Quantified Constraint Satisfaction Problem on semicomplete digraphs

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Abstract

We study the (non-uniform) quantified constraint satisfaction problem QCSP($\mathcal{H}$) as $\mathcal{H}$ ranges over semicomplete digraphs. We obtain a complexity-theoretic trichotomy: QCSP($\mathcal{H}$) is either in P, is NP-complete or is Pspace-complete. The largest part of our work is the algebraic classification of precisely which semicomplete digraphs enjoy only essentially unary polymorphisms, which is combinatorially interesting in its own right.

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1 Introduction

The *quantified constraint satisfaction problem* QCSP($\mathcal{B}$), for a fixed *template* (structure) $\mathcal{B}$, is a popular generalisation of the *constraint satisfaction problem* CSP($\mathcal{B}$). In the latter, one asks if a primitive positive sentence (the existential quantification of a conjunction of atoms) Φ is true on $\mathcal{B}$, while in the former this sentence may be positive Horn (where universal quantification is also permitted). Much of the theoretical research into CSPs is in respect of a large complexity classification project – it is conjectured that CSP($\mathcal{B}$) is always either in P or NP-complete [15]. This *dichotomy* conjecture remains unsettled, although dichotomy is now known on substantial classes (e.g. structures of size ≤ 3 [31, 9] and smooth digraphs [17, 2]). Various methods, combinatorial (graph-theoretic), logical and universal-algebraic have been brought to bear on this classification project, with many remarkable consequences. A conjectured delineation for the dichotomy was given in the algebraic language in [10].

Complexity classifications for QCSPs appear to be harder than for CSPs. Indeed, a classification for QCSPs will give a fortiori a classification for CSPs (if $\mathcal{B} \uplus \mathcal{K}_1$ is the disjoint union of $\mathcal{B}$ with an isolated element, then QCSP($\mathcal{B} \uplus \mathcal{K}_1$) and CSP($\mathcal{B}$) are polynomially equivalent). Just as CSP($\mathcal{B}$) is always in NP, so QCSP($\mathcal{B}$) is always in Pspace. However, no overarching polychotomy has been conjectured for the complexities of QCSP($\mathcal{B}$), as $\mathcal{B}$ ranges over finite structures, but the only known complexities are P, NP-complete and Pspace-complete. It seems plausible that these complexities are the only ones that can be so obtained (for more on this see [13]).

In this paper we study the complexity of QCSP($\mathcal{H}$), where $\mathcal{H}$ is a semicomplete digraph, i.e. an irreflexive graph so that for each distinct vertices x_i and x_j at least one of $x_i x_j$ or $x_j x_i$ (and possibly both) is in $E(\mathcal{H})$. We prove that each such problem is either in P, is NP-complete or is Pspace-complete. In some respects, our paper is a companion to the classifications for partially reflexive forests [26] and partially reflexive cycles [23], however our work here differs in two important ways. Firstly, our classification is a complete trichotomy instead of a partial classification between P and NP-hard. Secondly, our classification uses the algebraic method to derive hardness results, whereas in [26, 23] the main algebraic tool, surjective polymorphisms, appear only for tractability. Indeed, we believe our use of the algebraic method here is the most complex so far for any QCSP trichotomy complexity classification. The first published QCSP trichotomy appeared in (the preprints of) [7] and used relatively straightforward application of the algebraic method pioneered in the same paper. Subsequently, a combinatorial QCSP trichotomy appeared, essentially for irreflexive pseudoforests, in [27]. The task to unite [27, 26, 23], with the spirit of [14], to a QCSP trichotomy for partially reflexive pseudoforests, remains open-ended and ambitious. Two other notable trichotomies have appeared in the QCSP literature in the form of [3] and [4], though both are slightly unorthodox. The former deals with a variant of the QCSP, which allows for relativisation of the universal quantifier, and the latter deals with infinite equality languages.

Our work follows in the spirit of the CSP dichotomy for semicomplete digraphs given long ago in [1]. What we uncover is that the semicomplete digraphs with at most one cycle, whose CSPs are in P as per [1], beget QCSPs which remain in P. However, of the semicomplete digraphs with more than one cycle, whose CSPs are NP-complete, some produce QCSPs of maximal complexity while others remain no more than NP-complete. Our classification is as follows: if $\mathcal{H}$ is a semicomplete digraph then either

- $\mathcal{H}$ contains at most one cycle and QCSP($\mathcal{H}$) is in P, or
- $\mathcal{H}$ contains at least two cycles, a source and a sink and QCSP($\mathcal{H}$) is NP-complete, or
- $\mathcal{H}$ contains at least two cycles, but not both a source and a sink, and QCSP($\mathcal{H}$) is Pspace-complete.

The tractability results, membership for both P and NP, are relatively straightforward and date back to the last author's 2006 Ph.D. thesis [25]. Together with the complexity classification of the CSP for semicomplete digraphs, which was proved in [1], they justify the first two items of the above complexity classification of the QCSP.

The natural trichotomy conjecture for the complexity of QCSP of semicomplete digraphs was made (not in print), but repeated efforts to settle it combinatorially failed. The present work arose from a discussion in Dagstuhl about two related, more specific conjectures involving the algebraic approach, which had always been deemed appropriate as semicomplete digraphs are cores for which all polymorphisms are surjective.

The first of these specific conjectures sought to deal with a large subclass of the semicomplete digraphs, those with neither source nor sink (termed smooth). The conjecture stated that all polymorphisms of smooth semicomplete digraphs with multiple cycles are essentially unary. The largest part of our paper is in proving this conjecture. From the proof of our Theorem 3.3 it follows that the only smooth semicomplete finite digraphs with one cycle are the 2-cycle and the 3-cycle. When this first conjecture is proved, applying [7] we get that for any smooth semicomplete digraph $\mathcal{H}$ which is not the 2-cycle nor the 3-cycle, QCSP($\mathcal{H}$) is Pspace-complete.

The remaining cases, after removing those in NP and the smooth ones, are where there is more than one cycle and no source (dually resp., sink) but there is a sink (dually resp., source) in the graph. W.l.o.g. we assume that there is no source, but there is a sink in the graph. The remaining case is thus reduced to the digraph $\mathcal{H}$ built by iteratively adding m sinks to a smooth semicomplete digraph $\mathcal{H}'$ with multiple cycles. Suppose $\mathcal{K}_n$ is the irreflexive n-clique and let $\mathcal{K}_n^{\to m}$ be the same graph with m sinks iteratively added. From the first conjecture and [30], Lemma 1.3.1 (b) follows that $\mathrm{Pol}(\mathcal{H}')$ are contained in $\mathrm{Pol}(\mathcal{K}_n)$, where $n = |\mathcal{H}'|$. The second Dagstuhl conjecture held that perhaps $\mathrm{Pol}(\mathcal{H})$ should be contained in $\mathrm{Pol}(\mathcal{K}_n^{\to m})$, and that would be enough to prove Pspace-completeness for the corresponding QCSP (using our Corollary 6.2 (*ii*), which was already known to us at the time). This conjecture turned out to be false, but two substitute digraphs for $\mathcal{K}_n$ in this position were found and between these three they cover all cases. Thus, the Pspace-completeness follows in all remaining cases.

As previously stated, the bulk of our work is in proving all smooth semicomplete digraphs with more than one cycle have only essentially unary polymorphisms. It is easy to see this is not true for semicomplete digraphs which have a source and/or a sink; for each of which a simple ternary essential polymorphism may be given. Thus, we give a classification of the semicomplete digraphs all of whose polymorphisms are essentially unary. This could be the first part of a larger research program, beginning with semicomplete digraphs, which may continue to larger classes. For example, it is known precisely which smooth core digraphs have a weak near unanimity polymorphism [2] and which digraphs enjoy Mal'cev [11].

An extended abstract of this paper, omitting most of the proofs, appeared as [28]. We have significantly simplified the proofs for this journal version. The paper is organised as follows: After this introductory section, we give the definitions and terminology in the second section. The third section proves the upper bounds of complexity for all cases which are not Pspace-complete. The next three sections prove that the remaining cases are Pspace-complete, by dealing first with the strongly connected semicomplete digraphs in Section 4, then with smooth semicomplete digraphs in Section 5, and finally with all semicomplete digraphs in Section 6.

