega \in \Omega$ and $\varepsilon \in (0,1)$ will be fixed.

We consider problem (22)-(23) given by the representatives:

$$(\partial_t^2 - \triangle)U_\varepsilon + U_\varepsilon \cdot S_\varepsilon + U_\varepsilon^3 = 0, \tag{26}$$

$$U_\varepsilon|_{\{t=0\}} = A_\varepsilon,\ \partial_t U_\varepsilon|_{\{t=0\}} = B_\varepsilon, \tag{27}$$

where $A_\varepsilon, B_\varepsilon \in \mathcal{E}_{2,2}^{\Omega}(\mathbb{R}^3)$ satisfy (24) and $S_\varepsilon \in \mathcal{E}_b^{\Omega}([0,T) \times \mathbb{R}^3)$ satisfies (25).

Again, by using a procedure like in the proof of Theorem 3, we obtain

$$\begin{aligned}
&\max\left(\|\partial_t U_\varepsilon(t)\|_{L^2}, \|U_\varepsilon(t)\|_{H^1}\right) \\
\leq\ & \gamma\Big(\|\partial_t U_\varepsilon\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + \int_0^T \|U_\varepsilon(s) S_\varepsilon(s)\|_{L^2} ds\Big) \\
\leq\ & \gamma\Big(\|\partial_t U_\varepsilon\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + \int_0^T \|U_\varepsilon(s)\|_{L^2} \|S_\varepsilon(s)\|_{L^\infty} ds\Big) \\
\leq\ & \gamma\Big(\|\partial_t U_\varepsilon\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + \int_0^T \|U_\varepsilon(s)\|_{H^1} \|S_\varepsilon(s)\|_{L^\infty} ds\Big).
\end{aligned}$$

Since (25) holds, Gronwall's inequality implies the moderate bound for $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1}$. Moreover, $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1} = o\left(\left(\log \varepsilon^{-1}\right)^{1/2}\right)$.

In order to obtain moderate bounds for L^2-norms of higher order derivatives of U_ε, we differentiate equation (26) with respect to some spatial variable and obtain

$$(\partial_t^2 - \triangle)\nabla U_\varepsilon + \nabla U_\varepsilon S_\varepsilon + U_\varepsilon \nabla S_\varepsilon + 3U_\varepsilon^2 \nabla U_\varepsilon = 0. \tag{28}$$

Energy inequality and Sobolev embedding theorems give

$$\begin{aligned}
&\|(\partial_t\nabla U_\varepsilon, \nabla^2 U_\varepsilon)(t)\|_{L^2} \\
\le\ & \|(\partial_t\nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla U_\varepsilon(s) S_\varepsilon(s)\|_{L^2} ds \\
+\ & \int_0^T \|U_\varepsilon(s)\nabla S_\varepsilon(s)\|_{L^2} ds + 3\int_0^T \|U_\varepsilon^2(s)\nabla U_\varepsilon(s)\|_{L^2} ds \\
\le\ & \|(\partial_t\nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla U_\varepsilon(s)\|_{L^2}\|S_\varepsilon(s)\|_{L^\infty} ds \\
+\ & \int_0^T \|U_\varepsilon(s)\|_{L^2}\|\nabla S_\varepsilon(s)\|_{L^\infty} ds + C\int_0^T \|U_\varepsilon(s)\|_{H^1}^2\|\nabla U_\varepsilon(s)\|_{H^1} ds.
\end{aligned}$$

Since $\sup_{t\in[0,T)}\|U_\varepsilon(t)\|_{H^1} = o\Big(\big(\log\varepsilon^{-1}\big)^{1/2}\Big)$ Gronwall's inequality gives the moderate bound for $\sup_{t\in[0,T)}\|\nabla^2 U_\varepsilon(t)\|_{L^2}$.

By another differentiation with respect to some spatial variable we obtain

$$(\partial_t^2 - \triangle)\nabla^2 U_\varepsilon + \nabla^2 U_\varepsilon S_\varepsilon + 2\nabla U_\varepsilon \nabla S_\varepsilon + U_\varepsilon\nabla^2 S_\varepsilon + 6U_\varepsilon(\nabla U_\varepsilon)^2 + 3U_\varepsilon^2\nabla^2 U_\varepsilon = 0.$$

