THE VARIETY GENERATED BY TOURNAMENTS

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1 Introduction

By a tournament we mean a directed graph $(T, \to)$ such that whenever x, y are two distinct elements of T, then precisely one of the two cases, either $x \to y$ or $y \to x$, takes place. There is a one-to-one correspondence between tournaments and commutative groupoids satisfying $ab \in \{a, b\}$ for all a and b: set $ab = a$ if and only if $a \to b$. This makes it possible to identify tournaments with their corresponding groupoids and employ algebraic methods for their investigation.

So, an equivalent definition is: A tournament is a commutative groupoid, every subset of which is a subgroupoid. For two elements a and b of a tournament, we set $a \to b$ if and only if $ab = a$.

The aim of this paper is to investigate the the variety of groupoids generated by tournaments. This variety will be denoted by $\mathbf{T}$. We have started the investigation in our previous paper [9], in which it is proved that the variety is not finitely based. Here we will find a four-element base for the three-variable equations of $\mathbf{T}$, and proceed to investigate subdirectly irreducible algebras in $\mathbf{T}$. Our main effort will be focused on an attempt to find a positive solution to a conjecture, which has several equivalent formulations:

Conjecture 1. (1) *Every subdirectly irreducible algebra in* $\mathbf{T}$ *is a tournament.*

(2) *Every finite, subdirectly irreducible algebra in* $\mathbf{T}$ *is a tournament.*

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(3) *If A is a subalgebra of a direct product of finitely many finite tournaments, then every subdirectly irreducible homomorphic image of A is a tournament.*
(4) $\mathbf{T}$ *is the same as the quasi-variety* $\mathbf{T}_q$ *generated by tournaments.*
(5) *For every quasi-equation ϕ which is valid in all tournaments, there is a finite set Γ of equations true in all tournaments such that $\Gamma \vdash \phi$.*
(6) *For every $A \in \mathbf{T}_q$ and $a, b \in A$ and congruence ψ of A, we have $(\theta(a, ab) \vee \psi) \wedge (\theta(b, ab) \vee \psi = \psi$. (Here $\theta(a, b)$ denotes the congruence generated by (a, b).)*

The equivalence of these various formulations is easy to see. (Use the fact that the variety $\mathbf{T}$ is locally finite; this has been proved in [9].) We have not been able to prove the conjecture. We will prove here that it is true in various special cases.

For any $n \geq 1$, let $\mathbf{T}_n$ denote the variety generated by all n-element tournaments, and let $\mathbf{T}^n$ denote the variety determined by the at most n-variable equations of tournaments. So, $\mathbf{T}_n \subseteq \mathbf{T}_{n+1} \subseteq \mathbf{T} \subseteq \mathbf{T}^{n+1} \subseteq \mathbf{T}^n$ for all n.

Our proof in [9] relied on the construction of an infinite sequence $\mathbf{M}_n$ ($n \geq 3$) with the following properties: $\mathbf{M}_n$ is subdirectly irreducible, $|\mathbf{M}_n| = n + 2$ and $\mathbf{M}_n \in \mathbf{T}^n - \mathbf{T}^{n+1}$. These algebras will play an important role also in the present paper. They are defined as follows. $\mathbf{M}_n = \{a, b_0, \ldots, b_n\}$; the commutative and idempotent multiplication is defined by

$$\begin{aligned}
&ab_1 = b_0, \\
&ab_i = b_i \text{ for } i \leq n-1 \text{ and } i \neq 1, \\
&ab_n = a, \\
&b_i b_{i+1} = b_i \text{ for } i < n-1, \\
&b_n b_{n-1} = b_n, \\
&b_i b_j = b_{\max(i,j)} \text{ for } |i-j| \geq 2 \text{ and } i, j < n, \\
&b_n b_i = b_i \text{ for } i < n-1.
\end{aligned}$$

Here we will need to take one more similar algebra under consideration. We denote it by $\mathbf{J}_3$. It is defined as follows. $\mathbf{J}_3 = \{a, b_0, b_1, b_2, b_3\}$; $au_1 = u_0$; $u_0 \to u_1 \to u_2 \to u_0$, $a \to u_3 \to u_1 \to u_2 \to a$ and $u_2 \to u_3 \to u_0$.

We will prove later that $\mathbf{J}_3$ is a subdirectly irreducible algebra belonging to $\mathbf{T}^3 - \mathbf{T}^4$.

Conjecture 2. *Every subdirectly irreducible algebra from* $\mathbf{T}^3$ *is either a tournament or contains a subalgebra isomorphic to either* $\mathbf{J}_3$ *or* $\mathbf{M}_n$ *for some* $n \geq 3$.

This conjecture is even stronger than the more interesting Conjecture 1. However, it may happen that it would be easier to prove it in this form. We are also going to confirm this stronger conjecture in some special cases. We were able to verify, making use of a computer program, that it is true for all algebras with at most ten elements. (It turns out that there are 18399858 isomorphism types of subdirectly irreducible ten-element algebras in $\mathbf{T}^3$; 8874054 of them are not tournaments.)

We denote by $\mathbf{F}_n$ the free groupoid in $\mathbf{T}$ on n generators.

Theorem 3. $\mathbf{F}_n$ *is a free groupoid on* n *generators in* $\mathbf{T}_n$, *as well as in* $\mathbf{T}^n$.

Proof. Denote by A the free groupoid in $\mathbf{T}_n$ on n generators, by B the free groupoid in $\mathbf{T}^n$ on n generators and by h the canonical homomorphism of B onto A. All we need to do is to check that h is an isomorphism. Let a, b be two elements of B such that $h(a) = h(b)$. If f is a homomorphism of B into a tournament, then $f(B)$ is an at most n-element tournament, so that there exists a homomorphism g of A into $f(B)$ with $f = gh$; we get $f(a) = f(b)$. This means that the equation $a \approx b$ is satisfied in all tournaments, and thus $a = b$. □

2 Three-variable equations of tournaments

Theorem 4. *The following five equations are a base for the equational theory of* $\mathbf{T}^3$:

(1)	$xx = x$
(2)	$xy = yx$
(3)	$xy \cdot x = xy$
(4)	$(xy \cdot xz)(xy \cdot yz) = xyz$

The free groupoid $\mathbf{F}_3$ *has 15 elements*

$a = x$	$d = xy$	$g = yzx$	$j = xy \cdot xz$	$m = yxzx = yzxz$
$b = y$	$e = xz$	$h = xzy$	$k = yx \cdot yz$	$n = zxyx = zyxy$
$c = z$	$f = yz$	$i = xyz$	$l = zx \cdot zy$	$o = xyzy = xzyz$

The commutative multiplication is shown in the table given below. Moreover, the following equations are consequences of (1),...,(4):

$$(xy \cdot xz)x = xy \cdot xz \tag{5}$$
$$(xy \cdot xz) \cdot yz = xyzy \tag{6}$$
$$xyzy = xzyz \tag{7}$$
$$(yzx)(xy \cdot xz) = xy \cdot xz \tag{8}$$
$$xzyxz = xyz \tag{9}$$
$$yzx \cdot xyzy = yzx \tag{10}$$
$$yzx \cdot xzy = zyxy \tag{11}$$
$$yzx \cdot xy = zyxy \tag{12}$$
$$(xy \cdot xz)(zyxy) = xy \cdot xz \tag{13}$$
$$yxzx \cdot zyxy = xy \cdot xz \tag{14}$$
$$(xy \cdot xz)(xyzy) = xyzy \tag{15}$$
$$xy \cdot zxyx = zxyx \tag{16}$$
$$(xy \cdot xz)(yxzx) = xy \cdot xz \tag{17}$$
$$x(xy \cdot yz) = yzx \tag{18}$$
$$(yzx)(yx \cdot yz) = yzx \tag{19}$$
$$xy \cdot yxzx = xy \cdot xz \tag{20}$$

	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o
a	a	d	e	d	e	g	g	n	m	j	g	g	m	n	g
b	d	b	f	d	h	f	n	h	o	h	k	h	h	n	o
c	e	f	c	i	e	f	m	o	i	i	i	l	m	i	o
d	d	d	i	d	j	k	n	n	i	j	k	n	j	n	k
e	e	h	e	j	e	l	m	h	m	j	m	l	m	j	l
f	g	f	f	k	l	f	g	o	o	o	k	l	l	k	o
g	g	n	m	n	m	g	g	n	m	j	g	g	m	n	g
h	n	h	o	n	h	o	n	h	o	h	k	h	h	n	o
i	m	o	i	i	m	o	m	o	i	i	i	l	m	i	o
j	j	h	i	j	j	o	j	h	i	j	i	h	j	j	o
k	g	k	i	k	m	k	g	k	i	i	k	g	m	k	k
l	g	h	l	n	l	l	g	h	l	h	g	l	l	n	l
m	m	h	m	j	m	l	m	h	m	j	m	l	m	j	l
n	n	n	i	n	j	k	n	n	i	j	k	n	j	n	k
o	g	o	o	k	l	o	g	o	o	o	k	l	l	k	o

*Proof.*Put $X = xy$, $Y = xz$ and $Z = yz$; LS is the left and RS is the right side of the equation to be proved.