2 Preliminaries

Let $[n] := \{1, \ldots, n\}$. All graphs in what follows are directed, that is just a binary relation on a set. We denote *digraphs* by $\mathcal{G}$, $\mathcal{H}$, etc. and their vertex and edge sets by $V(.)$ and $E(.)$, respectively, where we might omit the (.) if this is clear. Note that any directed graph G has its dual G^{∂}. This is obtained by reversing all edges. Then sources become sinks and vice versa, in-degree of a vertex becomes its out-degree and so on. In fact, any statement and any proof can also be dualized. Rather than writing the same proof twice, we just say "dually" to indicate we use the dual argument to prove the dual statement.

A digraph $\mathcal{H}$ is *semicomplete* if it is irreflexive (loopless) and for any two distinct vertices i and j, at least one of ij and ji is an edge of $\mathcal{H}$. If $E(\mathcal{H})$ never contains both ij and ji, then it is furthermore a *tournament*. The equivalence relation of strong connectedness is defined in the usual way and its equivalence classes will be called strong components. If the strong component has one element, it is trivial, otherwise nontrivial. We start by noting that, just like in the case of tournaments, in semicomplete graphs the strong components can be linearly ordered, so that there is an edge out of every vertex in a smaller strong component into every vertex of a larger strong component (but never an edge going the other way, obviously).

The problems CSP($\mathcal{H}$) and QCSP($\mathcal{H}$) each take as input a sentence Φ, and ask whether this sentence is true on $\mathcal{H}$. For the former, the sentence involves the existential quantification of a conjunction of atoms – *primitive positive* (pp) logic. For the latter, the sentence involves the arbitrary quantification of a conjunction of atoms – *positive Horn* (pH) logic. It is well-known, for finite $\mathcal{H}$, that CSP($\mathcal{H}$) and QCSP($\mathcal{H}$) are in NP and Pspace, respectively.

The *direct product* $\mathcal{G}\times\mathcal{H}$ of two digraphs $\mathcal{G}$ and $\mathcal{H}$ has vertex set $\{(x,y) : x \in V(\mathcal{G}), y \in V(\mathcal{H})\}$ and edge set $\{((x,u),(y,v)) : x,y \in V(\mathcal{G}), u,v \in V(\mathcal{H}), xy \in E(\mathcal{G}), uv \in E(\mathcal{H})\}$. Direct products are (up to isomorphism) associative and commutative. The kth power $\mathcal{G}^k$ of a graph $\mathcal{G}$ is $\mathcal{G}\times\ldots\times\mathcal{G}$ (k times). A homomorphism from a graph $\mathcal{G}$ to a graph $\mathcal{H}$ is a function $h : V(\mathcal{G}) \to V(\mathcal{H})$ such that, if $xy \in E(\mathcal{G})$, then $h(x)h(y) \in E(\mathcal{H})$. A *$k$-ary polymorphism* of a graph $\mathcal{H}$ is a homomorphism from $\mathcal{H}^k$ to $\mathcal{H}$. A polymorphism f is idempotent when, for all x, $f(x,\ldots,x) = x$. We write $Pol(\mathcal{G})$ ($Pol_{id}(\mathcal{G})$) for the set of all (idempotent) polymorphisms of $\mathcal{G}$. A function $f(x_1,\ldots,x_n)$ *depends* on x_i if there exist $a_1,\ldots,a_n,a'_i$ such that $f(a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_n) \neq f(a_1,\ldots,a_{i-1},a'_i,a_{i+1},\ldots,a_n)$. The essential arity of f is the number of variables on which it depends.

A digraph is a *core* if all of its endomorphisms are automorphisms. All finite semicomplete digraphs are cores, for which all polymorphisms are surjective. For cores it is well-known the constants are pp-definable up to automorphism. That is, if $\mathcal{H}^c$ is $\mathcal{H}$ with all constants named, and $\mathcal{H}$ is a core, then CSP($\mathcal{H}$) and CSP($\mathcal{H}^c$) are poly time equivalent; and the same applies to the QCSP. A similar argument, given in the algebraic language, is in our Proposition 4.2, and the implication is that we may as well assume all the polymorphisms of a semicomplete digraph $\mathcal{H}$ are idempotent (because this is true for $\mathcal{H}^c$ which is actually the structure we will be working on).

The now-celebrated *algebraic approach* to CSP rests on one half of a Galois correspondence [5, 6, 16], where it is observed that the relations that are invariant under (preserved by) the polymorphisms of $\mathcal{H}$ are precisely the relations that are pp-definable in $\mathcal{H}$. For QCSP, in [7], Theorem 3.16 and Proposition 3.12, was obtained a similar characterisation substituting surjective polymorphisms for polymorphisms and pH for pp. The consequence of this is that if the polymorphisms (resp., surjective polymorphisms) of $\mathcal{H}$ are a subset of those of $\mathcal{H}'$, then there is a poly time reduction from CSP($\mathcal{H}'$) to CSP($\mathcal{H}$) (resp., QCSP($\mathcal{H}'$) to QCSP($\mathcal{H}$)); that is, the polymorphisms control the complexity. We will use another well-known special case of [5, 6, 16]: a relation is invariant under all idempotent polymorphisms of $\mathcal{H}$ (i.e., it is invariant under all polymorphisms of $\mathcal{H}^c$, the digraph $\mathcal{H}$ augmented with all one-element unary relations) iff it is pp-definable via the edge relation and the constants.

Certain types of polymorphisms are important in the algebraic approach, or are going to play a role in our paper, so we define them here. An operation $t : V^n \to V$, where $n \geq 3$ is a *near-unanimity* operation if, for all $x, y \in V$, $t(x, x, \ldots, x, y) = t(x, x, \ldots, x, y, x) = \ldots = t(y, x, x, \ldots, x) = x$. A ternary ($n = 3$) near-unanimity operation is called a *majority* operation. An operation $d : V^3 \to V$ is a *Mal'cev* operation if, for all $x, y \in V$, $d(x, y, y) = d(y, y, x) = x$. The main result of [20] proves that digraphs which enjoy a Mal'cev polymorphism must also admit a majority polymorphism, a property of digraphs not true in finite relational structures with more complicated language than digraphs. Finally, $w : V^n \to V$, where $n \geq 2$ is a *weak near-unanimity* operation if, for all $x, y \in V$, $w(x, x, \ldots, x, y) = w(x, x, \ldots, x, y, x) = \ldots = w(y, x, x, \ldots, x)$ and $w(x, x, \ldots, x) = x$. If $\mathcal{H}$ is a core digraph with no weak near-unanimity polymorphisms then CSP($\mathcal{H}$) is NP-complete [10, 24]. Note that a near-unanimity operation is a weak near-unanimity operation, so by the result of [20], if a digraph has no weak near-unanimity polymorphisms, it has neither a Mal'cev nor near-unanimity polymorphisms. That statement actually holds in all finite models, though we care only about digraphs here. If the finite model has a Mal'cev polymorphism, then it has a weak near-unanimity polymorphism, though it might have no near-unanimity polymorphism, by [19] and [24].

We summarize the impact of existence and non-existence of various polymorphisms:

Proposition 2.1. *Let $\mathcal{H}$ be a core digraph. If $\mathcal{H}$ has a Mal'cev or a near-unanimity polymorphism, then QCSP($\mathcal{H}$) is in P. If $\mathcal{H}$ has no weak near-*

unanimity polymorphism, then QCSP($\mathcal{H}$) is NP-hard and CSP($\mathcal{H}$) is NP-complete. If $\mathcal{H}$ has only essentially unary polymorphisms, then QCSP($\mathcal{H}$) is Pspace-complete. All these results also hold for $\mathcal{H}^c$.

Proof. All of these follow from [7, 10, 24, 20]. $\square$

If Φ is an input for QCSP($\mathcal{H}$) with quantifier-free part φ, then with this we associate the digraph $\mathcal{G}_\varphi$ whose vertices are variables of φ and edges are given by the atoms in φ. If Φ is existential, i.e. also an input to CSP($\mathcal{H}$), then the relationship between Φ and $\mathcal{G}_\Phi$ is that of canonical query to canonical database [21].