Similarly as above one can get

$$\begin{aligned}
&\|(\partial_t\nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(t)\|_{L^2} \\
\le\ & \|(\partial_t\nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla^2 U_\varepsilon(s)\|_{L^2}\|S_\varepsilon(s)\|_{L^\infty} ds \\
+\ & 2\int_0^T \|\nabla U_\varepsilon(s)\|_{L^2}\|\nabla S_\varepsilon(s)\|_{L^\infty} ds + \int_0^T \|U_\varepsilon(s)\|_{L^2}\|\nabla^2 S_\varepsilon(s)\|_{L^\infty} ds \\
+\ & 6\int_0^T \|U_\varepsilon(s)\|_{L^6}\|\nabla U_\varepsilon(s)\|_{L^6}^2 ds + 3\int_0^T \|U_\varepsilon(s)\|_{L^\infty}^2\|\nabla^2 U_\varepsilon(s)\|_{L^2} ds \\
\le\ & \|(\partial_t\nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla^2 U_\varepsilon(s)\|_{L^2}\|S_\varepsilon(s)\|_{L^\infty} ds \\
+\ & 2\int_0^T \|\nabla U_\varepsilon(s)\|_{L^2}\|\nabla S_\varepsilon(s)\|_{L^\infty} ds + \int_0^T \|U_\varepsilon(s)\|_{L^2}\|\nabla^2 S_\varepsilon(s)\|_{L^\infty} ds \\
+\ & C\Big(\int_0^T \|U_\varepsilon(s)\|_{H^1}\|\nabla U_\varepsilon(s)\|_{H^1}^2 ds + \int_0^T \|U_\varepsilon(s)\|_{H^2}^2\|U_\varepsilon(s)\|_{H^2} ds\Big).
\end{aligned}$$

Since $\sup_{t\in[0,T)}\|U_\varepsilon(t)\|_{H^2}$, $\sup_{t\in[0,T)}\|U_\varepsilon(t)\|_{H^1}$ and $\sup_{t\in[0,T)}\|\nabla^2 U_\varepsilon(t)\|_{L^2}$ have moderate bounds, $\sup_{t\in[0,T)}\|\nabla^3 U_\varepsilon(t)\|_{L^2}$ has a moderate bound, too.

Similarly, one can obtain the moderate bounds for L^2-norms of higher order derivatives of U_ε. Thus, we have just proved that $U_\varepsilon \in \mathcal{E}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$, i.e., $U = [U_\varepsilon] \in \mathcal{G}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$ is solution to problem (22)-(23).

Let us show that this solution is unique in $\mathcal{G}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$, i.e. that for given two solutions to equation (26), $U_{1\varepsilon}, U_{2\varepsilon} \in \mathcal{E}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$, their difference $\bar U_\varepsilon := U_{1\varepsilon} - U_{2\varepsilon}$ belongs to $\mathcal{N}^\Omega_{2,2}([0,T)\times\mathbb{R}^3)$.

The following holds:

$$\begin{aligned}
&(\partial_t^2 - \triangle)\bar U_\varepsilon + \bar U_\varepsilon S_\varepsilon + U_{1\varepsilon}^3 - U_{2\varepsilon}^3 + N_\varepsilon = 0, \\
&\bar U_\varepsilon|_{t=0} = N_{1\varepsilon},\ \partial_t \bar U_\varepsilon|_{t=0} = N_{2\varepsilon},
\end{aligned} \tag{29}$$

where $N_{1\varepsilon}, N_{2\varepsilon} \in \mathcal{N}^{\Omega}_{2,2}(\mathbb{R}^3)$ and $N_\varepsilon \in \mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$.

Using the fact that

$$\begin{aligned}\|U_{1\varepsilon}^3 - U_{2\varepsilon}^3\|_{L^2} &\leq \|U_{1\varepsilon} - U_{2\varepsilon}\|_{L^6}\|U_{1\varepsilon}^2 + U_{1\varepsilon}U_{2\varepsilon} + U_{2\varepsilon}^2\|_{L^3} \\ &\leq \|\bar{U}_\varepsilon\|_{H^1}\left(\|U_{1\varepsilon}\|_{H^1}^2 + \|U_{1\varepsilon}\|_{H^1}\|U_{2\varepsilon}\|_{H^1} + \|U_{2\varepsilon}\|_{H^1}^2\right),\end{aligned}$$

similarly as above we obtain

$$\begin{aligned}&\max\left(\|(\partial_t\bar{U}_\varepsilon\|_{L^2}, \|\bar{U}_\varepsilon\|_{H^1}\right) \\ \leq\ & \gamma\Big(\|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon(s)\|_{L^2}ds + \int_0^T \|\bar{U}_\varepsilon(s)\|_{L^2}\|S_\varepsilon(s)\|_{L^\infty}ds\Big) \\ +\ & C\int_0^T \left(\|U_{1\varepsilon}(s)\|_{H^1}^2 + \|U_{1\varepsilon}(s)\|_{H^1}\|U_{2\varepsilon}(s)\|_{H^1} + \|U_{2\varepsilon}(s)\|_{H^1}^2\right)\|\bar{U}_\varepsilon(s)\|_{H^1}ds.\end{aligned}$$

Gronwall's inequality implies that $\sup_{t\in[0,T)}\|\bar{U}_\varepsilon(t)\|_{H^1}$ is negligible. Similarly, one can show that the L^2-norms of all derivatives of $\bar{U}_\varepsilon$ are negligible.

Again, in both the existence and the uniqueness proof, derivatives of U_ε with respect to the time variable t can be estimated by derivatives of U_ε with respect to spatial variables by using the equation which we solve and by differentiating it. Thus, the proof is completed.