(5) $LS =_{(4)} ((xy \cdot xz)(xy \cdot x))((xy \cdot xz)(xz \cdot x)) =_{(3)} ((xy \cdot xz) \cdot xy)((xy \cdot xz) \cdot xz) =_{(3)} (xy \cdot xz)(xy \cdot xz) =_{(1)} RS$.

(6) $LS = XYZ =_{(4)} (XY \cdot XZ)(XY \cdot YZ) =_{(3)} ((XY \cdot XZ)(XY \cdot YZ))(XY \cdot XZ) =_{(3)} ((XY \cdot XZ)(XY \cdot YZ))((XY \cdot XZ) \cdot XZ) =_{(1,3)} ((XY \cdot XZ)(XY \cdot (XY \cdot YZ)))((XY \cdot XZ)(XZ \cdot XZ)) =_{(3)} ((XY \cdot XZ)((XY \cdot X)(XY \cdot (XY \cdot YZ))))((XY \cdot XZ)((XZ \cdot X)(XZ \cdot Z))) =_{(3,4)} ((XY \cdot XZ)((XY \cdot xyy)(XY \cdot xzy)))((XY \cdot XZ)((XZ \cdot xyy)(XZ \cdot yzy))) =_{(4)} (((xy \cdot xz)(xy \cdot yz))((xy \cdot xz)y))(((xy \cdot xz)(xy \cdot yz))((xy \cdot yz)y)) =_{(4)} ((xy \cdot xz)(xy \cdot yz))y =_{(4)} RS$.

(7) $LS =_{(6)} (xy \cdot xz) \cdot yz =_{(2)} (xz \cdot xy) \cdot zy =_{(6)} RS$.

(8) $LS =_{(4)} (yz \cdot yx)(yz \cdot zx) \cdot (xy \cdot xz) =_{(2)} (ZX \cdot ZY) \cdot XY =_{(6)} ZYXY = (xy \cdot (xz \cdot yz)) \cdot xz =_{(6)} zyxy \cdot xz =_{(7)} zxyx \cdot xz =_{(7)} (yx \cdot zx)x =_{(5)} RS$.

(9) $LS =_{(4)} (xzyx \cdot xz)(xzyx \cdot xzyz) =_{(7)} (x(xz \cdot xy))(x(xz \cdot y) \cdot z(xz \cdot y)) =_{(5)} (xz \cdot xy)(x(xz \cdot y) \cdot z(xz \cdot y)) =_{(4)} (xz \cdot xy)(((xz \cdot y)(xz \cdot x) \cdot ((xz \cdot y) \cdot yx))((xz \cdot y)(z \cdot xz) \cdot ((xz \cdot y) \cdot yz))) =_{(3)} (xz \cdot xy)(((xz \cdot y) \cdot yx)((xz \cdot y) \cdot yz)) =_{(4)} (xy \cdot xz)((xy \cdot (xy \cdot xz)(yz \cdot xz))(yz \cdot (xy \cdot xz)(yz \cdot xz))) = XY \cdot (X(XY \cdot YZ) \cdot (XY \cdot YZ)Z) =_{(7)} XY \cdot ((YZ \cdot (X(Y \cdot YZ)))(XY \cdot (Z(Y \cdot XY)))) =_{(3)} XY \cdot ((X \cdot YZ)(Z \cdot XY)) = (xy \cdot xz)((xy \cdot (xz \cdot yz))(yz \cdot (xy \cdot xz))) =_{(7)} (X \cdot YZ)(XY \cdot (Z(X \cdot YZ))) =_{(7)} ((X \cdot YZ) \cdot XY) \cdot X(Z \cdot XY) =_{(7)} (X \cdot YZ) \cdot X(XY \cdot XZ) =_{(5)} (X \cdot YZ)(XY \cdot XZ) =_{(8)} XY \cdot XZ = (xy \cdot xz)(xy \cdot yz) =_{(4)} RS$.

(10) $LS =_{(4,6)} (zy \cdot xz)(zy \cdot yx) \cdot (zy \cdot (xz \cdot xy)) = (ZY \cdot ZX)(Z \cdot YX) =_{(8)} ZY \cdot XZ = (yz \cdot xz)(xy \cdot yz) =_{(5)} RS$.

(11) $LS =_{(4)} (yx \cdot yz)(zx \cdot zy) \cdot (xy \cdot xz)(zx \cdot zy) = (XZ \cdot YZ)(XY \cdot YZ) =_{(5)} ZY \cdot X = (yz \cdot xz) \cdot xy =_{(6)} RS$.

(12) $LS =_{(9)} yzxxyx \cdot yzx =_{(3)} yzxyx \cdot yzx =_{(7)} zxyxx \cdot yzx =_{(3)} zxyx \cdot yzx =_{(7)} zyxy \cdot yzx =_{(3)} RS$.

(13) $LS =_{(6)} (xy \cdot (xz \cdot zy))(xy \cdot xz) =_{(12)} ((yz \cdot xz) \cdot xy) \cdot xz =_{(6)} zyxy \cdot xz =_{(7)} zxyx \cdot xz =_{(7)} (yx \cdot zx)x =_{(5)} RS$.

(14) $LS =_{(6)} (zx \cdot (yz \cdot yx))(xy \cdot (zx \cdot zy)) =_{(11)} ((yz \cdot xz) \cdot xy) \cdot xz =_{(12)} ((yz \cdot xz) \cdot xy)(xy \cdot xz) =_{(6)} (zyxy)(xy \cdot xz) =_{(13)} RS$.

(15) $LS =_{(6)} (xy \cdot xz)(yz \cdot (xy \cdot xz)) =_{(3)} yz \cdot (xy \cdot xz) =_{(6)} RS$.

(16) $LS =_{(9)} zxyxxyx \cdot zxyx =_{(3)} zxyxyx \cdot zxyx =_{(3,4)} zxyx \cdot zxyx =_{(1)} RS$.

(17) $LS = (X \cdot xz)(Xz \cdot x) =_{(11)} X(x \cdot zX) =_{(7)} (zx \cdot X)x = (zx \cdot xy)x =_{(5)} RS$.

(18) $LS =_{(7)} ((y \cdot yz)x) \cdot yz =_{(3)} RS$.

(19) $LS =_{(12)} (yx \cdot yz)x =_{(18)} RS$.

(20) $LS =_{(7)} (zx \cdot xy)x =_{(5)} RS$.

Now that the equations are proved, we can start to build the free groupoid on three generators x, y, z. Equations (1),…,(20) imply that the fifteen terms $a, \dots, o$ multiply among each other, with respect to the equational theory of $\mathbf{T}^3$, as in the table. Consequently, the free groupoid can have no more than fifteen elements. Clearly, $a, \dots, f$ are distinct from each other and from each of the elements $g, \dots, o$. The last nine elements are also distinct from each other: one can easily check that the terms behave differently on the three-element cycle. □

Lemma 5. *Let $A \in \mathbf{T}^3$ and let $a, b, c \in A$. Then:*

(1) *If $ab \to c$, then a, b, c generate a semilattice.*
(2) *If $ab \to c \to a$, then $bc = ab$.*
(3) *If $a \to c \to ab$, then $c \to b$.*
(4) *If $a \to c$ and $b \to c$, then $ab \to c$.*
(5) *If $a \to c \to b$ and a, b, c, ab are four distinct elements, then the subgroupoid generated by a, b, c either contains just these four elements and $c \to ab$, or else it contains precisely five elements $a, b, c, ab, ab \cdot c$ and $a \to ab \cdot c \to b$.*

Proof. Each of these situations generates a congruence in $\mathbf{F}_n$, and the congruence can be easily described from the multiplication table of the fifteen element free groupoid given above. □

Lemma 6. *In every algebra from $\mathbf{T}$, if $a \to c_1 \to \dots \to c_n \to b$, then $a \to abc_n \dots c_1$.*

Proof. We have to prove the quasiequation

$$xz_1 = x \& z_1 z_2 = z_1 \& \dots \& z_n y = z_n \Longrightarrow xyz_n \dots z_1 x = x.$$

As it is easy to see, the quasiequation is equivalent to the equation

$$(yz_n \dots z_1 x)y(yz_n)(yz_n z_{n-1}) \dots (yz_n \dots z_1)(yz_n \dots z_1 x) = yz_n \dots z_1 x$$

in all algebras from $\mathbf{T}$. It is easy to check that the quasiequation is satisfied by all tournaments. From this the result follows. □

3 Quasitournaments

By a quasitournament we mean a graph $(A, \rightarrow)$ where $\rightarrow$ is a binary relation satisfying the following three conditions:

(1) $a \rightarrow a$ for all $a \in A$;
(2) if $a \rightarrow b$ and $b \rightarrow a$, then $a = b$;
(3) for any pair a, b of elements of A there exists an element $c \in A$ such that $c \rightarrow a$, $c \rightarrow b$ and whenever c' is an element with $c' \rightarrow a$ and $c' \rightarrow b$, then $c' \rightarrow c$.

Clearly, the element c in the last condition is uniquely determined. We will denote it by ab. In this way, every quasitournament becomes a groupoid satisfying

(1) $xy = yx$,
(2) $xx = x$,
(3) $x \cdot xy = xy$,
(4) $xz = z \& yz = z \Longrightarrow xy \cdot z = z$.