In a digraph, a *source* (resp., *sink*) is a vertex with out-degree (resp. in-degree) 0. A digraph with no sources or sinks is called *smooth*. In a semicomplete graph, a source s (resp., sink t) satisfies, for all $x \neq s$ (resp., $x \neq t$), $xs \notin E(\mathcal{H})$ and $sx \in E(\mathcal{H})$ (resp., $tx \notin E(\mathcal{H})$ and $xt \in E(\mathcal{H})$). A digraph may have multiple sources or sinks, but a semicomplete may have at most one of each. If $\mathcal{H}$ is a digraph, then let $\mathcal{H}^{\to j}$ be $\mathcal{H}$ with, iteratively, j sinks added (i.e. each time we add a sink we make it forward-adjacent to each existing vertex). Let us label these added sinks, in order, $t_1, \ldots, t_j$ (thus t_j is the unique sink of $\mathcal{H}^{\to j}$). Similarly, let $\mathcal{H}^{\leftarrow j}$ be $\mathcal{H}$ with j sources added. We write $\mathcal{H}^{\to}$ (resp., $\mathcal{H}^{\leftarrow}$) for $\mathcal{H}^{\to 1}$ (resp., $\mathcal{H}^{\leftarrow 1}$). We denote by a^+ and a^- the sets $\{x \in V : ax \in E\}$ and $\{x \in V : xa \in E\}$, respectively. Also, for $S \subseteq V$, we write S^+ for the union $\bigcup\{a^+ : a \in S\}$ and dually $S^- = \bigcup\{a^- : a \in S\}$. The notation $S^{\forall +}$ (resp. $S^{\forall -}$) will stand for $\bigcap\{a^+ : a \in S\}$ (resp. $\bigcap\{a^- : a \in S\}$). By $\preceq_{\mathcal{H}}$ we denote the relation on V defined by $x \preceq_{\mathcal{H}} y$ iff $x^- \subseteq y^-$.

Proposition 2.2. *Let $\mathcal{H} = (V, \to)$ be semicomplete. Then $\preceq_{\mathcal{H}}$ is a partial order, $\preceq_{\mathcal{H}}$ has the largest element t iff t is a sink, and dually for least elements and sources.*

Proof. The relation $\preceq_{\mathcal{H}}$ is always reflexive and transitive since $\subseteq$ is. In semicomplete graphs, if $x, y \in V(\mathcal{H})$ are distinct, then we have $x \in y^- \setminus x^-$ or $y \in x^- \setminus y^-$, so $x^- \neq y^-$ and $\preceq_{\mathcal{H}}$ is antisymmetric.

If t is a sink, then $t^- = V \setminus \{t\}$, and t is in no set of the form x^-, so t is clearly the greatest element in $\preceq_{\mathcal{H}}$. Conversely, if $t \to x$, then $t \in x^-$ and since $t \notin t^-$, thus $\neg x \preceq_{\mathcal{H}} t$, implying that t is not the largest element with respect to $\preceq_{\mathcal{H}}$. $\square$

We mention some special semicomplete graphs that will appear in the paper. $\mathcal{K}_n$ is the irreflexive complete graph (clique) on vertex set $[n]$. $\mathcal{DC}_3$ is the directed 3-cycle. Let $\mathcal{T}_n$ be the transitive tournament on $[n]$ with the natural order $<$ corresponding to the edge relation (i.e. $ij \in E(\mathcal{T}_n)$ iff $i < j$). Let $\overline{\mathcal{T}_n}$ be $\mathcal{T}_n$ with the extant edge $1n$ augmented by $n1$, i.e. this becomes a double-edge.

3 Complexity upper bounds

The results of this section date back to the third author's Ph.D. [25] (available from his website) and are presented there combinatorially and in much fuller detail. The first is very straightforward.

Proposition 3.1. *Let $\mathcal{H}$ be a digraph with both a source s and a sink t, then QCSP($\mathcal{H}$) is in NP.*

Proof. Let Φ be an input to QCSP($\mathcal{H}$) with quantifier-free part φ. Suppose φ has an atom v_iv_j so that Φ quantifies v_i universally, then Φ is a no-instance since φ will never be satisfied when v_i is evaluated as t. Dually, we may assume φ has no atom v_iv_j so that Φ quantifies v_j universally; and we find that Φ can not contain universally quantified variables involved in atoms of φ. Thus, we may evaluate Φ as an input to CSP($\mathcal{H}$) in NP. $\square$

We now turn our attention to the poly time cases. It is well-known that QCSP($\mathcal{K}_2$) and QCSP($\mathcal{DC}_3$) are in P, and there are various ways to see this. One is to note that both $\mathcal{K}_2$ and $\mathcal{DC}_3$ admit a majority polymorphism (which is the first projection in all non-majority evaluations of variables) and then appeal to [7]. We are now interested in the semicomplete graphs $\mathcal{K}_2^{\rightarrow j}$, $\mathcal{K}_2^{\leftarrow j}$, $\mathcal{DC}_3^{\rightarrow j}$ and $\mathcal{DC}_3^{\leftarrow j}$ (for $j > 0$).

Proposition 3.2. *For $j \geq 0$, each of QCSP($\mathcal{K}_2^{\rightarrow j}$), QCSP($\mathcal{K}_2^{\leftarrow j}$), QCSP ($\mathcal{DC}_3^{\rightarrow j}$) and QCSP($\mathcal{DC}_3^{\leftarrow j}$), is tractable.*

Proof. For $j \geq 0$, we will give polynomial time reductions from QCSP ($\mathcal{K}_2^{\rightarrow j+1}$) to QCSP($\mathcal{K}_2^{\rightarrow j}$) and for QCSP($\mathcal{DC}_3^{\rightarrow j+1}$) to QCSP($\mathcal{DC}_3^{\rightarrow j}$). The general result for QCSP($\mathcal{K}_2^{\rightarrow j}$) and QCSP($\mathcal{DC}_3^{\rightarrow j}$) follows by induction, and the arguments for the other cases are clearly analogous.

We will make use of the game-theoretic interpretation of the QCSP. Let Φ be an input for QCSP($\mathcal{H}$) with quantifer-free part φ; then the $(\Phi, \mathcal{H})$-game pitches Universal (male) against Existential (female). They play their own type of variables according to the quantifier order of Φ, each evaluating those variables on $\mathcal{H}$. Once all the variables are evaluated, Existential wins iff the resulting assignment is true of φ on $\mathcal{H}$. It is plain to see that $\mathcal{H} \models \Phi$, i.e. Φ is a yes-instance of QCSP($\mathcal{H}$), iff Existential has a winning strategy in the $(\Phi, \mathcal{H})$-game.

(QCSP($\mathcal{K}_2^{\to j+1}$) to QCSP($\mathcal{K}_2^{\to j}$).) Let Φ be an input for QCSP($\mathcal{K}_2^{\to j+1}$) and φ be its quantifier-free part. Suppose that x_i is universally quantified in Φ and that x_i is not a sink in $\mathcal{G}_\varphi$. Clearly, Φ is a no-instance of QCSP($\mathcal{K}_2^{\to j+1}$) (witnessed when x_i is evaluated as the sink). In this case we set Ψ to be a fixed no-instance of QCSP($\mathcal{K}_2^{\to j}$) (e.g. stating for all x, y there is an edge xy). Otherwise, from φ we will build ψ by removing all atoms $x_i x_j$ where the vertex of $\mathcal{G}_\varphi$ associated with x_j is a sink and both x_i and x_j are existentially quantified in Φ. We now return the quantifiers of Φ to ψ, omitting any variable that has fully disappeared, to create Ψ. We claim $\mathcal{K}_2^{\to j+1} \models \Phi$ iff $\mathcal{K}_2^{\to j} \models \Psi$ and we will use a game-theoretic argument to show this.

(Forwards: $\mathcal{K}_2^{\to j+1} \models \Phi$ implies $\mathcal{K}_2^{\to j} \models \Psi$.) Recall from the definitions the vertices $t_1, \ldots, t_j(, t_{j+1})$ which were added to $\mathcal{K}_2$ to make $\mathcal{K}_2^{\to j}$ ($\mathcal{K}_2^{\to j+1}$) and that t_{j+1} is the sink of $\mathcal{K}_2^{\to j+1}$ and t_j is the sink of $\mathcal{K}_2^{\to j}$. Suppose Existential has a winning strategy in the $(\Phi, \mathcal{K}_2^{\to j+1})$-game. We claim Existential may win with exactly the same strategy in the $(\Psi, \mathcal{K}_2^{\to j})$-game, and to see this it is enough to see that any vertex $x \in \mathcal{G}_\varphi$ for which Existential played t_{j+1} in the $(\Phi, \mathcal{K}_2^{\to j+1})$-game was removed when building Ψ.

(Backwards: $\mathcal{K}_2^{\to j} \models \Psi$ implies $\mathcal{K}_2^{\to j+1} \models \Phi$.) Here, Existential builds a winning strategy in the $(\Phi, \mathcal{K}_2^{\to j+1})$-game by augmenting her winning strategy in the $(\Psi, \mathcal{K}_2^{\to j})$-game with the rule that any existential variable that subsists in Φ but not in Ψ may be played as t_{j+1}.