4.3 Klein-Gordon equation with additive stochastic process

We consider the problem

$$(\partial_t^2 - \triangle)U + U + U^3 + S = 0, \tag{30}$$

$$U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B, \tag{31}$$

where the stochastic processes $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R}^3)$ satisfy

$$\|B_\varepsilon\|_{L^2} + \|A_\varepsilon\|_{H^1} = o\left(\left(\log\varepsilon^{-1}\right)^{1/2}\right), \tag{32}$$

and $S \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$ is such that

$$\|S_\varepsilon\|_{L^\infty} = o\left(\left(\log\varepsilon^{-1}\right)^{1/2}\right) \tag{33}$$

and

(34) S_ε has a compact support.

Theorem 5 *Let the $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R}^3)$ and $S \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$ satisfy conditions (32) and (33)-(34), respectively. Then, for $T > 0$, the problem (30)-(31) has a unique solution almost surely in $\mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$.*

Proof. In the sequel $\omega \in \Omega$ and $\varepsilon \in (0,1)$ will be fixed.

We consider problem (30)-(31) given by the representatives:

$$(\partial_t^2 - \triangle)U_\varepsilon + U_\varepsilon + U_\varepsilon^3 + S_\varepsilon = 0 \tag{35}$$

$$(36) \qquad U_\varepsilon|_{\{t=0\}} = A_\varepsilon,\ \partial_t U_\varepsilon|_{\{t=0\}} = B_\varepsilon,$$

where $A_\varepsilon, B_\varepsilon \in \mathcal{E}^\Omega_{2,2}(\mathbb{R}^3)$ satisfy (32) and $S_\varepsilon \in \mathcal{E}^\Omega_{2,2}([0,T) \times \mathbb{R}^3)$ satisfies conditions (33) and (34).

Like in the beginning of the proof of Theorem 3

$$\max\left(\|\partial_t U_\varepsilon(t)\|_{L^2}, \|U_\varepsilon(t)\|_{H^1}\right) \le \gamma\Big(\|\partial_t U_\varepsilon(0)\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + \int_0^T \|S_\varepsilon(s)\|_{L^2} ds\Big).$$

The moderate bound for $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1}$ immediately follows. Moreover, $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1} = o\left(\left(\log \varepsilon^{-1}\right)^{1/2}\right)$.

In order to obtain moderate bounds for L^2-norms of higher order derivatives of U_ε, we differentiate equation (35) with respect to some spatial variable and obtain

$$(37) \qquad (\partial_t^2 - \triangle)\nabla U_\varepsilon + \nabla U_\varepsilon + 3U_\varepsilon^2 \nabla U_\varepsilon + \nabla S_\varepsilon = 0.$$

Energy inequality and Sobolev embedding theorems give

$$\begin{aligned}
&\|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(t)\|_{L^2} \le \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla U_\varepsilon(s)\|_{L^2} ds \\
+ \quad & 3\int_0^T \|U_\varepsilon^2(s)\nabla U_\varepsilon(s)\|_{L^2} ds + \int_0^T \|\nabla S_\varepsilon(s)\|_{L^2} ds \\
\le \quad & \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla U_\varepsilon(s)\|_{L^2} ds \\
+ \quad & C\int_0^T \|U_\varepsilon(s)\|^2_{H^1} \|\nabla U_\varepsilon(s)\|_{H^1} ds + \int_0^T \|\nabla S_\varepsilon(s)\|_{L^2} ds.
\end{aligned}$$

Using $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1} = o\left(\left(\log \varepsilon^{-1}\right)^{1/2}\right)$ one can apply Gronwall's inequality and obtain the moderate bound for $\sup_{t\in[0,T)} \|\nabla^2 U_\varepsilon(t)\|_{L^2}$.

By another differentiation with respect to some spatial variable we obtain

$$(\partial_t^2 - \triangle)\nabla^2 U_\varepsilon + \nabla^2 U_\varepsilon + 6U_\varepsilon(\nabla U_\varepsilon)^2 + 3U_\varepsilon^2 \nabla^2 U_\varepsilon + \nabla^2 S_\varepsilon = 0.$$

Similarly as above one can get

$$\begin{aligned}
&\|(\partial_t \nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(t)\|_{L^2} \le \|(\partial_t \nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(0)\|_{L^2} \\
+ \quad & \int_0^T \|\nabla^2 U_\varepsilon(s)\|_{L^2} ds + C\Big(\int_0^T \|U_\varepsilon(s)\|_{H^1} \|\nabla U_\varepsilon(s)\|^2_{H^1} ds \\
+ \quad & \int_0^T \|U_\varepsilon(s)\|^2_{H^1} \|\nabla^2 U_\varepsilon(s)\|_{H^1} ds\Big) + \int_0^T \|\nabla^2 S_\varepsilon(s)\|_{L^2} ds.
\end{aligned}$$

Using the same argument as above and applying that $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1}$ and $\sup_{t\in[0,T)} \|\nabla U_\varepsilon(t)\|_{H^1}$ have moderate bounds we obtain the moderate bound for $\sup_{t\in[0,T)} \|\nabla^3 U_\varepsilon(t)\|_{L^2}$.