On the other hand, it is easy to check that every groupoid satisfying these four quasiequations is a quasitournament with respect to the relation $\rightarrow$ defined by $a \rightarrow b$ if and only if $ab = b$, and this is a one-to-one correspondence between quasitournaments and the groupoids satisfying the four quasiequations. We will identify the two classes. So, the class of quasitournaments is a quasivariety; it will be denoted by $\mathbf{Q}$.

Lemma 7. *We have $\mathbf{T} \subset \mathbf{T}^3 \subset \mathbf{T}' \subset \mathbf{Q}$, where $\mathbf{T}'$ is the variety determined by the following four equations:*

(1) $xy = yx$,
(2) $xx = x$,
(3) $x \cdot xy = xy$,
(4) $((xz \cdot y)x)z = xy \cdot z$.

The variety generated by $\mathbf{Q}$ is equal to $\mathbf{T}^2$.

Proof. The first assertions are easy to see. In order to prove the last, it is sufficient to show that the free groupoids in $\mathbf{T}^2$ are quasitournaments.

Clearly, $\mathbf{T}^2$ is the variety of commutative idempotent groupoids satisfying $(xy)y = xy$.

Let X be a nonempty set. Denote by F the free commutative groupoid over X. If u, v are two elements of F, we say that u is a subterm of v if $v = uw_1 \dots w_n$ for some $w_1, \dots, w_n \in F$ ($n \geq 0$). Denote by G the set of all elements of F that contain no subterm uu or $(uv)v$ (for any $u, v \in F$). Define a binary operation $*$ on G as follows: if $u = v$, then $u * v = u$; if $v = uw$ for some w, then $u * v = v$; if $u = vw$ for some w, then $u * v = u$; in all other cases, put $u * v = uv$. It is easy to prove that G is a commutative and idempotent groupoid satisfying $(xy)y = xy$ with respect to $*$. From this it follows that the groupoid is free in the variety determined by the three equations, i.e., in $\mathbf{T}^2$. By the construction of G, G is a quasitournament. $\square$

Two elements a, b of a quasitournament A are said to be comparable if either $a \to b$ or $b \to a$. So, a quasitournament is a tournament if and only if it contains no pair of incomparable elements.

For a quasitournament A and two elements $a, b \in A$, write $a \leq b$ if there exists a path from a to b; write $a \sim b$ if $a \leq b$ and $b \leq a$. So, $\leq$ is a quasiordering and $\sim$ is an equivalence on A.

Lemma 8. *Let $A \in \mathbf{T}^3$. Then $\leq$ is a compatible quasiordering, $\sim$ is a congruence of A and the factor $A/\sim$ is a semilattice; actually, $\sim$ is just the least congruence of A such that the factor is a semilattice.*

Proof. Compatible means that $a \leq b$ implies $ac \leq bc$; for this, it is sufficient to prove that $a \to b$ implies $ac \leq bc$. If $ab = a$, then $ac = aca = abca = baca = bcac \to bca \to bc$.

Consequently, $\sim$ is a congruence. Due to the equation (9), the factor $A/\sim$ satisfies $xy \cdot z = xz \cdot y$; together with commutativity, this implies associativity. We have proved that $A/\sim$ is a semilattice. Clearly, every congruence, the factor by which is a semilattice, contains $\sim$. $\square$

Lemma 9. *Let A be a quasitournament and θ be a congruence of A such that whenever x, y are two incomparable elements with $x\theta xy$, then $x\theta y$. Let B be a block of θ, $a, b \in B$, and $c \in A - B$. Then:*

(1) *$a \to c$ if and only if $b \to c$;*
(2) *$c \to a$ if and only if $c \to b$;*
(3) *if a, c are incomparable, then $ac = bc$.*

Proof. (1) Let $a \to c$. We have $ac\theta bc$, so that $a\theta bc$ and $bc \notin B$; consequently, $(b, c) \notin \theta$. If b, c are incomparable, we get a contradiction by the assumption. So, b, c are comparable and then $b\theta bc$ implies $b = bc$.

(2) Let $c \to a$. We have $ac\theta bc$, so that $c\theta bc$ and $bc \notin B$. The rest is similar as in case 1.

(3) We have $ac \notin B$, $ac\theta bc$ and $ac \to a$, so $bc \to a$ by (1). Since also $bc \to c$, we get $bc \to ac$. We have $ac \to a$, so $ac \to b$ by (2). Since also $ac \to c$, we get $ac \to bc$. Now $bc \to ac \to bc$ imply $ac = bc$. $\square$

Theorem 10. *Let A be a subdirectly irreducible quasitournament which is not a tournament, and θ be its monolith. Then only two cases are possible: Either there are two incomparable elements $a, b \in A$ with $(a, ab) \in \theta$ or else θ has a single non-singleton block B and B is a simple quasitournament.*

Proof. Let the first case not apply, so that the assumptions of Lemma 9 are satisfied. Take a nontrivial block B of θ. By Lemma 9, it can be easily verified that $B^2 \cup \mathrm{id}$ is a congruence of A and also that if α is a congruence of B, then $\alpha \cup \mathrm{id}$ is a congruence of A. From this it follows that B is the only non-singleton block of θ and that B is simple. $\square$

4 Subdirectly irreducibles come in quadruples

Lemma 11. *Let A be a finite subdirectly irreducible algebra in $\mathbf{T}^3$, and let α be its monolith. Then either A contains a zero element 0 and $A - \{0\}$ is a subdirectly irreducible subalgebra of A, or else $(a, b) \in \alpha$ and $a \neq b$ imply $a \leq x$ for any $x \in A$.*

Proof. If $\sim = \mathrm{id}$, then A is a semilattice, so it is a two-element semilattice and we have the first case. Now assume that $\sim$ is not the identity; hence $\alpha \subseteq \sim$. Denote by B the least block of $\sim$. If $|B| > 1$, then $B^2 \cup \mathrm{id}$ is a nontrivial congruence, $\alpha \subseteq B^2 \cup \mathrm{id}$ and we have the second case. Let $B = \{0\}$ for an element 0. Clearly, 0 is the zero element of A. If $A/\sim$ contains two different atoms C and D, then $(C \cup \{0\})^2 \cup \mathrm{id}$ and $(D \cup \{0\})^2 \cup \mathrm{id}$ are two congruences contradicting the subdirect irreducibility. Hence, there exists precisely one atom C of $A/\sim$. But then, $A - \{0\}$ is a subalgebra and the restriction of α to $A - \{0\}$ is the monolith of $A - \{0\}$. $\square$

Given a quasitournament A, we denote by A_* the quasitournament obtained from A by adding a new zero element (element 0 such that $x0 = 0$ for all x) and we denote by A^* the quasitournament obtained from A by adding a unit.

Lemma 12. *There is a one-to-one correspondence, given by $A \mapsto A_*$, between all finite, at least three-element subdirectly irreducible algebras in $\mathbf{T}^3$ without zero and all finite, at least three-element subdirectly irreducible algebras in $\mathbf{T}^3$ with zero. The algebras A and A_* generate the same variety.*

Proof. Since A_* is a homomorphic image of the direct product of A with the two-element chain, the algebras A and A_* generate the same variety. The rest is an easy consequence of Lemma 11. □

It should be clear what we mean by the term obtained from a given term t by deleting all variables from a given proper subset X of $\mathbf{v}(t)$; by $\mathbf{v}(t)$ we denote the set of the variables contained in t. Let us denote this term by t^{-X}. One can easily prove that an equation $u \approx v$ is satisfied in A_* if and only if $\mathbf{v}(u) = \mathbf{v}(v)$ and $u^{-X} \approx v^{-X}$ is satisfied in A for any proper subset X of $\mathbf{v}(u)$.

Lemma 13. *There is a one-to-one correspondence, given by $A \mapsto A^*$, between all finite, at least three-element subdirectly irreducible algebras in $\mathbf{T}^3$ without unit and all finite, at least three-element subdirectly irreducible algebras in $\mathbf{T}^3$ with unit. We have $A \in \mathbf{T}$ if and only if $A^* \in \mathbf{T}$, and also $A \in \mathbf{T}^n$ if and only if $A^* \in \mathbf{T}^n$ for any $n \geq 3$.*

Proof. Let $A \in \mathbf{T}$ (or $A \in \mathbf{T}^n$, respectively); we are going to prove that the same holds for A^*. Let $u \approx v$ be an arbitrary equation (an equation in at most variables, respectively) which is satisfied in any tournament. Clearly, $\mathbf{v}(u) = \mathbf{v}(v)$. For any proper subset X of $\mathbf{v}(u)$, the equation $u^{-X} \approx v^{-X}$ is satisfied in all tournaments, because for any tournament T, T^* is also a tournament; consequently, these equations are satisfied in A. This means that $u \approx v$ is satisfied in A^*. But then, A^* belongs to $\mathbf{T}$ (or to $\mathbf{T}^n$, respectively). The rest is an easy application of Lemma 11. □

Theorem 14. *All subdirectly irreducible algebras of cardinality ≥ 3 in $\mathbf{T}$ (and also in $\mathbf{T}^n$ for any $n \geq 3$) can be partitioned into quadruples A, A_*, A^*, A_*^* where A is a subdirectly irreducible algebra without zero and without unit.*

Proof. It follows from the preceding lemmas. □

5 Subdirectly irreducible algebras with just one incomparable pair

Lemma 15. *The algebra* $\mathbf{J}_3$ *is subdirectly irreducible and belongs to* $\mathbf{T}^3 - \mathbf{T}^4$.