(QCSP($\mathcal{DC}_3^{\to j+1}$) to QCSP($\mathcal{DC}_3^{\to j}$).) This case reads exactly as the previous with $\mathcal{DC}_3$ substituted everywhere for $\mathcal{K}_2$. ☐

We will now prove which semicomplete digraphs are tractable and which are NP-complete. The remainder of the paper proves that all other cases are Pspace-complete.

Theorem 3.3. *Let $\mathcal{H}$ be a semicomplete digraph. If $\mathcal{H}$ has at most one directed cycle, then QCSP($\mathcal{H}$) is in P. If $\mathcal{H}$ has more than one directed cycle, but also a source and a sink, then QCSP($\mathcal{H}$) is NP-complete.*

Proof. If $\mathcal{H}$ has both a source and a sink, in particular if it has no cycles, then by Proposition 3.1, QCSP($\mathcal{H}$) reduces to CSP($\mathcal{H}$). The complexity of the CSP for semicomplete digraphs was classified in [1], and in the case when $\mathcal{H}$ contains both a source and a sink it coincides with our assertion. Assuming that $\mathcal{H}$ contains no sinks or no sources, it must contain at least one cycle. Note that any nontrivial strong component has at least one cycle, the Hamiltonian cycle for that component. Moreover, any cycle of length 4 or more has a diagonal by semicompleteness, and therefore a smaller cycle inside it. So, "at most one cycle" means "at most one nontrivial strong component, of size ≤ 3". The only semicomplete digraphs with just one cycle and without a source are $\mathcal{K}_2$, $\mathcal{DC}_3$, $\mathcal{K}_2^{\to j}$ and $\mathcal{DC}_3^{\to j}$, while the only semicomplete digraphs with just one cycle and a source, but without a sink are $\mathcal{K}_2^{\leftarrow j}$ and $\mathcal{DC}_3^{\leftarrow j}$, all of which were dealt with in Proposition 3.2 and the remarks preceding it. $\square$

4 Strongly connected case

This section proves that all strongly connected semicomplete digraphs not covered by Theorem 3.3 induce Pspace-complete QCSP when they are templates. The section is divided in three parts. The initial part establishes useful preliminary lemmas and states the actual result on polymorphisms which we will prove and which will imply the desired Pspace-hardness result. The first subsection is devoted to a subclass of strongly connected semicomplete digraphs which we call the P-graphs. P-graphs will serve both as a part of our inductive base in the main proof, and also in the second subsection we will use various ways the assumption that the digraph under consideration is not a P-graph, since those have been dealt with in the first subsection.

The following easy lemma will be used a few times in the paper. It was used in [2], but probably is folklore. Before we state it, we define the following notation for tuples: $(x^i y^j z^k) = (\underbrace{x, \ldots, x}_{i}, \underbrace{y, \ldots, y}_{j}, \underbrace{z, \ldots, z}_{k})$. Moreover, if $f(x_1, \ldots, x_n)$ is an operation and $1 \leq i \leq n$, then $f_i(x, y)$ will denote $f(x^{i-1} y x^{n-i})$.

Lemma 4.1. *Let a set* $\mathbf{C}$ *of idempotent operations on the set* A, $|A| > 1$, *be closed under identification of variables and contain no near-unanimity nor Mal'cev operations and only the two projections among its binary operations. Then for all* $f \in \mathbf{C}$ *with arity* $n > 0$, *there exists precisely one* i *such that* $1 \leq i \leq n$ *and* $f_i(x, y) = y$ *(and thus,* $f_j(x, y) = x$ *for all* $j \neq i$, $1 \leq j \leq n$*).*

Proof. Assume $n > 2$. If no such i exists, then f is a near-unanimity operation. On the other hand, if there were two such $i' < i''$, then $m(x, y, z) := f(y^{i'-1} x y^{i''-i'-1} z y^{n-i''})$ would be a Mal'cev operation, since $m(y, x, x) = f_{i'}(x, y)$ and $m(x, x, y) = f_{i''}(x, y)$. The case $n = 1$ is trivial. If $n = 2$, the opposite assumption yields $x = y$, contradicting $|A| > 1$. The parenthesized remark follows from $f_j(x, y) = x$ or $f_j(x, y) = y$ for all j. $\square$

All finite semicomplete digraphs are cores, since any endomorphism must be injective by semicompleteness, and therefore an automorphism by finiteness. In the case of core digraphs, we can easily strengthen Theorem 5.2 of [7], which states that for all finite digraphs $\mathcal{G}$ which have only essentially unary surjective polymorphisms, QCSP($\mathcal{G}$) is Pspace-complete.

Proposition 4.2. *For all finite core digraphs $\mathcal{G}$ which have no idempotent polymorphisms other than projections, QCSP($\mathcal{G}$) is Pspace-complete.*

Proof. If a core digraph has a k-ary polymorphism f, then $\alpha(x) := f(x, x, \ldots, x)$ is a unary polymorphism, so α is an automorphism (in particular, this means that all polymorphisms of core digraphs are surjective). Define $g(x_1, \ldots, x_k)$ to be $\alpha^{-1}(f(x_1, \ldots, x_k))$. Clearly, g is an idempotent polymorphism of $\mathcal{G}$. Moreover, g has the same essential arity as f, since α is a bijective map, and thus any pair of n-tuples are mapped to distinct elements f iff they are mapped to distinct elements by $\alpha^{-1} \circ f = g$. Since g is a projection, which is essentially unary, then f is also essentially unary, so by Theorem 5.2 of [7], QCSP($\mathcal{G}$) is Pspace-complete. □

When $\mathcal{G}$ is smooth and semicomplete, we will investigate the idempotent polymorphisms of $\mathcal{G}$ and those are precisely the polymorphisms of $\mathcal{G}^c$. So, the new structure we will be working on is $\mathcal{G}^c$, as announced in the Preliminaries section. From the polymorphisms side, the idempotent polymorphisms of $\mathcal{G}$ are the same as polymorphisms of $\mathcal{G}^c$, so it makes no difference whether we speak about one or the other. However, if we are trying to compute relations compatible with all idempotent polymorphisms of $\mathcal{G}$, those are precisely the relations definable via primitive positive formulae (pp-definable) from all one-element unary relations (constants) and the edge relation, i.e. from $\mathcal{G}^c$.

In this section our goal is to prove

Theorem 4.3. *If $\mathcal{G}$ is a strongly connected semicomplete digraph with more than one cycle, then QCSP($\mathcal{G}$) is Pspace-complete.*

and we will do it by proving that all strongly connected semicomplete digraphs with more than one cycle have no idempotent polymorphisms other than the projections and then invoking Proposition 4.2.

We start by noting that, just like in the case of tournaments, in semicomplete digraphs the strong components can be linearly ordered, so that there is an edge out of every vertex in a smaller strong component into every vertex of a larger strong component (but never an edge going the other way, obviously). In case of strongly connected digraphs, this seems like a non-issue since there is a single strong component, but it will arise in some subgraphs. For the rest of this section, $\mathcal{G} = (V, E)$ is a strongly connected semicomplete digraph which is not a cycle.

Definition 4.4. Let L be a subset of V. We define the relation $\equiv_L$ on V by: $u \equiv_L v$ iff

1. $u^+ \cap L = v^+ \cap L$ and $u^- \cap L = v^- \cap L$, or
2. $\{u, v\} \subseteq L$.

The relation defined by just (1) is clearly an equivalence relation since the equality is an equivalence relation. Thus, $\equiv_L$ is the union of the relation defined by (1) and $L \times L$. To prove that $\equiv_L$ is an equivalence relation, we need to prove that no elements $u \in L$ and $v \notin L$ can be $\equiv_L$-related. We know that $uv \in E$ or $vu \in E$, so u is either in $v^- \cap L$ or in $v^+ \cap L$ (or both), but u is in neither of the sets $u^- \cap L$ and $u^+ \cap L$. In particular, L is a $\equiv_L$-class, which will be useful presently.

Lemma 4.5. *Let L be a subset of V such that the induced subgraph on L is strongly connected and let v be a vertex such that $v^+ \cap L \neq \emptyset \neq v^- \cap L$. If f is an n-ary idempotent polymorphism of $\mathcal{G}$ which is the first projection on L, then $f(v, a_2, \ldots, a_n) = v$, where $a_i \in L \cup \{v\}$ for all $2 \leq i \leq n$.*

Proof. Assume that $f(v, a_2, \ldots, a_n) \neq v$ and that v is selected to have maximal $|v^- \cap L| + |v^+ \cap L|$ among the vertices in $V \setminus L$ which satisfy that $v^+ \cap L \neq \emptyset \neq v^- \cap L$ and that there exist vertices $a_i \in L \cup \{v\}$, for all $2 \leq i \leq n$, such that $f(v, a_2, \ldots, a_n) \neq v$.

