THE BERMAN CONJECTURE FOR NILPOTENT EXTENSIONS OF REGULAR SEMIGROUPS

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Abstract

For a semigroup S, $p_n(S)$ denotes the number of all operations on S, induced by words, which depend on all their variables. In this note we prove that any finite semigroup which can be represented as a nilpotent ideal extension of a regular semigroup satisfies the Berman conjecture, i.e. that its p_n-sequence is either bounded above by a constant, or eventually strictly increasing.

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Let S be any semigroup. By a *term operation of S* we mean an n-ary operation of S which is induced by some nonempty word w by substitution of letters. Such an operation is *essentially* n-ary if it depends on all of its variables. For $n \geq 1$, we denote the set of all essentially n-ary term operations of S by $\mathcal{E}_n^S$, while $\mathcal{E}_0^S$ is the set of all constant unary term operations of S. Let $p_n(S) = |\mathcal{E}_n^S|$ for all $n \geq 0$. Since it is not difficult to show that the semigroup of all term operations of S over some given set of variables X is isomorphic to the (relatively) free semigroup on X of the variety generated by S, it follows that all $p_n(S)$, $n \geq 0$, are finite numbers if and only if S generates a locally finite semigroup variety. Of course, all these concepts can be generalized for arbitrary algebras, cf. the survey paper by Grätzer and Kisielewicz [7].

The p_n-sequence is an important invariant of a semigroup (or of an algebra), and it contains a lot of information on its structural features. Actually, the principal goal of the theory of p_n-sequences of general algebras, formulated back in 1966 by E. Marczewski [9], is to characterize those sequences of non-negative integers which are representable as the p_n-sequence of some algebra. Hence, investigations aiming to establish the way in which the numerical properties of sequences influence the structure of corresponding algebras, have a central importance in this theory. We refer to [7] for some related examples.

In this environment, it is even more interesting to restrict the study of p_n-sequences to finite algebras (semigroups), and to look for some regularities following from the finiteness condition. One of the most intriguing hypotheses in this field was given by J. Berman, who conjectured that the p_n-sequence of any finite algebra is either bounded above by a constant, or eventually strictly increasing. While this assertion, referred to as the *Berman conjecture*, is easily

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shown to be true for finite monoids, groups, rings, modules, lattices, Boolean algebras, etc., R. Willard refuted the Berman conjecture in [10] by constructing a 4-element algebra with finitely many basic operations whose p_n-sequence is $(0, 3, 2, 5, 4, 7, 6, 9, 8, \ldots)$.

Our main interest is in investigating the question whether the Berman conjecture holds in the class of all finite semigroups. In [5], all finite semigroups whose p_n-sequences have constant bounds are determined. The Berman conjecture for all globally idempotent finite semigroups ($S^2 = S$) is proved in [6], and for some other particular classes, in [4]. It may be noted that the principal results of [4, 5] are formulated in terms of nilpotent ideal extensions of globally idempotent semigroups of some specific kind. This is so because if S is finite, then the sequence $S \supseteq S^2 \supseteq \ldots \supseteq S^m \supseteq S^{m+1} \supseteq \ldots$ (where $S^m = \{s_1 s_2 \ldots s_m : \ s_1, \ldots, s_m \in S\}$) of ideals of S must eventually stabilize, say at $S^m = S^{m+1}$. If we denote such S^m by S^ω, then S^ω is globally idempotent, while the Rees quotient S/S^ω is nilpotent. So, [4] covers the cases when S^ω is either a union of groups, or left (right) reductive (which, for example, includes all finite commutative semigroups and all finite semigroups in which S^ω is inverse). This one-theorem note deals with the case when S^ω is regular.

Theorem. *Let S be a finite semigroup such that S^ω is regular. Then either $p_n(S)$ is bounded above by a constant, or $p_n(S) < p_{n+1}(S)$ for n large enough.*

Proof. Let $S^\omega = S^m$. We prove that the required inequality holds for all $n \geq m$ (except for the cases described in [5]). In the sequel, we may assume that every term operation under consideration depends on all of its variables. Otherwise, S satisfies a heterotypical identity, implying S to be a nilpotent extension of a completely simple semigroup (since every finite nil-semigroup is nilpotent), by Lemma 2.3 of [6], or by a result of Chrislock [2]. However, in such a case, our result follows from Theorem 2.2 of [4].

First, choose an integer $r \geq 1$ such that for each $a \in S$ belonging to a subgroup of S, a^r is the identity element of that group (this can be achieved, for example, by taking r to be the l.c.m. of orders of all subgroups of S). Now we define a mapping $\phi : \mathcal{E}_n^S \to \mathcal{E}_{n+1}^S$ by

$$(\phi f)(x_1, \ldots, x_n, x_{n+1}) = (f x_{n+1})^r f.$$

By the above remarks, ϕ is well defined. We proceed by proving that ϕ is injective.