Proof. Define terms s_i, t_i in variables x, y_1, y_2, y_3 by

(1) $s_1 = xy_1$ and $t_1 = y_1$;
(2) $s_2 = t_1 y_2$ and $t_2 = s_1 y_2$;
(3) $s_3 = t_2 y_3 y_1 x y_3$ and $t_3 = s_2 y_3 y_1 x y_3$;
(4) $s_4 = xy_1 t_3 s_3 (xt_3)$ and $t_4 = s(xy_1 t_3)$.

Making use of the fact that in a tournament we must have either $xy_1 = x$ or $xy_1 = y_1$, it is easy to see that the equation $s = t$ is true in all tournaments and hence in any algebra of $\mathbf{T}$. On the other hand, it is not true in $\mathbf{J}_3$: under the interpretation $x \mapsto a$ and $y_i \mapsto b_i$, we have $s \mapsto a$ while $t \mapsto u_0$. Consequently, $\mathbf{J}_3$ does not belong to $\mathbf{T}$. Since it is generated by four elements, it cannot belong to $\mathbf{T}^4$. By Theorem 4, it belongs to $\mathbf{T}^3$. □

Theorem 16. *Let* $A \in \mathbf{T}^3$ *be a subdirectly irreducible algebra containing precisely one two-element subset* $\{a, b\}$ *with* $ab \notin \{a, b\}$. *Then* A *contains a subalgebra isomorphic to either* $\mathbf{J}_3$ *or* $\mathbf{M}_n$ *for some* $n \geq 3$. *Consequently,* A *does not belong to* $\mathbf{T}$.

Proof. Suppose that A contains neither $\mathbf{J}_3$ nor $\mathbf{M}_n$. By an a-sequence we will mean a finite sequence $u_0, u_1, \ldots, u_n$ of elements of A such that $n \geq 0$, $u_0 = ab$, $u_1 = a$ (if $n \geq 1$) and for every $i \geq 2$, one of two cases takes place: either $u_{i-2} \to u_i \to u_{i-1}$ or $u_{i-1} \to u_i \to u_{i-2}$. A sequence with the same properties, except that $u_1 = b$, will be called a b-sequence. An a-sequence is said to be minimal if there is no shorter a-sequence with the same endpoint.

Claim1. *Let* $u_0, u_1, \ldots, u_n$ *be a minimal* a*-sequence with* $n \geq 2$. *Then* $u_1 \to u_2 \to u_0$ *and* $u_2 \to b$.

Proof. Since the a-sequence is minimal, $u_2 \neq a$ and hence either $b \to u_2$ or $u_2 \to b$. If $u_0 \to u_2 \to u_1$, then $bu_2 = u_0$ by Lemma 5(2), so that u_0 is either b or u_2, a contradiction. Consequently, the other case, $u_1 \to u_2 \to u_0$, must take place. By Lemma 5(3), $u_2 \to b$.

Claim2. *Let* $u_0, \ldots, u_n$ *be a minimal* a*-sequence such that* $n \geq 2$ *and* $u_0, \ldots, u_i, b$ *is not* a*-admissible for any* $i \leq n$. *Then* $u_{n-1} \to u_n$, $u_n \to u_i$ *for all* $i \leq n-2$, *and* $u_n \to b$.

Proof. For $n = 2$ this is due to Claim 1. Let $n > 2$ and suppose that the assertion has been proved for all numbers in $\{2, \ldots, n-1\}$. If $b \to u_n$, then $u_{n-1} \to b \to u_n$, so that the sequence $u_0, \ldots, u_n, b$ is a-admissible, a contradiction with the assumption. We get $u_n \to b$. If $a \to u_n$, then $ab \to u_n$ by the minimality of n and $a = bu_nau_n = bau_na = ab$ gives us a contradiction. Hence $u_n \to a$. By the minimality of n, $u_n \to u_0, \ldots, u_n \to u_{n-2}$. From $u_n \to u_{n-2}$ we get $u_{n-1} \to u_n$, since either $u_{n-1} \to u_n \to u_{n-2}$ or $u_{n-2} \to u_n \to u_{n-1}$ must take place.

Claim3. *The element b is not an endpoint of any a-sequence.*

Proof. Suppose, on the contrary, that there exists a minimal a-sequence $u_0, \ldots, u_n, b$. Clearly, $n \geq 3$. By Claim 2, we have $u_0 \to u_1 \to \ldots \to u_{n-1}$, $u_j \to u_i$ whenever $0 \leq i < i+2 \leq j \leq n-1$, and $u_i \to b$ for all $2 \leq i < n$. From $u_{n-1} \to b$ we get $u_{n-1} \to b \to u_n$. If $u_{n-2} \to u_n \to u_{n-1}$, then $u_i \to u_n$ for all $i \leq n-2$ by the minimality of n, and $\{b, u_0, \ldots, u_n\}$ is a subalgebra of A isomorphic to $\mathbf{M}_n$, a contradiction, So, only the case $u_{n-1} \to u_n \to u_{n-2}$ remains to be considered. By the minimality of n, $u_n \to u_i$ for all $i \leq n-2$. But then, the elements b, u_0, u_1, u_2, u_3 form a subalgebra isomorphic to $\mathbf{J}_3$, a contradiction.

Claim4. *Where C_a denotes the set of endpoints of all a-sequences, $E_a = C_a^2 \cup \mathrm{id}$ is a congruence of A; we have $b \notin C_a$.*

Proof. By Claim 3, $b \notin C_a$. Let $p, q \in C_a$ and r be an element such that $p \to r \to q$. We need to prove that $r \in C_a$. There exist two minimal a-sequences $u_0, \ldots, u_n = p$ and $v_0, \ldots, v_m = q$. If neither $u_0, \ldots, u_n, r$ nor $v_0, \ldots, v_m, r$ is an a-sequence, then $u_n \to r, u_{n-1} \to r, \ldots, u_0 \to r$ and $r \to v_m, r \to v_{m-1}, \ldots, r \to v_0$, a contradiction.

Claim5. *Where C_b denotes the set of endpoints of all b-sequences, $E_b = C_b^2 \cup \mathrm{id}$ is a congruence of A; we have $a \notin C_b$.*

Proof. It is a symmetric version of Claim 4.

Claim6. *The two congruences E_a and E_b of A are nontrivial, while their intersection is the identity. Consequently, A is not subdirectly irreducible.*

Proof. We have $(a, ab) \in E_a$ and $(b, ab) \in E_b$. If $E_a \cap E_b \neq \mathrm{id}$, then there is an element in $C_a \cap C_b$. Suppose there is such an element r. There are a minimal a-sequence $u_0, \ldots, u_n = r$ and a minimal b-sequence $v_0, \ldots, v_m = r$. Clearly, we cannot have $n = m = 2$. Hence $\{u_{n-1}, v_{m-1}\} \neq \{a, b\}$ and we have either $u_{n-1} \to v_{m-1}$ or $v_{m-1} \to u_{n-1}$. Consider, for example, the case $u_{n-1} \to v_{m-1}$. Then it is easy to see that $u_0, \ldots, u_m = r = v_n, v_{n-1}, \ldots, v_1$ is an a-sequence ending with b, a contradiction. □

6 A few lemmas

Lemma 17. *Let $A \in \mathbf{T}^3$, let B be a subgroupoid of A such that $B^2 \cup \mathrm{id}_A$ is a congruence of A, and let θ be a congruence of B. Then $\theta \cup \mathrm{id}_A$ is a congruence of A.*

Proof. We must prove that $a\theta b$ implies either $ac = bc$ or $ac\theta bc$ for any $c \in A$. Let $ac \neq bc$. Then the elements a, b, ac, bc all belong to B. Put $e = ac$, $f = bc$, $g = af$, $h = be$ and $l = ac \cdot bc$. Since all these elements belong to B, we have $ae\theta be$, $af\theta bf$, $ael\theta bel$ and $afl\theta bfl$, i.e., $e\theta h$, $g\theta f$, $l\theta h$ and $g\theta l$ (this can be checked from the multiplication table of $\mathbf{F}_3$). By transitivity of θ, $e\theta f$, i.e., $ac\theta bc$. $\square$

Lemma 18. *Let $A \in \mathbf{T}^3$ be a subdirectly irreducible algebra such that the monolith of A is $B^2 \cup \mathrm{id}_A$ for a block B. Then B is simple.*

Proof. Use Lemma 17. $\square$

Lemma 19. *Let A be a finite, subdirectly irreducible algebra in $\mathbf{T}^3$, with monolith μ. Let S be a union of non-singleton blocks of A, and denote by U the union of all non-singleton blocks of A. Suppose that for any $x \in A$ and $s \in S$, either $xs \in S$ or $|x(s/\mu)| = 1$. Then either $s = \emptyset$ or $S = U$.*

Proof. The condition says that $S^2 \cup \mathrm{id}$ is a congruence. This congruence must be either identity, or contain μ. $\square$

Lemma 20. *There is no finite, subdirectly irreducible algebra in $\mathbf{T}^3$ with precisely two non-singleton blocks of its monolith.*

Proof. Suppose the monolith has precisely two non-singleton blocks S_1 and S_2. If $S_1S_2 = S_i$ for some i, then Lemma 19 gives contradiction with $S = S_i$. If $S_1S_2 \notin \{S_1, S_2\}$, then Lemma 19 gives contradiction with $S = S_1$. $\square$

7 Strongly connected algebras

A quasitournament A is said to be strongly connected if for any $a, b \in A$ there exists a path $a = a_0 \to a_1 \to \ldots \to a_n = b$.