Similarly, one can obtain the moderate bounds for L^2-norms of higher order derivatives of U_ε. Thus, we have just proved that $U_\varepsilon \in \mathcal{E}^\Omega_{2,2}([0,T) \times \mathbb{R}^3)$, i.e., $U = [U_\varepsilon] \in \mathcal{G}^\Omega_{2,2}([0,T) \times \mathbb{R}^3)$ is a solution to problem (30)-(31).

Let us show that this solution is unique in $\mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$, i.e. that for given two solutions to equation (35), $U_{1\varepsilon}, U_{2\varepsilon} \in \mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$, their difference $\bar{U_\varepsilon} := U_{1\varepsilon} - U_{2\varepsilon}$ belongs to $\mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$. The following holds:

$$\begin{aligned}(38)\qquad &(\partial_t^2 - \triangle)\bar{U_\varepsilon} + \bar{U_\varepsilon} + U_{1\varepsilon}^3 - U_{2\varepsilon}^3 + N_\varepsilon = 0,\\ &\bar{U_\varepsilon}|_{t=0} = N_{1\varepsilon},\ \partial_t\bar{U_\varepsilon}|_{t=0} = N_{2\varepsilon},\end{aligned}$$

where $N_{1\varepsilon}, N_{2\varepsilon} \in \mathcal{N}^{\Omega}_{2,2}(\mathbb{R}^3)$ and $N_\varepsilon \in \mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$.

As above

$$\begin{aligned}&\max\left(\|\partial_t\bar{U_\varepsilon}\|_{L^2}, \|\bar{U_\varepsilon}\|_{H^1}\right)\\ \leq\ &\gamma\Big(\|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon(s)\|_{L^2}ds + \int_0^T \|\bar{U_\varepsilon}(s)\|_{L^2}ds\Big)\\ +\ &C\int_0^T \left(\|U_{1\varepsilon}(s)\|^2_{H^1} + \|U_{1\varepsilon}(s)\|_{H^1}\|U_{2\varepsilon}(s)\|_{H^1} + \|U_{2\varepsilon}(s)\|^2_{H^1}\right)\|\bar{U_\varepsilon}(s)\|_{H^1}ds.\end{aligned}$$

Since $\sup_{t\in[0,T)}\|U_{i\varepsilon}(t)\|_{H^1} = o\left(\left(\log\varepsilon^{-1}\right)^{1/2}\right)$, $i=1,2$, one can again apply Gronwall's inequality and obtain that $\sup_{t\in[0,T)}\|\bar{U_\varepsilon}(t)\|_{H^1}$ is negligible. Similarly, one can show that L^2-norms of all derivatives of $\bar{U_\varepsilon}$ are negligible.

As in the previous cases, in both the existence and the uniqueness proof, derivatives of U_ε with respect to the time variable t can be estimated by derivatives of U_ε with respect to spatial variables by using the equation which we solve and by differentiating it. Thus, the proof is completed.

4.4 Stochastic wave equation with Lipschitz nonlinearities

Let f and g be globally Lipschitz functions, polynomially bounded together with all their derivatives and such that $f(0) = g(0) = 0$.

Consider the problem

$$(\partial_t^2 - \triangle)U + f(U)S_1 + g(U) + S_2 = 0, \tag{39}$$

$$U|_{\{t=0\}} = A,\ \partial_t U|_{\{t=0\}} = B, \tag{40}$$

where $A, B \in \mathcal{G}^{\Omega}_{2,2}(\mathbb{R}^3)$ are $\mathcal{G}_{2,2}$-Colombeau generalized stochastic processes. The $\mathcal{G}_b$-Colombeau generalized stochastic process $S_1 \in \mathcal{G}^{\Omega}_b([0,T)\times\mathbb{R}^3)$ is such that

$$\|S_{1\varepsilon}\|_{L^\infty} = o\left(\log\varepsilon^{-1}\right), \tag{41}$$

and the $\mathcal{G}_{2,2}$-Colombeau generalized stochastic process $S_2 \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$ has compact support.

Theorem 6 *Suppose that the stochastic processes A and B belong to $\mathcal{G}^{\Omega}_{2,2}(\mathbb{R}^3)$, $S_1 \in \mathcal{G}^{\Omega}_b([0,T)\times\mathbb{R}^3)$ satisfies condition (41) and that $S_2 \in \mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$ has compact support. Let f and g be globally Lipschitz functions, polynomially bounded together with all their derivatives and such that $f(0) = g(0) = 0$. Then, for every $T > 0$, problem (39)-(40) has a unique solution almost surely in $\mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$.*

Proof. In the sequel $\omega \in \Omega$ and $\varepsilon \in (0,1)$ will be fixed.