So, let $u = f(v, a_2, \ldots, a_n) \neq v$. First we prove that $u \equiv_L v$. Let $a \in L$. If $v \to a$, then pick $a'_2, \ldots, a'_n \in L$ such that $a'_i = a$ if $a_i = v$ and $a_i \to a'_i$ if $a_i \in L$. We get $u = f(v, a_2, \ldots, a_n) \to f(a, a'_2, \ldots, a'_n) = a$. Similarly, if $a \to v$, then pick $a'_2, \ldots, a'_n \in L$ such that $a'_i = a$ if $a_i = v$ and $a'_i \to a_i$ if $a_i \in L$. We get $a = f(a, a'_2, \ldots, a'_n) \to f(v, a_2, \ldots, a_n) = u$. So, we proved that $v^+ \cap L \subseteq u^+ \cap L$ and $v^- \cap L \subseteq u^- \cap L$. Assume that one of those subsets is proper. As we know $u \neq v$, and suppose that $u \to v$. Select $b_2, \ldots, b_n \in L$ such that $b_i \to a_i$ for all $2 \leq i \leq n$. Then $f(u, b_2, \ldots, b_n) \to f(v, a_2, \ldots, a_n) = u$, so we get that $f(u, b_2, \ldots, b_n) \neq u$. But this contradicts the choice of v, since $\emptyset \neq v^+ \cap L \subseteq u^+ \cap L$, $\emptyset \neq v^- \cap L \subseteq u^- \cap L$ and $|u^- \cap L| + |u^+ \cap L| > |v^- \cap L| + |v^+ \cap L|$. The alternative is that $u^- \cap L = v^- \cap L$ and $u^+ \cap L = v^+ \cap L$. The case when $v \to u$ is proved dually.

Let $v_1 := f(v, a_2, \ldots, a_n)$ and we know from above considerations that $v_1 \equiv_L v$, and $v_1 \neq v$. So, we may assume without loss of generality that

$v \to v_1$. We denote by U the equivalence class $v/\equiv_L$. Let us define a sequence $v_0, v_1, v_2, \ldots$ of elements of U recursively, and together with it $n-1$ more auxiliary sequences $a_2^{(i)}$, $a_3^{(i)}$,..., $a_n^{(i)}$. We start with setting $v_0 := v$, fixing a Hamiltonian cycle C going through L, and define $a_i^{(0)} := a_i$ for all $2 \leq i \leq n$. We define $v_{i+1} := f(v_i, a_2^{(i)}, \ldots, a_n^{(i)})$, and once v_{i+1} is known, the auxiliary sequences for each j, $2 \leq j \leq n$, are defined like this:

- If $a_j^{(i)} = v_i$, then $a_j^{(i+1)} := v_{i+1}$,
- if $a_j^{(i)} \in L \cap v^-$, then $a_j^{(i+1)} := v_{i+1}$ and
- if $a_j^{(i)} \in L \setminus v^-$, then we select $a_j^{(i+1)}$ to be the next element along the fixed Hamiltonian cycle for L, that is, the edge $a_j^{(i)} a_j^{(i+1)}$ is in the Hamiltonian cycle C.

To give a more informal idea of the proof in order to avoid getting lost in notation, we are walking through U by the sequence $v_0, v_1, ...$ while simultaneously walking through L along the Hamiltonian cycle with the parameters until we get a chance to jump with a parameter to the next v. We know that initially all positions are evaluated as elements of $L \cup \{v\}$ ($v = v_0$), and this property continues, at the ith iteration all positions in $f(x_1, x_2, ..., x_n)$ are evaluated as elements of L or as v_i. However, we gradually make more and more of them equal to v_i. Eventually, we are going to get that $v_{k+1} = f(v_k, a_2^k, a_3^k, ..., a_n^k) = f(v_k, v_k, ..., v_k) = v_k$, which is a contradiction since there should be an edge between them.

Now more formally, we prove by the induction on i that $v_i \in U$, that all $a_j^{(i)} \in L \cup \{v_i\}$ and that $v_i \to f(v_i, a_2^{(i)}, \ldots, a_n^{(i)}) = v_{i+1}$ (in particular, $v_i \neq v_{i+1}$). All three claims hold for $i = 0$ by our choice of v and $a_2, \ldots, a_n$ and from the fact that $v \to v_1$.

Now assume that $v_k \in U$, that all $a_j^{(k)} \in L \cup \{v_k\}$ and that $v_k \to v_{k+1}$. We see that the proof of $v_1 \equiv_L v$ from the second paragraph applies in proving that $v_k^+ \cap L \subseteq v_{k+1}^+ \cap L$ and $v_k^- \cap L \subseteq v_{k+1}^- \cap L$, so $v_{k+1} \in U$. We need to prove that $a_j^{(k)} \to a_j^{(k+1)}$. If $a_j^{(k)} = v_k$, then $a_j^{(k+1)} = v_{k+1}$ and $a_j^{(k)} \to a_j^{(k+1)}$ follows from $v_k \to v_{k+1}$ which is true by the inductive assumption. If $a_j^{(k)} \in v^- \cap L$, then from $v_k \in U$ (in other words, $v_k \equiv_L v$) and from $v_k^- \cap L \subseteq v_{k+1}^- \cap L$ it follows that $a_j^{(k)} \in v_{k+1}^- \cap L$, hence $a_j^{(k)} \to v_{k+1} = a_j^{(k+1)}$. Finally, if $a_j^{(k)} \in L \setminus v^-$, then by definition $a_j^{(k)} \to a_j^{(k+1)}$ along the Hamiltonian cycle. From the assumption that f is a polymorphism, we obtain that $v_{k+1} = f(v_k, a_2^{(k)}, \dots, a_n^{(k)}) \to f(v_{k+1}, a_2^{(k+1)}, \dots, a_n^{(k+1)}) = v_{k+2}$. Therefore, $v_{k+1} \neq f(v_{k+1}, a_2^{(k+1)}, \dots, a_n^{(k+1)}) = v_{k+2}$ by semicompleteness and since all $a_i^{k+1} \in L \cup \{v_{k+1}\}$, then by the maximality of $|v^- \cap L| + |v^+ \cap L|$ we get that $v_{k+1}^- \cap L = v^- \cap L$ and $v_{k+1}^+ \cap L = v^+ \cap L$, so $v_{k+1} \in U$, which completes the inductive proof.

We know that if $a_i \neq v$, then the sequence $a_i^{(0)}, a_i^{(1)}, a_i^{(2)}, \dots$ will contain an element $a_i^{(k)}$ which is in $v^- \cap L$, as this sequence is initially moving along the Hamiltonian cycle for L, and at some point it must reach elements of the nonempty subset $L \cap v^-$. Then $a_i^{(k+1)} = v_{k+1}$ and that auxiliary sequence will from that point on be equal to the main sequence, that is $a_i^{(j)} = v_j$ for all $j > k$. This will eventually happen with all auxiliary sequences, with k at most $|L| - 1$ (as by that time all members of L will have occurred in the Hamiltonian cycle). So, we know that $a_i^{(|L|)} = v_{|L|}$ for all i. But then we derive the final contradiction from $v_{|L|+1} = f(v_{|L|}, v_{|L|}, \dots, v_{|L|}) = v_{|L|}$ by idempotence of f, which is impossible since we proved that $v_{|L|} \to v_{|L|+1}$. $\square$

The following definition shortens our notation and makes terminology a little easier.

Definition 4.6. A subset $L \subseteq V$ is *nice* if the induced subgraph on L is strongly connected and all idempotent polymorphisms of $\mathcal{G}$ restrict to L as projections.