Lemma 21. *Let $A \in \mathbf{T}^3$, let B be a subgroupoid of A such that $b \in B$ and $b \leq a$ imply $a \in B$, and let θ be a congruence of B. Then $\theta \cup \mathrm{id}_A$ is a congruence of A.*

Proof. Clearly, $B^2 \cup \mathrm{id}_A$ is a congruence, so we can apply Lemma 21. $\square$

Lemma 22. *Let $A \in \mathbf{T}^3$ be subdirectly irreducible. Then every subgroupoid B of A such that $b \in B$ and $b \leq a$ imply $a \in B$, is subdirectly irreducible. In particular, the least block of $\sim_A$ is subdirectly irreducible.*

Proof. It follows from Lemma 21. $\square$

Lemma 23. *Let $A \in \mathbf{T}^3$ be such that the least block of $\sim_A$ is a tournament. Then for every element $a \in A - B$, such that a is incomparable with at least one element of B, there exists a unique element $a' \in B$ with the following two properties:*

(1) *$ax = a'$ for any $x \in B$ incomparable with a (in particular, $a' \to a$);*
(2) *$y \to a'$ for any $y \in B$ such that $y \to a$.*

Proof. Suppose $ax_1 \neq ax_2$ for two elements $x_1, x_2 \in B$ incomparable with a. We have either $ax_1 \to x_2$ or $x_2 \to ax_1$. If $ax_1 \to x_2$, then $ax_1 \to x_2$ and $ax_1 \to a$ imply $ax_1 \to ax_2$ by the properties of a product. If $x_2 \to ax_1$, then $x_2 \to ax_1 \to a$ implies $ax_1 \to ax_2$ by Lemma 5(5). So, $ax_1 \to ax_2$ in any case. But then $ax_2 \to ax_1$ by symmetry, and we get $ax_1 = ax_2$.

Take an arbitrary element $x \in B$ which is incomparable with a, and put $a' = ax$. Let $y \in B$ be such that $y \to a$. The only alternative to $y \to a'$ could be $a' \to y$, so suppose that. Since $a' = ax$ and $a' \to y \to a$, we have $xy = a'$. But xy is either x or y, so $y = a'$. $\square$

Lemma 24. *Let $A \in \mathbf{T}^3$ be a finite, subdirectly irreducible algebra such that the least block B of $\sim_A$ is a tournament. Then $x \to a$ for any $x \in B$ and any $a \in A - B$.*

Proof. Suppose, on the contrary, that some element of $A - B$ is incomparable with at least one element of B, and take a minimal (with respect to $\leq$) such element a. Take an element $x \in B$ incomparable with a and put $a' = ax$. If there is an element b such that $B < b/\sim < a/\sim$, then there is one such element with $b \to a$ (replace b with ab if necessary); we have $x \to b$ by the minimality of a, so that $b \to a'$ by Lemma 5(5), a contradiction. This proves that $a/\sim$ is an atom in $A/\sim$. So by Lemma 22, it is sufficient to assume that $A = B \cup (a/\sim)$.

The set $A - B$ can be partitioned into two subsets: the (possibly empty) subset C of the elements c satisfying $x \to c$ for all $x \in B$, and the subset

D of the elements a for which the element $a' \in B$, as in Lemma 23, exists. Denote by θ the equivalence on A with blocks $\{x\} \cup \{a \in D : a' = x\}$ for $x \in B$ (and singletons, corresponding to the elements of C). The following two observations will imply that θ is a congruence of A.

Claim1. If $c \in C$ and $a \in D$, then $(ac)' = a'$. *Proof.* Since $a' \to a$ and $a' \to c$, we have $a' \to ac$. Since $a' \to ac \to a$, we have $x \cdot ac = xa = a'$ by Lemma 5(2). If $x \to ac$, then $x \to ac \to a$ implies $ac \to a'$ by Lemma 5(5), a contradiction.

Claim2. If $a, b \in D$ and $a' \to b'$, then $(ab)' = a'$. *Proof.* Since $a' \to b'$, we have $ab' = a'$. By the definition of b', $a' \to b'$ implies $a' \to b$. By Theorem 4(9) we get $ab \cdot b' = ab'bab' = a'bab' = a'ab' = a'b' = a'$. It remains to prove that ab and b' are incomparable. If $b' \to ab$, then $b' \to ab \to a$ gives $ab \to a'$, a contradiction.

We conclude that θ is a congruence of A. This gives us a contradiction with Lemma 11, since θ is nontrivial and $\theta \cap B^2 = \mathrm{id}$. □

Lemma 25. *Let $A \in \mathbf{T}^3$ be a finite, subdirectly irreducible algebra without zero, such that the least block of $\sim_A$ is a tournament. Then A is a tournament.*

Proof. Suppose that A contains a pair of incomparable elements. By Lemma 24, both elements must belong to $A - B$, where B is the least block of $\sim_A$. If $A - B$ is a subgroupoid, then $(A - B)^2 \cup \mathrm{id}_A$ is a congruence, which is not possible. So, let $ab \in B$ for some $a, b \in A - B$. For every $x \in B$ we have $x \to a$ and $x \to b$ and hence $x \to ab$. Hence ab is the unit element 1_B of B. Hence $((A-B) \cup \{1_B\})^2 \cup \mathrm{id}_A$ is a congruence, and we get a contradiction. □

Theorem 26. *Every finite, subdirectly irreducible algebra in $\mathbf{T}^3$ which is not a tournament contains a strongly connected, subdirectly irreducible subalgebra which is again not a tournament.*

Proof. By Lemma 12 we can assume that the algebra has no zero element. By Lemma 22 the least block of $\sim$ does the work, unless it is a tournament. However, it is not a tournament by Lemma 25. □

A quasitournament A is said to be rich if $a \to b \to c$ implies that the elements a, c are comparable, i.e., either $a \to c$ or $c \to a$.

Lemma 27. *Let A be a rich, strongly connected quasitournament and let a, b be two incomparable elements of A. Then there exist elements c, d such that $a \to c \to d \to b$, $d \to a$, $b \to c$, $ab \to c$ and $d \to ab$.*

Proof. Denote by n the length of a shortest path leading either from a to b or from b to a.

Let there be a path $a = u_0 \to u_1 \to \ldots \to u_n = b$. We have $n \geq 3$, because A is rich. Since A is rich, the elements u_i, u_{i+2} are comparable for any $0 \leq i < i+2 \leq n$; by the minimality of n, $u_{i+2} \to u_i$. Suppose $n \geq 4$. If n is even, then $u_n \to u_{n-2} \to \ldots \to u_2 \to u_0$ is a path from b to a, contradicting the minimality of n. If n is odd, then $u_n \to u_{n-2} \to \ldots \to u_1 \to u_2 \to u_0$ gives the same contradiction. So, $n = 3$.

There is also a path of length 3 from b to a, namely, $b \to u_1 \to u_2 \to a$. So, the assumption that the path of length n went from a to b, was inessential.

Since $u_2 \to a$ and $u_2 \to b$, we have $u_2 \to ab$. Since $a \to u_1$ and $b \to u_1$, we have $ab \to u_1$. □

Lemma 28. *Let A be a rich, strongly connected quasitournament and let $a_1, \ldots, a_n$ be an n-tuple of pairwise incomparable elements of A. Then there exist elements b and c such that $b \to a_i \to c$ for all i.*

Proof. By induction on n. For $n = 1$ this is clear. Let $n \geq 2$. By the induction assumption, there are elements b', c' such that $b' \to a_i \to c'$ for all $i = 1, \ldots, n-1$.

Let us prove first that there is an element c with $c' \to c$ and $a_n \to c$. If c' and a_n are incomparable, it follows from Lemma 27. We cannot have $c' \to a_n$, since that would give, together with $a_1 \to c'$, the comparability of a_n with a_1 by the richness of A. So, if c' and a_n are comparable, then $a_n \to c'$ and we can take $c = c'$.

For $i < n$ we have $a_i \to c' \to c$, so that c and a_i are comparable. If $c \to a_i$, then it follows from $a_n \to c \to a_i$ that a_i, a_n are comparable, a contradiction. Hence $a_i \to c$.