We consider problem (39)-(40) given by the representatives:

$$(\partial_t^2 - \triangle)U_\varepsilon + f(U_\varepsilon)S_{1\varepsilon} + g(U_\varepsilon) + S_{2\varepsilon} = 0, \tag{42}$$

$$U_\varepsilon|_{\{t=0\}} = A_\varepsilon,\ \partial_t U_\varepsilon|_{\{t=0\}} = B_\varepsilon, \tag{43}$$

where $A_\varepsilon, B_\varepsilon \in \mathcal{E}^\Omega_{2,2}(\mathbb{R}^3)$, $S_{1\varepsilon} \in \mathcal{E}^\Omega_b([0,T) \times \mathbb{R}^3)$ and $S_{2\varepsilon} \in \mathcal{E}^\Omega_{2,2}([0,T) \times \mathbb{R}^3)$ satisfy the conditions given in the statement of the Theorem.

Using Lemma 1

$$\begin{aligned}
& \max\left(\|\partial_t U_\varepsilon(t)\|_{L^2}, \|U_\varepsilon(t)\|_{H^1}\right) \\
\leq\ & \gamma\Big(\|\partial_t U_\varepsilon(0)\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + \int_0^T \|f(U_\varepsilon(s))\|_{L^2}\|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+\ & \int_0^T \|g(U_\varepsilon(s))\|_{L^2} ds + \int_0^T \|S_{2\varepsilon}(s)\|_{L^2} ds\Big) \\
\leq\ & \gamma\|\partial_t U_\varepsilon(0)\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + C\Big(\int_0^T \|U_\varepsilon(s)\|_{L^2}\|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+\ & \int_0^T \|U_\varepsilon(s)\|_{L^2} ds + \int_0^T \|S_{2\varepsilon}(s)\|_{L^2} ds\Big) \\
\leq\ & \gamma\|\partial_t U_\varepsilon(0)\|_{L^2} + \|U_\varepsilon(0)\|_{H^1} + C\Big(\int_0^T \|U_\varepsilon(s)\|_{H^1}\|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+\ & \int_0^T \|U_\varepsilon(s)\|_{H^1} ds + \int_0^T \|S_{2\varepsilon}(s)\|_{L^2} ds\Big),
\end{aligned}$$

where we have used the Lipschitz property of the functions f and g and Sobolev embedding theorems.

Since the regularized stochastic process $S_{1\varepsilon}$ satisfies (41) one can apply Gronwall's inequality and obtain the moderate bound for $\sup_{t\in[0,T)} \|U_\varepsilon(t)\|_{H^1}$.

In order to obtain moderate bounds for L^2-norms of higher order derivatives of U_ε, we differentiate equation (42) with respect to some spatial variable and obtain

$$(\partial_t^2 - \triangle)\nabla U_\varepsilon + f'(U_\varepsilon)\nabla U_\varepsilon S_{1\varepsilon} + f(U_\varepsilon)\nabla S_{1\varepsilon} + g'(U_\varepsilon)\nabla U_\varepsilon + \nabla S_{2\varepsilon} = 0.$$

Energy inequality and Sobolev embedding theorems give

$$\begin{aligned}
& \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(t)\|_{L^2} \\
\leq\ & \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} + \int_0^T \|f'(U_\varepsilon(s))\|_{L^\infty}\|\nabla U_\varepsilon(s)\|_{L^2}\|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+\ & \int_0^T \|f(U_\varepsilon(s))\|_{L^2}\|\nabla S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+\ & \int_0^T \|g'(U_\varepsilon(s))\|_{L^\infty}\|\nabla U_\varepsilon(s)\|_{L^2} ds + \int_0^T \|\nabla S_{2\varepsilon}(s)\|_{L^2} ds \\
\leq\ & \|(\partial_t \nabla U_\varepsilon, \nabla^2 U_\varepsilon)(0)\|_{L^2} + C\Big(\int_0^T \|\nabla U_\varepsilon(s)\|_{L^2}\|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+\ & \int_0^T \|U_\varepsilon(s)\|_{L^2}\|\nabla S_{1\varepsilon}(s)\|_{L^\infty} ds + \int_0^T \|\nabla U_\varepsilon(s)\|_{L^2} ds + \int_0^T \|\nabla S_{2\varepsilon}(s)\|_{L^2} ds\Big).
\end{aligned}$$

Applying that $\sup_{t\in[0,T)}\|U_\varepsilon(t)\|_{L^2}$ and $\sup_{t\in[0,T)}\|\nabla U_\varepsilon(t)\|_{L^2}$ have moderate bounds we obtain the moderate bound for $\sup_{t\in[0,T)}\|\nabla^2 U_\varepsilon(t)\|_{L^2}$.