Lemma 4.7. *Let L be a nice subset of V and let v be a vertex such that $v^+ \cap L \neq \emptyset \neq v^- \cap L$. Then $L \cup \{v\}$ is nice.*

Proof. It suffices to prove that if f is an n-ary idempotent polymorphism of $\mathcal{G}$ which is the first projection on L, then f is the first projection on $L \cup \{v\}$. From Lemma 4.5 we know that we only have to prove that $f(a_1, a_2, \dots, a_n) = a_1$ where $a_1 \in L$ and the other a_i are in $L \cup \{v\}$. We will denote $f(a_1, \dots, a_n)$ by u for shorter notation, and prove that $u = a_1$. Let b_1 be any vertex in $L \cap a_1^+$ and $b_2, \dots, b_n \in L$ be such that $a_i \to b_i$ (they exist since the induced subgraph on L is strongly connected, hence smooth, and since $L \cap v^+ \neq \emptyset$). Therefore, $f(a_1, a_2, \dots, a_n) \to f(b_1, b_2, \dots, b_n) = b_1$, and hence $\emptyset \neq a_1^+ \cap L \subseteq u^+ \cap L$. Dually we prove that $\emptyset \neq a_1^- \cap L \subseteq u^- \cap L$.

Now if $u \neq a_1$, we already know that u is not equal to any other vertices of L, since $L \setminus \{a_1\} = L \cap (a_1^+ \cup a_1^-) = (L \cap a_1^+) \cup (L \cap a_1^-) \subset (L \cap u^+) \cup (L \cap u^-) \subseteq u^+ \cup u^-$. From $u \neq a_1$ we get that $a_1 \to u$ or $u \to a_1$. If $a_1 \to u$, we select $b_2, \dots, b_n \in L$ such that $a_i \to b_i$, and then we get $u = f(a_1, a_2, \dots, a_n) \to f(u, b_2, \dots, b_n) = u$ (the last equality holds by

Lemma 4.5, since $u^+ \cap L \neq \emptyset \neq u^- \cap L$). This is a contradiction with the assumption that $\mathcal{G}$ is loopless. The case when $u \to a$ is dealt with dually. $\square$

Lemma 4.8. *Let $L = \{a, b\}$ be compatible with (i. e. closed under) the idempotent polymorphisms of $\mathcal{G}$ and let $a \leftrightarrow b$. If $v \in V \setminus L$ is such that $v^+ \cap L \neq \emptyset \neq v^- \cap L$ then $\{a, b, v\}$ is nice.*

Proof. Let f be an n-ary idempotent polymorphism of $\mathcal{G}$ and let $i \leq n$ be such that $f(a^i b^{n-i}) = a$ and $f(a^{i-1} b^{n-i+1}) = b$. Such an i must exist since, by idempotence, $f(a^n b^0) = a$ and $f(a^0 b^n) = b$. Since $(a^i b^{n-i}) \to (b^i a^{n-i})$ in the digraph $\mathcal{G}^n$, and $\{a, b\}$ is closed under f, our assumption means also that $f(b^i a^{n-i}) = b$ and $f(b^{i-1} a^{n-i+1}) = a$. Without loss of generality, we can assume that $a \to v$ and $v \to b$.

Claim 1: $f(a^{i-1} v b^{n-i}) = v$. Also, (dually) $f(b^{i-1} v a^{n-i}) = v$.

Let $u := f(a^{i-1} v b^{n-i})$. Notice that $a = f(b^{i-1} a^{n-i+1}) \to f(a^{i-1} v b^{n-i}) = u$ and $u = f(a^{i-1} v b^{n-i}) \to f(b^i a^{n-i}) = b$, so we have proved $a \to u \to b$. Let us consider cases:

Case 1: Let $v \to u$. Define $w := f(u^i a^{n-i})$. We get $u = f(a^{i-1} v b^{n-i}) \to f(u^i a^{n-i}) = w$ and $a = f(a^i b^{n-i}) \to f(u^i a^{n-i}) = w$. Thus, $w = f(u^i a^{n-i}) \to f(w^n) = w$, a contradiction.

Case 2: Suppose that $u \to v$. Define $w := f(b^{i-1} u^{n-i+1})$. We get $w = f(b^{i-1} u^{n-i+1}) \to f(a^{i-1} v b^{n-i}) = u$ and also $w = f(b^{i-1} u^{n-i+1}) \to f(a^{i-1} b^{n-i+1}) = b$. Hence, $w = f(w^n) \to f(b^{i-1} u^{n-i+1}) = w$, a contradiction.

Since neither $u \to v$ nor $v \to u$, it must be that $u = v$, that is $v = f(a^{i-1} v b^{n-i})$. If we reverse all edges and transpose a and b, we obtain the proof of the other statement. $\square$

Claim 2: For any tuple $\overline{c} \in \{b, v\}^{i-1}$,

$$f(\overline{c} b a^{n-i}) = b. \tag{1}$$

Denote $u := f(\overline{c} b a^{n-i})$. We know that $a = f(a^i b^{n-i}) \to f(\overline{c} b a^{n-i}) = u$ and $v = f(a^{i-1} v b^{n-i}) \to f(\overline{c} b a^{n-i}) = u$. Assume that $u \neq b$.

If $b \to u$, then $u = f(\overline{c} b a^{n-i}) \to f(u^n) = u$, a contradiction. The remaining possibility is that $u \to b$. But then $a \to u \to b$ and we can apply Claim 1 to u in place of v to obtain $u = f(a^{i-1} u b^{n-i}) \to f(\overline{c} b a^{n-i}) = u$, again a contradiction. $\square$

Dually, we also have for all $\overline{c} \in \{a, v\}^{i-1}$ that

$$f(\overline{c} a b^{n-i}) = a. \tag{2}$$

Claim 3: For any tuple $\overline{c} \in \{a, b\}^{i-1}$,

$$f(\overline{c}ab^{n-i}) = a \text{ and } f(\overline{c}ba^{n-i}) = b. \tag{3}$$

To see the first equation in Equation (3), fix any $\overline{c} \in \{a, b\}^{i-1}$ and let $\overline{d} \in \{b, v\}^{i-1}$ be such that $\overline{d}(j) = b$ whenever $\overline{c}(j) = a$, while $\overline{d}(j) = v$ whenever $\overline{c}(j) = b$. Now $b \stackrel{(1)}{=} f(\overline{d}ba^{n-i}) \to f(\overline{c}ab^{n-i})$, so from $f(\overline{c}ab^{n-i}) \in \{a, b\}$ follows $f(\overline{c}ab^{n-i}) = a$. The proof of the second equation in Equation (3) is analogous, we just take the proof of the first equation, transpose a and b, reverse all edges and use Equation (2) in place of Equation (1). $\square$

Claim 4: $\{a, b\}$ is nice.

First note that if we take $\overline{c} = a^{i-1}$ in Equation (3), then we obtain $f(a^{i-1}ba^{n-i}) = b$. Take $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_i)$ (g is obtained from f by cyclically permuting the last $n - i + 1$ variables). g is also an idempotent polymorphism of $\mathcal{G}$ and $g(a^{n-1}b) = f(a^{i-1}ba^{n-i}) = b$, while $g(a^n) = a$ by idempotence. Hence, by Claim 3 applied to g we get that for any $\overline{c} \in \{a, b\}^{n-1}$, $g(\overline{c}a) = a$ and $g(\overline{c}b) = b$. In other words, g restricts on $\{a, b\}$ as the nth projection. But then, since f is obtained from g by a permutation of coordinates, f restricts to $\{a, b\}$ as the ith projection. Since f was an arbitrarily chosen idempotent polymorphism of $\mathcal{G}$, and since the induced subgraph on $\{a, b\}$ is strongly connected, this means that $\{a, b\}$ is nice. $\square$

Now Lemma 4.8 follows from Claim 4, $v^+ \cap \{a, b\} \neq \emptyset \neq v^- \cap \{a, b\}$ and Lemma 4.7. $\square$ $\square$

4.1 P-graphs

We start with some well-known definitions. A *tournament* is an irreflexive digraph $\mathcal{T}$ such that for all distinct vertices x and y, *exactly* one of $x \to y$, $y \to x$ is an edge of $\mathcal{T}$. That is, a tournament is a semicomplete digraph without 2-cycles. A tournament is *transitive* (or a *chain*) if the edge relation is a transitive relation, which means it is a strict linear order on the set of all vertices. An intransitive tournament is *locally transitive* if for every vertex v of the tournament the induced subgraphs on v^+ and on v^- are transitive tournaments. We changed this definition from the standard one by adding the word "intransitive" (usually, but not in our paper, transitive tournaments are locally transitive), since we are chiefly interested in the intransitive locally transitive tournaments in this paper. A *congruence* of a tournament $(V, \to)$ is an equivalence relation ρ on V such that for all (x_1, x_2), $(y_1, y_2) \in \rho$ such that $(x_1, y_1) \notin \rho$, $x_1 \to y_1$ iff $x_2 \to y_2$. If ρ is a congruence of the tournament $\mathcal{T} = (V, \to)$, then the *factor tournament* $\mathcal{T}/\rho$ is the tournament $(V/\rho, \Rightarrow)$, where $a/\rho \Rightarrow b/\rho$ iff $a/\rho \neq b/\rho$ and $a \to b$. More generally, in all semicomplete digraphs, we will write $A \Rightarrow B$, where A, B are sets of vertices, to denote that $a \to b$ and $\neg b \to a$ for all $a \in A$ and $b \in B$.