Put $b = b'a_n$. For $i < n$ we have $b \to b' \to a_i$, so that a_i, b are comparable. If $a_i \to b$, then $a_i \to b \to a_n$ implies that a_i, a_n are comparable, a contradiction. Hence $b \to a_i$. □

Lemma 29. *Let A be a finite, rich, strongly connected quasitournament. Then there exist two distinct, comparable elements $a, b \in A$ such that every element of A is comparable to either a or b.*

Proof. Let $a_1, \ldots, a_n$ be a maximal n-tuple of pairwise incomparable elements of A. By Lemma 28 there exist elements a, b such that $a \to a_i \to b$ for all i. Since $a \to a_1 \to b$, the elements a, b are comparable. Let x be any element of A. By the maximality, x is comparable with a_i for some i. If $x \to a_i$, then $x \to a_i \to b$ implies that x, b are comparable. If $a_i \to x$, then $a \to a_i \to x$ implies that x, a are comparable. $\square$

8 Polynomials on pairs of elements

For a groupoid A, define a quasiordering $\leq$ on the set of ordered pairs of elements of A in the following way: $(c,d) \leq (a,b)$ iff there is a unary polynomial p of A such that $p(a) = c$ and $p(b) = d$. Write $(a,b) \sim (c,d)$ if $(a,b) \leq (c,d) \leq (a,b)$. Write $(c,d) < (a,b)$ if $(c,d) \leq (a,b)$ but not $(a,b) \leq (c,d)$.

Theorem 30. *Let $A \in \mathbf{T}$ and let a, b be two incomparable elements of A. Then $(a, ab) < (a,b)$ and $(b,a) \not\leq (a, ab)$.*

Proof. Clearly, $(a, ab) \leq (a,b)$. Suppose that there is a polynomial p with $p(a) = a$ and $p(ab) = b$. There are a term $t(x, x_2, \dots, x_n)$ and elements $c_2, \dots, c_n \in A$ such that $p(x) = t(x, c_2, \dots, c_n)$ for all $x \in A$. Define terms r_i, s_i $(i = 0, 1, \dots)$ in this way: $r_0 = x$, $s_0 = y$, $r_{i+1} = t(r_i, x_2, \dots, x_n)$ and $s_{i+1} = t(r_i s_i, x_2, \dots, x_n)$.

Let T be a tournament and $x \mapsto a_0$, $y \mapsto b_0$, $x_i \mapsto d_i$ be an interpretation in T. Define $a_i, b_i \in T$ by $r_i \mapsto a_i$ and $s_i \mapsto b_i$. Since all a_i, b_i belong to $\{a_0, b_0, d_2, \dots, d_n\}$, there exist i, j with $0 \leq i < j \leq (n+1)^2$ and $(a_i, b_i) = (a_j, b_j)$. Observe that if $a_k b_k = a_k$ for some k, then $a_m = b_m$ for all $m > k$, so that $a_i = b_i$ and hence $a_{n^2} b_{n^2} = b_{n^2}$. Since T is a tournament, the only alternative to $a_k b_k = a_k$ for some k is $a_k b_k = b_k$ for all k, in which case we also have $a_{n^2} b_{n^2} = b_{n^2}$. Hence $a_{n^2} b_{n^2} = b_{n^2}$ in any case. This proves that the equation $r_{n^2} s_{n^2} = s_{n^2}$ is satisfied in all tournaments, and hence in A. But in A, under the interpretation $x, y, x_2, \dots, x_n \mapsto a, b, c_2, \dots, c_n$ we have $r_i s_i \mapsto ab$ and $s_i \mapsto b$ for all i, a contradiction.

In order to prove $(b,a) \not\leq (a, ab)$, it is enough to replace the definition of r_i, s_i with $r_0 = x$, $s_0 = y$, $r_{i+1} = t(r_i s_i, x_2, \dots, x_n)$ and $s_{i+1} = t(r_i, x_2, \dots, x_n)$. $\square$

9 Subdirectly irreducible algebras in $\mathbf{T}_3$

We denote by $\mathbf{C}_2$ the two-element semilattice, and by $\mathbf{C}_3$ the three-element tournament cycle.

Theorem 31. *The variety $\mathbf{T}_3$ has just three subdirectly irreducible algebras, namely, $\mathbf{C}_2$, $\mathbf{C}_3$, and $(\mathbf{C}_3)_*$.*

Proof. It is sufficient to prove that every finite, subdirectly irreducible algebra S in $\mathbf{T}_3$ is isomorphic to one of the three algebras. Since $\mathbf{T}_3$ is generated by $\mathbf{C}_3$, there exist a positive integer n, a subalgebra D of $\mathbf{C}_3^n$ and a congruence θ of D such that S is isomorphic to the factor D/θ. Take n to be minimal with this property. Suppose $n \geq 2$.

Denote by β the only cover of θ in the congruence lattice of D. We will make use of tame congruence theory, as developed in [7]. It is clear that the type of β/θ is either **4** or **5**. Let U be a (β, θ)-minimal set and $(c, d) \in (\beta - \theta) \cap U^2$. Then either $(c, cd) \in \beta - \theta$ or $(d, cd) \in \beta - \theta$. Applying either the polynomial $x \mapsto xc$ or the polynomial $x \mapsto xd$, we obtain a minimal set Uc or Ud. Hence we can assume that $d = cd$.

Now $U = e(D)$ for an idempotent polynomial e of D. Since $(x, y) \mapsto x * y = e(xy)$ maps U^2 onto U and $c * c = c$ and $c * d = d * c = d * d = d$, it follows that there is a pseudo-meet operation $\wedge$ on U with $c \wedge d = d$. Thus we have $(c/\theta) \cap U = \{c\}$ and also $cx = x$ for all $x \in U$.

Denote the elements of $\mathbf{C}_3$ by 0,1,2, with $0 \to 2 \to 1 \to 0$. We can assume that $c = 0^n$ and thus $U \subseteq \{0, 1\}^n$. It follows from tame congruence theory that for any $x, y \in D$, $(x, y) \in \theta$ iff for every unary polynomial p, $ep(x) = c$ is equivalent to $ep(y) = c$.

Denote by $d_1, \ldots, d_z$ all the elements of D. Put $c' = (cd_1)(cd_2)\ldots(cd_z)$ and $c'' = (c'd_1)(c'd_2)\ldots(c'd_z)$. For some k and l (and some ordering of the indexes) we have $c = 0^n$, $c' = 0^k 1^{n-k}$, $c'' = 0^k 1^l 2^{n-k-l}$ and $D \subseteq \{0, 2\}^k \times \{0, 1\}^l \times \{0, 1, 2\}^{n-k-l}$.

Claim1. *If $x, y \in D$ and $x(i) = y(i)$ for all $i \geq k$, then $x\theta y$.* Indeed, if $(x, y) \notin \theta$ then there is a polynomial p such that, e.g., $ep(x) = c$ and $ep(y) \neq c$. Put $q = ep(y)$. Now $q(i) = c(i)$ for $i \geq k$. Also, $q \in U$ and hence $q(i) \in \{0, 1\}$ for $i < k$. Since $q \in D$, $q(i) \in \{0, 2\}$ for $i < k$. Thus $q = c$, a contradiction.

Claim2. *$k = 0$, $c' = 1^n$ and $c'' = 1^l 2^{n-l}$.* This follows from Claim 1 by the minimality of n.

Claim3. *$D \supseteq \{0, 1\}^n \cup (\{1\}^l \times \{0, 1, 2\}^{n-l})$.* Since n was minimal, for $i < n$ there are $x, y \in D$ such that $(x, y) \notin \theta$ and x, y differ on i only. By the characterization of θ, there is an element $q_i \in U \subseteq \{0, 1\}^n$ such that q_i differs from c on i only. We must have $q_i = 0^i 1 0^{n-i-1}$. Then, by forming products, $D \supseteq \{0, 1\}^n$. Since 1^n, $1^l 2^{n-l}$ and 0^n belong to D, it is not hard to see that also $S \supseteq \{1\}^l \times \{0,1,2\}^{n-l}$.

Case 1: $l = 0$. Then $D = \mathbf{C}_3^n$. We can assume that the number of components on which c differs from d is as small as possible. In that case we are going to show that the two elements differ on one component only. Indeed, suppose that there is an element $q \in \{0,1\}^n$ different from both c and d and such that $cq = q$ and $dq = d$. We have $q = qc \ \beta \ qd = d$. If $q\theta d$ then $e(q)\theta d$ and $(c, e(q)) \in \beta - \theta$ while $e(q) \in \{0,1\}^n$ agrees with $e(c) = c$ at all i with $q(i) = c(i)$, and agrees with $e(d) = d$ at all i with $q(i) = d(i)$; i.e., $e(q) = q$ and the minimality is contradicted. On the other hand, if $(q, d) \in \beta - \theta$ then there exists a polynomial p such that $ep(q) = c$ iff $ep(d) \neq c$. These two elements agree more often than c, d do, and since $(c/\theta) \cap U = \{c\}$, they are related by $\beta - \theta$, again a contradiction. Hence c, d differ on one component only and we can assume that $d = 10^{n-1}$.