By another differentiation with respect to some spatial variable we obtain

$$
\begin{aligned}
&(\partial_t^2-\triangle)\nabla^2 U_\varepsilon + f''(U_\varepsilon)(\nabla U_\varepsilon)^2 S_{1\varepsilon} + f'(U_\varepsilon)\nabla^2 U_\varepsilon S_{1\varepsilon} + 2f'(U_\varepsilon)\nabla U_\varepsilon \nabla S_{1\varepsilon}\\
&+f(U_\varepsilon)\nabla^2 S_{1\varepsilon} + g''(U_\varepsilon)(\nabla U_\varepsilon)^2 + g'(U_\varepsilon)\nabla^2 U_\varepsilon + \nabla^2 S_{2\varepsilon} = 0.
\end{aligned}
$$

Similarly as above one can get

$$
\begin{aligned}
&\|(\partial_t\nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(t)\|_{L^2} \le \|(\partial_t\nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(0)\|_{L^2}\\
+ \;& \int_0^T \|f''(U_\varepsilon(s))\|_{L^\infty}\|\nabla U_\varepsilon(s)\|^2_{H^1}\|S_{1\varepsilon}(s)\|_{L^\infty}ds\\
+ \;& \int_0^T \|f'(U_\varepsilon(s))\|_{L^\infty}\|\nabla^2 U_\varepsilon(s)\|_{L^2}\|S_{1\varepsilon}(s)\|_{L^\infty}ds\\
+ \;& 2\int_0^T \|f'(U_\varepsilon(s))\|_{L^\infty}\|\nabla U_\varepsilon(s)\|_{L^2}\|\nabla S_{1\varepsilon}(s)\|_{L^\infty}ds\\
+ \;& \int_0^T \|f(U_\varepsilon(s))\|_{L^2}\|\nabla^2 S_{1\varepsilon}(s)\|_{L^\infty}ds + \int_0^T \|g''(U_\varepsilon(s))\|_{L^\infty}\|\nabla U_\varepsilon(s))\|^2_{H^1}ds\\
+ \;& \int_0^T \|g'(U_\varepsilon(s))\|_{L^\infty}\|\nabla^2 U_\varepsilon(s)\|_{L^2}ds + \int_0^T \|\nabla^2 S_{2\varepsilon}(s)\|_{L^2}ds\\
\le \;& \|(\partial_t\nabla^2 U_\varepsilon, \nabla^3 U_\varepsilon)(0)\|_{L^2} + C\Big((1+\int_0^T \|U_\varepsilon(s)\|^{q_1}_{H^2})\|\nabla U_\varepsilon(s)\|^2_{H^1}\|S_{1\varepsilon}(s)\|_{L^\infty}ds\\
+ \;& \int_0^T \|\nabla^2 U_\varepsilon(s)\|_{L^2}\|S_{1\varepsilon}(s)\|_{L^\infty}ds + \int_0^T \|\nabla U_\varepsilon(s)\|_{L^2}\|\nabla S_{1\varepsilon}(s)\|_{L^\infty}ds\\
+ \;& \int_0^T \|U_\varepsilon(s)\|_{L^2}\|\nabla^2 S_{1\varepsilon}(s)\|_{L^\infty}ds + \int_0^T (1+\|U_\varepsilon(s)\|^{q_2}_{H^2})\|\nabla U_\varepsilon(s))\|^2_{H^1}ds\\
+ \;& \int_0^T \|\nabla^2 U_\varepsilon(s)\|_{L^2}ds + \int_0^T \|\nabla^2 S_{2\varepsilon}(s)\|_{L^2}ds\Big),
\end{aligned}
$$

for some $q_1, q_2 \in \mathbb{N}$.

Using similar arguments as we did above we obtain the moderate bound for $\sup_{t\in[0,T)}\|\nabla^3 U_\varepsilon(t)\|_{L^2}$. Similarly, one can obtain the moderate bounds for L^2-norms of higher order derivatives of U_ε. Thus, we have just proved that $U_\varepsilon \in \mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$, i.e., $U=[U_\varepsilon]\in\mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$ is a solution to problem (39)-(40).

Let us show that this solution is unique in $\mathcal{G}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$, i.e. that for given two solutions to equation (42), $U_{1\varepsilon}, U_{2\varepsilon}\in\mathcal{E}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$, their difference $\bar{U}_\varepsilon := U_{1\varepsilon}-U_{2\varepsilon}$ belongs to $\mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$. The following holds:

$$
\begin{aligned}
(44)\qquad &(\partial_t^2-\triangle)\bar{U}_\varepsilon + (f(U_{1\varepsilon})-f(U_{2\varepsilon}))\,S_{1\varepsilon} + g(U_{1\varepsilon}) - g(U_{2\varepsilon}) + N_\varepsilon = 0,\\
&\bar{U}_\varepsilon|_{t=0} = N_{1\varepsilon},\ \partial_t\bar{U}_\varepsilon|_{t=0} = N_{2\varepsilon},
\end{aligned}
$$

where $N_{1\varepsilon}, N_{2\varepsilon}\in\mathcal{N}^{\Omega}_{2,2}(\mathbb{R}^3)$ and $N_\varepsilon\in\mathcal{N}^{\Omega}_{2,2}([0,T)\times\mathbb{R}^3)$.