We also introduce the interval notation for a digraph $\mathcal{G} = (\{a_1, \ldots, a_n\}, \to)$ with the fixed Hamiltonian cycle $a_1 \to a_2 \to \ldots \to a_n \to a_1$: $[a_i, a_j]$ is the set of all vertices that are traversed by the path which starts at a_i, ends at a_j and uses only the directed edges of the Hamiltonian cycle, each edge at most once (it is not going the full circle or more). For instance, $[a_2, a_1] = \{a_1, a_2, \ldots, a_n\}$, while $[a_1, a_2] = \{a_1, a_2\}$. We also define $[a_i, a_j) := [a_i, a_j] \setminus \{a_j\}$, $(a_i, a_j] := [a_i, a_j] \setminus \{a_i\}$ and $(a_i, a_j) := [a_i, a_j] \setminus \{a_i, a_j\}$.

The following proposition can be found in [8]:

Proposition 4.9. *If $\mathcal{T}$ is a locally transitive tournament and a, b are vertices of $\mathcal{T}$ such that $b \in a^+$, then b^+ is a union of a terminal interval of the chain a^+ and an initial interval of the chain a^-.*

Proof. Since a^+ is a chain and $b \in a^+$, then $a \notin b^+$ and $b^+ \cap a^+$ is a terminal chain in a^+. If $b^+ \cap a^- = \emptyset$, then we are done (empty set is an interval!). Otherwise, if there exist $c, d \in a^-$ such that $c \to d$, $c \in b^-$ and $d \in b^+$, then $a \to b \to d \to a$ and $a, b, d \in c^+$, so $\mathcal{T}$ is not locally transitive. $\square$

Lemma 4.10. *Let $\mathcal{T} = (\{b_1, \ldots, b_n\}, \to)$ be a locally transitive tournament. Then $\mathcal{T}$ is strongly connected, there exists a Hamiltonian cycle $C = a_1 \to a_2 \to \ldots \to a_n \to a_1$, where $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\}$ and there exists a function $\varphi_{\mathcal{T}} : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that (all intervals are with respect to the cycle C):*

(i) *For all $i \in \{2, 3, ..., n\}$, $\varphi_{\mathcal{T}}(i) \notin \{i-1, i\}$ and also $\varphi_{\mathcal{T}}(1) \notin \{1, n\}$,*

(ii) *$a_i^+ = (a_i, a_{\varphi_{\mathcal{T}}(i)}]$ and*

(iii) *$a_{\varphi_{\mathcal{T}}(i+1)} \in [a_{\varphi_{\mathcal{T}}(i)}, a_i)$ and $a_{\varphi_{\mathcal{T}}(1)} \in [a_{\varphi_{\mathcal{T}}(n)}, a_n)$.*

Proof. First note that $\mathcal{T}$ must be smooth. If $\mathcal{T}$ has a sink b_i, then $\{b_1, \ldots, b_n\} = b_i^- \cup \{b_i\}$. Since the induced subgraph on b_i^- is transitive, then $\mathcal{T}$ must be transitive, too, because it is obtained from b_i^- by adding a new greatest element b_i. An analogous argument proves that the locally transitive tournaments can not have a source.

So assume that $\mathcal{T}$ is smooth. Let us define $a_1 = b_1$, then let $b_1^+ = \{a_2, a_3, \ldots, a_k\}$ so that the strict linear order induced by $\to$ on b_1^+ is $a_2 <$

$a_3 < \ldots < a_k$ (in particular, $a_1 \to a_2 \to \ldots \to a_k$). Finally, let $b_1^- = \{a_{k+1}, a_{k+2}, \ldots, a_n\}$ and let the linear order induced by $\to$ on b_1^- be $a_{k+1} < a_{k+2} < \ldots < a_n$ (in particular, $a_{k+1} \to a_{k+2} \to \ldots \to a_n \to a_1$). Since a_k is not a sink, then a_k^+ is a nonempty initial segment of the chain $a_{k+1} < a_{k+2} < \ldots < a_n$, by Proposition 4.9. Therefore, $a_k \to a_{k+1}$ and the a_is form a Hamiltonian cycle, as desired.

Now define $\varphi_{\mathcal{T}} : \{1, \ldots, n\} \to \{1, \ldots, n\}$ so that $a_{\varphi_{\mathcal{T}}(i)}$ is the greatest element (sink) in the transitive tournament on a_i^+. Since $a_{i-1} \in a_i^-$ and $a_{\varphi_{\mathcal{T}}(i)} \in a_i^+$, the condition (i) of the statement of the Lemma holds. (ii) holds by our choice of $\varphi_{\mathcal{T}}$. Finally, (iii) is a consequence of (i) and the fact that $(a_{i+1}, a_{\varphi(i)}] \subseteq a_{i+1}^+$. $\square$

In particular, since the locally transitive strongly connected tournament $\mathcal{T}$ is semicomplete and from the definition above, we get

$$(iv) \quad a_{\varphi_{\mathcal{T}}(i)+1} \to a_i, \;\; a_i \to a_{i+1}, \;\; \neg\, a_{\varphi_{\mathcal{T}}(i)} \to a_{i+1}, \text{ and } a_i^+ \setminus \{a_{i+1}\} \subseteq a_{i+1}^+$$

(where the addition here is modulo n, so $n + 1 = 1$).

We will use the easier notation for a locally transitive tournament $\mathcal{T}$ when the vertex set is $\{1, 2, \ldots, n\}$, where we will understand, unless otherwise stated, that the fixed Hamiltonian cycle is $1 \to 2 \to \ldots \to n \to 1$, and $a_i = i$, so we will have $(\varphi_{\mathcal{T}}(i) + 1) \to i$ instead of $a_{\varphi_{\mathcal{T}}(i)+1} \to a_i$ et cetera. We illustrate the locally transitive tournaments on Figure 1.

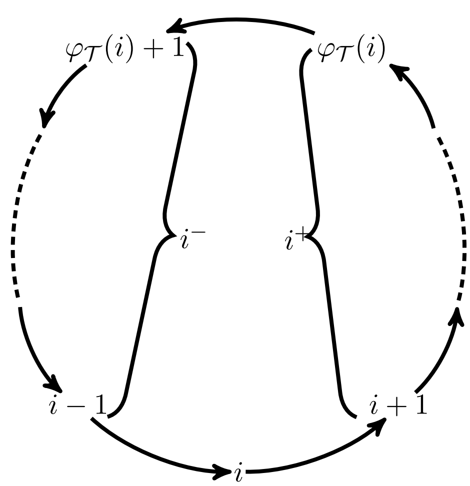

Figure 1: A locally transitive tournament

Definition 4.11. A locally transitive tournament $\mathcal{T} = (\{1,\dots,n\},\to)$ is *regular* iff $n = 2k+1$ for some positive integer k and for all $1 \le i < j \le 2k+1$, $i \to j$ iff $j - i \le k+1$ (otherwise $j \to i$). In other words, in the unique (up to isomorphism) regular locally transitive tournament with $2k+1$ vertices, $\varphi_{\mathcal{T}}(i) = i + k$ if $i \le k+1$, and $\varphi_{\mathcal{T}}(i) = i - k - 1$ if $i > k+1$.