Now we will show that whenever $x(0) = y(0)$ then $x\theta y$. Suppose that $x(0) = y(0)$ and $(x, y) \notin \theta$. Then (c, d) belongs to the join of θ with the congruence generated by (x, y) and so there exist elements $w_0, w_1, \ldots, w_m$ such that $w_0 = c$, $w_m = d$ and for every $i < m$ either $(w_i, w_{i+1}) \in \theta$ or $w_i(0) = w_{i+1}(0)$. Replace w_i by

$$w_i' = w_i c(01^{n-1})(02^{n-1})c.$$

Then $w_0' = c$, $w_m' = d$, $\{w_0', \ldots, w_m'\} \subseteq \{c, d\}$ and for every $i < m$ either $(w_i', w_{i+1}') \in \theta$ or $w_i' = w_{i+1}'$. This means $(c, d) \in \theta$, a contradiction.

We have shown that in Case 1, the homomorphism of D onto S factors through the (first) projection of D onto $\mathbf{C}_3$. This contradicts the minimality of $n > 1$.

Case 2: $l \neq 0$. Note that if $x = p(y)$ for some non-constant polynomial p, then $x = yy_1 \ldots y_z$ for some $y_1, \ldots, y_z \in D$. Thus if in addition, $y(i) = 1$ for some $i < l$, then $x(i) = 1$ for the same i, since the restriction of D to l is contained in $\{0,1\}^l$. Hence our characterization of θ implies that where $Q = D - \{0^l\} \times \{0,1,2\}^{n-l}$, we have $Q \times Q \subseteq \theta$.

It is impossible that $Q/\theta \cap \{0\}^l \times \{0,1,2\}^{n-l} \neq \emptyset$. Because if this happened, then every element of D would be θ-equivalent to some element of the subalgebra $P = \{0^l\} \times \{0,1,2\}^{n-l}$. Then S would be isomorphic to P/θ' where θ' is the restriction of θ to P, contradicting the minimality of n.

Hence S is isomorphic to $(P/\theta')_*$.

We have proved that every finite subdirectly irreducible algebra in $\mathbf{T}_3$ is either $\mathbf{C}_2$ or $\mathbf{C}_3$ or else contains zero. By Theorem 14 it follows that the only subdirectly irreducibles are the three claimed ones. $\square$

10 T is inherently non-finitely-generated

We use often the tournament $\mathbf{L}_n$, which consists of n elements $a_0, \ldots, a_{n-1}$ with $a_i \to a_j$ iff either $i = j$ or $j = i + 1$ or $i > j + 1$.

Let $\mathbf{N}_n$ be the tournament $\mathbf{L}_n$ with two elements a and b adjoined where $a_i \to a \to b$ for all $i < n$, $a_i \to b$ for all $i < n - 1$ and $b \to a_{n-1}$.

Theorem 32. *If A is any groupoid with $\mathbf{N}_n \in \boldsymbol{HSP}(A)$ then $|A| \geq n$. Hence the variety $\mathbf{T}$ is inherently non-finitely-generated.*

Proof. We can assume that A is finite, D is a subalgebra of A^k, φ is a homomorphism of D onto $\mathbf{N}_n$, and k is minimum for the existence of D and φ. Thus there exist $f, g \in D$ such that $\varphi(f) \neq \varphi(g)$ and $f|_{k-1} = g|_{k-1}$.

The crucial property of $\mathbf{L}_n$ is that for any $x \neq y$ and $u \neq v$ in $\mathbf{L}_n$ there is a translation (i.e., a polynomial p of the form $p(w) = wr_1r_2 \cdots r_t$) such that $\{p(x), p(y)\} = \{u, v\}$. In fact, $\mathbf{L}_n$ is a simple algebra of type **3** and it follows from a result in [7] that $\mathbf{L}_n$ must be a homomorphic image of a subalgebra of A (actually, $k = 1$). But let's just prove directly that $|A| \geq n$.

From the two remarks above, there must exist $f_i, g_i \in D$ such that $f_i|_{k-1} = g_i|_{k-1}$ and $\varphi(f_i) = a$ and $\varphi(g_i) = a_i$. Then put

$$f = f_0 f_1 \ldots f_{n-1}, \qquad h_i = f_0 \ldots f_{i-1} g_i f_{i+1} \ldots f_{n-1}$$

and we have that all elements $f, h_0, \ldots, h_{n-1}$ agree on $k - 1$ and $\varphi(f) = a$ while $\varphi(h_i) = a_i$.

These elements of D must all disagree at their last coordinate, hence A has at least $n + 1$ elements. $\square$

11 Simple algebras

Theorem 33. *Every finite simple algebra in $\mathbf{T}$ is a tournament.*

Proof. The relevant result from Hobby-McKenzie [7] is this. Let S be a finite simple algebra of type **3** or **4**. If $S \in \boldsymbol{HSP}(A_1, \ldots, A_k)$ and A_i and k are finite, then $S \in \boldsymbol{HS}(A_i)$ for some i. The proof is as follows. We have $S \cong D/\theta$ where D has congruences $\eta_0, \ldots, \eta_m$ such that $\bigwedge\{\eta_i : i \leq m\} = 0_D$ and $D/\eta_i \in \boldsymbol{S}(A_{j_i})$. Now θ is a maximal congruence of D and the type of $(\theta, 1_D)$ is **3** or **4**. Let M be any $(\theta, 1_D)$-minimal set in D. Then $M = \{a, b\}$ is a two-element set and $M = e(D)$ for some unary polynomial e with $e = e^2$.

Moreover, for every $(c,d) \in D^2 - \theta$, there is a unary polynomial f with $\{f(c), f(d)\} = \{a, b\}$. We can see that there is i with $\eta_i \leq \theta$. For if this fails, then for every i, picking $(c_i, d_i) \in \eta_i - \theta$ and $f_i(\{c_i, d_i\}) = M$, we see that $(a,b) \in \eta_i$. But this would hold for every i, forcing $a = b$. Thus for some i, $\eta_i \leq \theta$. So $S \in \boldsymbol{H}(D/\eta_i)$, i.e., $S \in \boldsymbol{HS}(A_{j_i})$.

Now when A is a tournament, we have that $\boldsymbol{HS}(A) = \boldsymbol{S}(A)$ and every member of $\boldsymbol{HS}(A)$ is a tournament. Thus we have completed a proof that every finite simple algebra of type **3** or **4** in $\mathbf{T}$ is a tournament. (Such a finite simple algebra must be a homomorphic image of a subdirect product of finitely many tournaments, since $\mathbf{T}$ is locally finite.)

The variety $\mathbf{T}$ omits types **1** and **2**, i.e., it has no non-trivial Abelian congruences, or again, equivalently, it is congruence meet-semi-distributive. All this follows in tame congruence theory since if a and b are two distinct elements of an algebra in $\mathbf{T}$ then either $\{a, ab\}$ or $\{b, ab\}$ becomes a two-element semilattice under the basic operation of the algebra.

Thus we have simple algebras only of types **3**, **4**, **5**. It remains to see that every finite simple algebra of type **5** in the variety $\mathbf{T}$ is a tournament. To do this, I will show first that such an algebra must have a zero element u, satisfying $ux = u$ for all x.

Thus let S be a finite simple algebra of type **5** in $\mathbf{T}$. This means that the minimal sets are two-element sets on which some polynomial induces the operation of a semilattice, but there is only one polynomial-induced semilattice operation on a minimal set. Let $\{a, b\}$ be one of the minimal sets for S. Without losing generality, assume that $a \neq ab$. Then $\{a, ab\}$ is a minimal set since it is the image under $f(x) = xa$ of $\{a, b\}$. Obviously $x \cdot y$ is a semilattice operation on $\{a, ab\}$ and so there is no polynomial $q(x, y)$ of S with $q(a, a) = a$, $q(ab, ab) = ab$, $q(a, ab) = q(ab, a) = a$. I claim that for every minimal set $\{c, d\}$ we have that $cd \in \{c, d\}$. The tool for proving this is the above assertion involved in type **5** and the fact that $\{c, d\} = f(\{a, b\})$ for some polynomial f. By induction on the complexity of f, we show that $f(a)f(ab) = f(ab)$.

So assume that f, g are polynomials with this property and that $h(x) = f(x)g(x)$. Note that for any element p we must have $f(ab)p \to f(a)$. For where $u = f(ab)p$ and $v = f(ab)pf(a)$ we have $(u, v) = (\lambda f(ab), \lambda f(a))$ with $\lambda(x) = f(ab)px$, but also (using the equation $x(yz) = ((xy)(yz))((xz)(yz))$

of $\mathbf{T}$:

$$v = \{[f(ab)f(a)][f(ab)p]\}\{[f(ab)p][f(a)p]\} \qquad = [f(ab)p][f(a)p]$$

$$u = f(ab)f(a)p = \{[f(ab)f(a)][f(ab)p]\}\{[f(ab)f(a)][f(a)p]\} \qquad = u[f(ab)(f(a)p)] = u\{(f(a$$

and so where $\gamma(x) = u\{f(ab)(f(x)p)v\}$ we have that $(u, v) = (\gamma(a), \gamma(ab))$. Now if $u \neq v$ then there is a polynomial τ with $\{\tau(u), \tau(v)\} = \{a, ab\}$. By composing either with λf or with γ, we have a polynomial g such that $g(a) = ab$ and $g(ab) = a$, which would give that the set $\{a, ab\}$ has the structure of a Boolean algebra induced by polynomials, contradicting that the type is **5**. Thus $u = v$ and we have that $f(ab)p \to f(a)$. Similary, $g(ab)p \to g(b)$. But then $f(ab)g(ab) \to f(a)$ and $f(ab)g(ab) \to g(a)$, implying that $h(ab) \to h(a)$ as desired.