By Lemma 1

$$
\max\left(\|\partial_t\bar{U}_\varepsilon(t)\|_{L^2}, \|\bar{U}_\varepsilon(t)\|_{H^1}\right) \le \gamma\Big(\|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} + \int_0^T \|N_\varepsilon(s)\|_{L^2}ds
$$

$$
\begin{aligned}
+ & \int_0^T \|f(U_{1\varepsilon}(s)) - f(U_{2\varepsilon}(s))\|_{L^2} \|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+ & \int_0^T \|g(U_{1\varepsilon}(s)) - g(U_{2\varepsilon}(s))\|_{L^2} ds\Big) \\
\leq & \gamma \|N_{2\varepsilon}\|_{L^2} + \|N_{1\varepsilon}\|_{H^1} \\
+ & C\Big(\int_0^T \|N_\varepsilon(s)\|_{L^2} ds + \int_0^T \|\bar{U}_\varepsilon(s)\|_{H^1} \|S_{1\varepsilon}(s)\|_{L^\infty} ds + \int_0^T \|\bar{U}_\varepsilon(s)\|_{H^1} ds\Big),
\end{aligned}
$$

where we have used the Lipschitz property of the functions f and g and Sobolev embedding theorems.

Since $S_{1\varepsilon}$ satisfies condition (41) one can apply Gronwall's inequality and obtain that $\sup_{t\in[0,T)} \|\bar{U}_\varepsilon(t)\|_{H^1}$ is negligible.

By differentiating (44) with respect to some spatial variable we obtain

$$
\begin{aligned}
&(\partial_t^2 - \triangle)\nabla \bar{U}_\varepsilon + \big(f'(U_{1\varepsilon})\nabla U_{1\varepsilon} - f'(U_{2\varepsilon})\nabla U_{2\varepsilon}\big)\, S_{1\varepsilon} \\
&+ \left(f(U_{1\varepsilon}) - f(U_{2\varepsilon})\right) \nabla S_{1\varepsilon} + g'(U_{1\varepsilon})\nabla U_{1\varepsilon} - g'(U_{2\varepsilon})\nabla U_{2\varepsilon} + \nabla N_\varepsilon = 0.
\end{aligned}
$$

Again, energy inequality and Sobolev embedding theorems give

$$
\begin{aligned}
& \|(\partial_t \nabla \bar{U}_\varepsilon, \nabla^2 \bar{U}_\varepsilon)(t)\|_{L^2} \leq \|(\partial_t \nabla \bar{U}_\varepsilon, \nabla^2 \bar{U}_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla N_\varepsilon(s)\|_{L^2} ds \\
+ & \int_0^T \|f'(U_{1\varepsilon}(s))\nabla U_{1\varepsilon}(s) - f'(U_{1\varepsilon}(s))\nabla U_{2\varepsilon}(s)\|_{L^2} \|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+ & \int_0^T \|f'(U_{1\varepsilon}(s))\nabla U_{2\varepsilon}(s) - f'(U_{2\varepsilon}(s))\nabla U_{2\varepsilon}(s)\|_{L^2} \|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+ & \int_0^T \|f(U_{1\varepsilon}(s)) - f(U_{2\varepsilon}(s))\|_{L^2} \|\nabla S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+ & \int_0^T \|g'(U_{1\varepsilon}(s))\nabla U_{1\varepsilon}(s) - g'(U_{1\varepsilon}(s))\nabla U_{2\varepsilon}(s)\|_{L^2} ds \\
+ & \int_0^T \|g'(U_{1\varepsilon}(s))\nabla U_{2\varepsilon}(s) - g'(U_{2\varepsilon}(s))\nabla U_{2\varepsilon}(s)\|_{L^2} ds \\
\leq & \|(\partial_t \nabla \bar{U}_\varepsilon, \nabla^2 \bar{U}_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla N_\varepsilon(s)\|_{L^2} ds \\
+ & \int_0^T \|f'(U_{1\varepsilon}(s))\|_{L^\infty} \|\nabla \bar{U}_\varepsilon(s)\|_{L^2} \|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+ & \int_0^T \|f'(U_{1\varepsilon}(s)) - f'(U_{2\varepsilon}(s))\|_{L^4} \|\nabla U_{2\varepsilon}(s)\|_{L^4} \|S_{1\varepsilon}(s)\|_{L^\infty} ds \\
+ & C_f \int_0^T \|\bar{U}_\varepsilon(s)\|_{L^2} \|\nabla S_{1\varepsilon}(s)\|_{L^\infty} ds + \int_0^T \|g'(U_{1\varepsilon}(s))\|_{L^\infty} \|\nabla \bar{U}_\varepsilon(s)\|_{L^2} ds \\
+ & \int_0^T \|g'(U_{1\varepsilon}(s)) - g'(U_{2\varepsilon}(s))\|_{L^4} \|\nabla U_{2\varepsilon}(s)\|_{L^4} ds \\
\leq & \|(\partial_t \nabla \bar{U}_\varepsilon, \nabla^2 \bar{U}_\varepsilon)(0)\|_{L^2} + \int_0^T \|\nabla N_\varepsilon(s)\|_{L^2} ds \\
+ & C\Big(\int_0^T \|\nabla \bar{U}_\varepsilon(s)\|_{L^2} \|S_{1\varepsilon}(s)\|_{L^\infty} ds
\end{aligned}
$$