To this end, we shall consider a fixed but arbitrary n-tuple $(a_1, \ldots, a_n)$ of elements of S. Assume that $\phi f = \phi g$ for two (essentially) n-nary term operations f, g of S. For brevity, denote $c = f(a_1, \ldots, a_n)$ and $d = g(a_1, \ldots, a_n)$. Since $n \geq m$, $c, d \in S^\omega$ and thus they are regular. Now, our assumption is that

$$(cx)^r c = (dx)^r d$$

for all $x \in S$. Let c', d' be any inverses for c, d, respectively. By setting $x \to c'$ and $x \to d'$, we obtain $c = (cc')^r c = (dc')^r d = d(c'd)^r$ and $d = (dd')^r d = (cd')^r c = c(d'c)^r$, showing that $c \,\mathcal{H}\, d$. Denote by H the $\mathcal{H}$-class of c and d.

If H is a group, then we immediately have $c = (cx)^r c = (dx)^r d = d$ for any $x \in H$ (since $cx, dx \in H$), by the choice of r. So, let H be a non-group $\mathcal{H}$-class. Then choose idempotents $e \in R_c$ and $h \in L_c$. It follows that the $\mathcal{H}$-class $R_h \cap L_e$ contains inverses $\bar{c}, \bar{d}$ of c, d, respectively. Since $h \in R_{\bar{d}} \cap L_c$ it follows by Theorem 2.17 of [3] that $c\bar{d} \in R_c \cap L_{\bar{d}} = H_e$. Hence, $(c\bar{d})^r = e$, and so $c = ec = (c\bar{d})^r c = (d\bar{d})^r d = d$.

Since the parameters $(a_1, \ldots, a_n)$ were chosen arbitrarily, we conclude that $f = g$. Therefore, it remains to exhibit a term operation of S which is not of the form ϕf for any $f \in \mathcal{E}_n^S$. We claim that the operation $x_1 \ldots x_n x_{n+1}$ will do. We may take for granted that S has a non-group $\mathcal{H}$-class H, for otherwise Theorem 2.2 of [4] provides the required result, as before. Let $a \in H$ be arbitrary, $e \in R_a$ and set $x_{n+1} \to a$, while all other variables are evaluated to e. The assumption that $x_1 \ldots x_n x_{n+1} = \phi f$ yields $a = ea = e^n a = (f(e, \ldots, e)a)^r f(e, \ldots, e) = (ea)^r e = a^r e = (ae)^r$. However, since $H_a = R_e \cap L_a$, by [3, Theorem 2.17] we have that $ae \notin R_a \cap L_e (= H_e)$. On the other hand, $R_{ae} \leq R_a$ and $L_{ae} \leq L_e$, and the preceding argument shows that at least one of these inequalities must be strict. Since the $\mathcal{R}$-classes ($\mathcal{L}$-classes) contained in the same $\mathcal{D}$-class of a finite semigroup must be incomparable (cf. Proposition 3.7 of [8]), $ae \notin D_a$, whence $a = (ae)^r \notin D_a$. The contradiction just obtained completes the proof of the theorem. $\square$

References

[1] Berman, J., *Free Spectra of Finite Algebras*, Lecture Notes, University of Illinois at Chicago, 1986.

[2] Chrislock, J. L., A certain class of identities on semigroups, *Proc. Amer. Math. Soc.* **21** (1969), 189–190.

[3] Clifford, A. H. and Preston, G. P., *The Algebraic Theory of Semigroups, Vol.I*, Amer. Math. Soc., Providence, 1961.

[4] Crvenković, S., Dolinka, I. and Ruškuc, N., Notes on the number of operations of finite semigroups, *Acta Sci. Math. (Szeged)* **66** (2000), 23–31.

[5] Crvenković, S., Dolinka, I. and Ruškuc, N., Finite semigroups with few term operations, *J. Pure Appl. Algebra* **157** (2001), 205–214.

[6] Crvenković, S., Dolinka, I. and Ruškuc, N., The Berman conjecture is true for finite surjective semigroups and their inflations, *Semigroup Forum* **62** (2001), 103–114.

[7] Grätzer G. and Kisielewicz, A., A survey of some open problems on p_n-sequences and free spectra of algebras and varieties, in eds. Romanowska, A. and Smith, J. D. H., *Universal Algebra and Quasigroup Theory*, 57–88, Heldermann Verlag, Berlin, 1992.

[8] Lallement, G., *Semigroups and Combinatorial Applications*, Wiley, New York, 1979.

[9] Marczewski, E., Independence in abstract algebras. Results and problems, *Colloq. Math.* **14** (1966), 169–188.

[10] Willard, R., Essential arities of term operations in finite algebras, *Discrete Math.* **149** (1996), 239–259.

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