Now by Hobby-McKenzie [7], we have a compatible partial order $\leq$ on S such that $ab \leq a$ (for the particular minimal set $\{ab, a\}$) and for every minimal set $\{f(ab), f(a)\}$ (f a polynomial), $f(ab) \leq f(a)$. Let $u \in S$ be a minimal element under this order. Let v be any element of S. We wish to show that $uv = u$. Suppose not. Then we can assume that $uv = v \neq u$. (Just replace v by uv.) There is a chain $x_0 = u, x_1, \ldots, x_s = v$ where for all $i < s$, $\{x_i, x_{i+1}\}$ is a minimal set. There is $i < s$ with $ux_i = u$ and $ux_{i+1} \neq u$. Then $\{u, ux_{i+1}\}$ is a minimal set $\{f(ab), f(a)\}$ and since we've seen that $f(ab) \to f(a)$, we have that $f(ab) = ux_{i+1}$, $f(a) = u$, implying that $u \neq f(ab) \leq u$, contradicting the minimality of u.

Thus our algebra S has a zero element u.

Case 1: $S - \{u\}$ is a subalgebra. Then since S is simple, it follows that $|S| = 2$, so certainly S is a tournament.

Case 2: We have $vw = u$ where $v \neq u \neq w$. Since S is simple, there must be a sequence $v = v_0 \to v_1 \to \cdots \to v_q = w$ (else the congruence generated by identifying w with u cannot make v equivalent to anything). We can assume that v, w and $v_0, \ldots$ are chosen so that q is minimal. Now $q > 2$ by Theorem 16. By minimality, $vv_{q-1} \neq u$. However $(yx)(yz) \to y(xz)$ is valid in our variety. Taking $x = w$, $y = v_{q-1}$, $z = v$ we get $vv_{q-1} \to u$. This is an obvious contradiction, showing that Case 2 cannot occur.

Thus the only finite simple algebra S in $\mathbf{T}$ of type **5** is the two-element one. $\square$

12 Conjecture: equivalent formulations

It seems to be a hard and interesting problem to determine whether the variety $\mathbf{T}$ generated by tournaments is the same as the quasi-variety $\mathbf{T}_q$ generated by all tournaments. Since both classes are locally finite, the problem can be formulated several ways: Is it true that whenever A is a subalgebra of $\prod\{\mathbf{T}_i : 1 \leq i \leq n\}$, $\mathbf{T}_i$ finite tournaments, then every subdirectly irreducible homomorphic image of A is a tournament? Is it true that every finite si algebra in $\mathbf{T}$ is a tournament? Is it true that for every quasi-equation ϕ which is valid in all tournaments, there is a finite set Γ of equations true in all tournaments such that $\Gamma \vdash \phi$?

Here we shall show that only very special ϕ need be considered. Write $\theta(x, y)$ for the congruence generated by a pair (x, y) in an algebra A. Another equivalent form of our problem: Is it true that for every $A \in \mathbf{T}_q$ and $a, b \in A$ and congruence ψ of A, we have that $(\theta(a, ab) \vee \psi) \wedge (\theta(b, ab) \vee \psi = \psi$?

By a *clog* I mean a system (a, b, c, d) of elements in an algebra A such that $a = ab \neq b$ and $c = ca = cb$ and $d = da = db$. If $A \in \mathbf{T}'$ and (a, b, c, d) is a clog, then obviously $(a, b) \in \theta(c, b) \wedge \theta(d, b)$. By a *linear polynomial* of A, I mean a function of the form $f(x) = xa_1 \cdots a_n$ for some $a_1, \cdots, a_n \in A$. Write $(a, b) \leq_1 (c, d)$ to denote that there exists a linear polynomial f for which $\{f(c), f(d)\} = \{a, b\}$. Given elements $a, b, c, d \in A$ and a clog (u, v, w, z) in A, we say that this clog is a *special clog for* (a, b, c, d) iff $(w, v) \leq_1 (a, b)$ and $(z, v) \leq_1 (c, d)$.

Lemma 34. *Let $A \in \mathbf{T}^3$ and $a, b, c, d \in A$. Then $\theta(a, b) \wedge \theta(c, d) \neq 0_A$ iff there exists a special clog for (a, b, c, d). In fact, if $(e, f) \in \theta(a, b) \wedge \theta(c, d)$ with $e \neq f$ then there exists a special clog (u, v, w, z) for (a, b, c, d) with $(u, v) \in \theta(e, f)$.*

Proof. Suppose that $0_A \neq \lambda \leq \theta(a, b) \wedge \theta(c, d)$. We first show that there exist u, v, w with $u = uv \neq v$, $w = wv = wu$, $(w, v) \leq_1 (a, b)$ and $(u, v) \in \lambda$. We begin with the observation that, choosing any pair $(u', v') \in \lambda$ with $u' = u'v' \neq v'$ (there exists such a pair), there must exist some $x_0 = v', x_1, \ldots, x_n = u'$ where $(x_i, x_{i+1}) \leq_1 (a, b)$ for all $i < n$. Replacing x_i by $x_i v'$, we can assume that $x_i = x_i v'$ for all i. We also assume that n is the least positive integer for which there exists such a system $x_0 = v', \cdots, x_n = u'$. with $u' = u'v' \neq v'$, $(u', v') \in \lambda$, $(x_i, x_{i+1}) \leq_1 (a, b)$.

If $n = 1$, then $(u', v') = (x_1, x_0) \leq_1 (a, b)$ and we can take $(u, v, w) = (u', v', u')$. Also, if $x_1 u' = x_1$ then we can take $(u, v, w) = (u', v', x_1)$. Now assume that $n > 1$ and $x_1u' \neq x_1$. Then replace u', v' by x_1u', x_1 and the sequence $x_0, \ldots, x_n$ by $y_0, \ldots, y_{n-1}$ where $y_i = x_1x_{i+1}$. Since $x_1 = x_1v' \equiv x_1u' (\mathrm{mod}\ \lambda)$, we have contradicted the minimality of n.

So let (u, v, w) satisfy $u = uv \neq v$, $(u, v) \in \lambda$, $w = wv = wu$, $(w, v) \leq_1 (a, b)$. Since $\lambda \leq \theta(c, d)$ there is a system $x_0 = v, x_1 \ldots, x_n = u$, for some n, where $(x_i, x_{i+1}) \leq_1 (c, d)$. Again, we can assume that $x_iv = x_i$ and that n is minimal for the existence of a system $(u, v, w, x_0, \ldots, x_n)$ satisfying all these conditions. If $x_1u = x_1$ then (u, v, w, x_1) is a special (a, b, c, d) clog with $(u, v) \in \lambda$, as desired. So assume that $x_1u \neq x_1$, which implies, of course, that $n > 1$.

Case 1: $x_1w = x_1w \cdot x_1u$. In this case, replace u, v, w by x_1u, x_1, x_1w (noting that $(x_1w, x_1) = (x_1w, x_1v)$ so that $(x_1w, x_1) \leq_1 (a, b)$), and replace $x_0, \ldots x_n$ by $y_0, \ldots, y_{n-1}$ where $y_i = x_1x_{i+1}$. This contradicts minimality of n.

Case 2: $x_1w \neq x_1w \cdot x_1u$. Now $x_1w = x_1w \cdot x_1v \equiv x_1w \cdot x_1u$ (mod λ). Also $(x_1w, x_1w \cdot x_1u) = (wux_1 \cdot wux_1, vux_1 \cdot wux_1)$ so that $(x_1w, x_1w \cdot x_1u) \leq_1 (a, b)$. Replace u, v, w by $x_1w \cdot x_1u, x_1w, x_1w \cdot x_1u$ and replace $x_0, \ldots, x_n$ by $y_0, \ldots, y_{n-1}$ where $y_i = x_{i+1}x_1 \cdots x_1w$. This contradicts the minimality of n. □

Theorem 35. *The following are equivalent:*

(1) $\mathbf{T} = \mathbf{T}_q$.

(2) *Let $x, y, z, x_1, x_1', \ldots, x_k, x_k', \ldots$ be distinct variables and for every positive integer n, let $t_n(w)$ denote $wx_1 \cdots x_n$ and $t_n'(w)$ denote $wx_1' \cdots x_n'$. Letting $\{u, v\} = \{y, yz\}$ and $\{r, s\} = \{z, yz\}$, then $\mathbf{T}$ satisfies the quasi-equations*

$$x = xt_n(v) \wedge t_n(v) = t_n'(s) \wedge t_n(u) = t_n(u)t_n(v) = t_n(u)x$$

$$\wedge t_n'(r) = t_n'(r)t_n'(s) = t_n'(r)x \longrightarrow x = t_n(v)\ .$$

(3) *With notation as above, for all n and $A \in \mathbf{T}_q$ and elements $x, y, z, x_1, x_1', \ldots$ in A, we have that the congruence on A generated by identifying the two sides of every equation to the left of the arrow in the quasi-equation above identifies x with $t_n(v)$.*

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