$$+ \int_0^T \|\bar{U_\varepsilon}(s))\|_{H^1} \|\nabla U_{2\varepsilon}(s)\|_{H^1} \|S_{1\varepsilon}(s)\|_{L^\infty} ds$$
$$+ \int_0^T \|\bar{U_\varepsilon}(s)\|_{L^2} \|\nabla S_{1\varepsilon}(s)\|_{L^\infty} ds + \int_0^T \|\nabla \bar{U_\varepsilon}(s)\|_{L^2} ds$$
$$+ \int_0^T \|\bar{U_\varepsilon}(s)\|_{H^1} \|\nabla U_{2\varepsilon}(s)\|_{H^1} ds\Big),$$

where C_f is Lipschitz constant for function f.

Similarly, one can show that the L^2-norms of all derivatives of $\bar{U_\varepsilon}$ are negligible. Derivatives of U_ε with respect to the time variable t can be estimated by derivatives of U_ε with respect to spatial variables by using the equation which we solve and differentiating it. Thus, the proof is completed.

5 Acknowledgment

The authors would like to express their gratitude to Michael Oberguggenberger for his interest and many valuable suggestions for improvement.

References

[1] S. Albeverio, Z. Haba, F. Russo, *Trivial solutions for a non-linear two-space-dimensional wave equation perturbed by space-time white noise*, Stochastics and Stochastics Reports 56, no. 1-2, 127-160, 1996.

[2] H. A. Biagioni, *A Nonlinear Theory of Generalized Functions*, Lect. Not. Math. 1421, Springer, Berlin, 1990.

[3] H. A. Biagioni, M. Oberguggenberger, *Generalized solutions to the Korteveg-de Vries and the regularized long-wave equations*, SIAM J. Math. Anal. 23, 923-940, 1992.

[4] J. F. Colombeau, *Elementary Introduction in New Generalized Functions*, Nort Holland, Amsterdam, 1985.

[5] T. Hida, H. H. Kuo, J. Potthoff, L. Streit, *White Noise, An Infinite Dimensional Calculus*, Kluwer, Dordrecht, 1993.

[6] H. Holden, B. Øksendal, J. Ubøe, T. S. Zhang, *Stochastic Partail Differential Equations*, Birkhäuser-Verlag, Basel, 1996.

[7] R. Leandre, F. Russo, *Small stochastic perturbation of a one-dimensional wave equation*, Koerezlioglu, H. (ed.) et al., *Stochastic analysis and related topics*. Boston: Birkhaeuser. Prog. Probab. 31, 285-332, 1992.

[8] M. Nedeljkov, M. Oberguggenberger, S. Pilipović, *Generalized solutions to nonlinear wave equations*, Preprint.

[9] M. Nedeljkov, S. Pilipović, D. Scarpalézos, *The Linear Theory of Colombeau Generalized Functions*, Pitman Research Notes in Mathematics Series, Longman, Essex, 1998.

[10] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations*, Pitman Res. Not. Math. 259, Longman Sci. Techn., Essex, 1992.

[11] M. Oberguggenberger, F. Russo, *Nonlinear stochastics wave equations*, Integral Transforms and Special Functions, Vol. 6, No. 1-4, 71-83, 1998.

[12] M. Oberguggenberger, F. Russo, *Nonlinear SPDE's: Colombeau solutions and pathwise limits*, In: L. Decreusefond, J. Gjerde, B. Øksendal, A. S. Üstünel (Eds.), *Stochastic Analysis and Related Topics VI*. Birkhäuser, Boston, 319-332, 1998.

[13] S. Peszat, J. Zabczyk, *Nonlinear stochastic wave and heat equations*, Probab. Th. Rel. Fields 116, 421-443, 2000.

[14] F. Russo, *Colombeau generalized functions and stochastic analysis*, In: A.I. Cardoso, M. de Faria, J. Potthoff, R. Sénéor, L. Streit (Eds.), *Analysis and Applications in Physics*. Kluwer, Dordrecht, 329-349, 1994.

[15] M. Struwe, *Semilinear wave equations*, Bull. Am. Math. Soc. 26, 53-85, 1992.

[16] J. B. Walsh, *An Introduction to Stochastic Partial Differential Equations*, In: R. Carmona, H. Kesten, J. B. Walsh (Eds), *École D'Été de Probabilités de Saint Flour XIV*, Springer Lecture Notes Vol. 1180, Springer-Verlag, 265-439, New York, 1980.