Lemma 4.12. *Let $\mathcal{T} = (\{1,\dots,n\},\to)$ be a locally transitive tournament such that $\varphi_{\mathcal{T}}$ is a permutation of $\{1,\dots,n\}$. Then $\mathcal{T}$ is regular.*

Proof. We first claim that $\varphi_{\mathcal{T}}(i+1) = \varphi_{\mathcal{T}}(i)+1$ (in this proof we repeatedly use the addition modulo n, that is $n+1$ is actually 1, et cetera). We know that $\varphi_{\mathcal{T}}(i+1)$ is not in the interval $[i, \varphi_{\mathcal{T}}(i)]$ by Lemma 4.10 and since $\varphi_{\mathcal{T}}$ is a permutation. Therefore, $\varphi_{\mathcal{T}}(i+1) \in (\varphi_{\mathcal{T}}(i), i)$ from which we conclude $(\varphi_{\mathcal{T}}(i)+1) \in (\varphi_{\mathcal{T}}(i), \varphi_{\mathcal{T}}(i+1)]$. Moreover, $(i+1) \notin (\varphi_{\mathcal{T}}(i), i)$ since $(i+1) \in (i, \varphi_{\mathcal{T}}(i)]$, so $(\varphi_{\mathcal{T}}(i)+1) \in ((i+1), \varphi_{\mathcal{T}}(i+1)]$ and thus $(i+1) \to (\varphi_{\mathcal{T}}(i)+1)$. Also, by (iv) proved after Lemma 4.10 we know that $\varphi_{\mathcal{T}}(i)+1 \to i$. We just proved that $\varphi_{\mathcal{T}}(\varphi_{\mathcal{T}}(i)+1) = i$. Analogously as above we get that $\varphi_{\mathcal{T}}(i)+2 \in (\varphi_{\mathcal{T}}(i)+1, \varphi_{\mathcal{T}}(\varphi_{\mathcal{T}}(i)+1)]$, but $\varphi_{\mathcal{T}}(\varphi_{\mathcal{T}}(i)+2) \neq \varphi_{\mathcal{T}}(\varphi_{\mathcal{T}}(i)+1) = i$, so $\varphi_{\mathcal{T}}(i)+2 \to (i+1)$. Therefore, $\varphi_{\mathcal{T}}(i+1) = \varphi_{\mathcal{T}}(i)+1$, as desired. This implies that the out-degrees of all vertices are the same number, say k, and since $\mathcal{T}$ is a tournament, the in-degree of any vertex is therefore $n-k-1$. Since the number of edges in any digraph is equal to the sum of all out-degrees and also to the sum of all in-degrees, therefore to kn and also to $(n-k-1)n$, we get that $k = n-k-1$, that is $n = 2k+1$. Therefore, we get that $1^+ = \{2,3,\dots,k+1\}$, so from $\varphi_S(1) = k+1$ and from $\varphi_S(i+1) = \varphi_S(i)+1$ for all i we get that S is the regular locally transitive tournament with $2k+1$ vertices. □

Definition 4.13. The semicomplete digraph $\mathcal{G}_{\mathcal{T}} = (V,E)$ will be called a *P-graph* parametrized by the locally transitive tournament $\mathcal{T} = (\{1,\dots,n\},\to$ $)$ if there exists a partition ρ of the vertex set V into nonempty subsets $A_1,\dots,A_n$ such that for all $i \neq j$ and all $a \in A_i$ and $b \in A_j$, $ab \in E$ iff $i \to j$ in $\mathcal{T}$.

Informally, a P-graph is obtained from the locally transitive tournament $\mathcal{T}$ by "expanding" each vertex i into a semicomplete digraph A_i, where between vertices a and b lying in distinct subgraphs A_i and A_j, respectively, the edge is $a \to b$ iff $i \to j$. In case a and b are in the same set A_i, no assumptions are taken (other than semicompleteness).

Lemma 4.14. *Let $\mathcal{T} = (\{1,\dots,n\},\to)$ be a locally transitive tournament. Then*

- *$\rho := \ker \varphi_{\mathcal{T}}$ is a congruence of $\mathcal{T}$,*
- *$\mathcal{T}/\rho$ is a regular locally transitive tournament $\mathcal{T}'$,*
- *$\mathcal{T}$ is a P-graph parametrized by $\mathcal{T}'$, and*
- *every P-graph parametrized by $\mathcal{T}$ is also a P-graph parametrized by $\mathcal{T}'$.*

Proof. ρ is an equivalence relation on $\{1,\ldots,n\}$ since it is the kernel of a function. Let the equivalence classes of ρ be the sets $A_1, A_2, \ldots, A_w$. From $\varphi_{\mathcal{T}}(i) \neq \varphi_{\mathcal{T}}(\varphi_{\mathcal{T}}(i))$ follows that $i/\rho \neq \varphi_{\mathcal{T}}(i)/\rho$, that is, $\varphi_{\mathcal{T}}(i)$ is always outside i/ρ.

Claim 1: Each A_i is $[a_i, b_i]$ for some $a_i, b_i \in \{1,2,\ldots,n\}$.

Assume that $i' < j'$ and $(i', j') \in \rho$, that is, $\varphi_{\mathcal{T}}(i') = m = \varphi_{\mathcal{T}}(j')$. We may find an isomorphic copy $\mathcal{T}_0 = (\{1,2,\ldots,n\}, \Rightarrow)$ by cyclically rotating the names of vertices of $\mathcal{T}$ until m becomes n, and the vertices $\{i', j'\}$ become $\{i, j\}$. We are in the case where $i < j$, $\varphi_{\mathcal{T}'}(i) = \varphi_{\mathcal{T}_0}(j) = n$, the new tournament is locally transitive and the Hamiltonian cycle $1 \to 2 \to \ldots \to n \to 1$ still satisfies the conclusions of Lemma 4.10. We have thus reduced the claim to proving for all integers $k \in (i, j)$ that $\varphi_{\mathcal{T}_0}(k) = n$, as this kind of "convexity" of the ρ-classes implies that all those classes are intervals with respect to the Hamiltonian cycle. From $i < k < j$ and $\varphi_{\mathcal{T}_0}(i) = \varphi_{\mathcal{T}_0}(j) = n$ follows that $i \Rightarrow k \Rightarrow j$ and therefore we know at least that $\varphi_{\mathcal{T}_0}(k) \notin [i, j)$.

First assume for some integer $k \in (i, j)$ that $j \leq \varphi_{\mathcal{T}_0}(k) < n$ and let us select the least such k. Then either $\varphi_{\mathcal{T}_0}(k) < n = \varphi_{\mathcal{T}_0}(k-1)$, or $\varphi_{\mathcal{T}_0}(k-1) < k-1$, either of which implies that $\varphi_{\mathcal{T}_0}(k) \notin [\varphi_{\mathcal{T}_0}(k-1), (k-1))$, contradicting Lemma 4.10 (*iii*). Now we assume that $1 \leq \varphi_{\mathcal{T}_0}(k) < i$ for some integer $k \in (i, j)$, and select the greatest integer $k \in (i, j)$ for which this condition holds. Therefore, $\varphi_{\mathcal{T}_0}(k+1) = n$ (since we just proved that $\varphi_{\mathcal{T}_0}(k+1) \notin (k, n)$). But then $\varphi_{\mathcal{T}_0}(k+1) \notin [\varphi_{\mathcal{T}_0}(k), k)$, which once again contradicts Lemma 4.10 (*iii*). This final contradiction finishes the proof of Claim 1. □

Claim 2: If $i \to j$ for some $i, j \in \{1,2,\ldots,n\}$ and $(i, j) \notin \rho$, then for all $i' \in i/\rho$, $i' \to j$.

Indeed, from Claim 1 and $\varphi_{\mathcal{T}}(i) \notin i/\rho$ we obtain for all $i' \in i/\rho$ that $(i, \varphi_{\mathcal{T}}(i)] \setminus (i/\rho) = (i', \varphi_{\mathcal{T}}(i)] \setminus (i/\rho)$. Therefore, $j \in (i, \varphi_{\mathcal{T}}(i)] \setminus (i/\rho)$ implies that also $j \in (i', \varphi_{\mathcal{T}}(i)] \setminus (i/\rho)$, and since $\varphi_{\mathcal{T}}(i') = \varphi_{\mathcal{T}}(i)$, then $i' \to j$ □

Now we can prove that ρ is a congruence. If $(i, j) \notin \rho$, and (i, i'), $(j, j') \in \rho$, then from Claim 2 follows that if $i \to j$, we get $i' \to j$. Now if $j' \to i'$, by Claim 2 it would follow that $j \to i'$, which contradicts the assumption that $\mathcal{T}$ is a tournament, so the only remaining possibility is that $i' \to j'$. Therefore, ρ is a congruence of $\mathcal{T}$.

Next, from the fact that ρ is a congruence follows that ρ is a partition of the vertex set $\{1, 2, \ldots, n\}$ which satisfies all requirements of Definition 4.13, except that we must show that $\mathcal{T}/\rho$ is a locally transitive tournament. We prove that for any i, if $\varphi_{\mathcal{T}}(i)/\rho = [a, b]$, then $\varphi_{\mathcal{T}}(i) = b$, which follows from $i \to \varphi_{\mathcal{T}}(i)$, the fact that ρ is a congruence and Lemma 4.10 (ii). From $|\{\varphi_{\mathcal{T}}(i) : 1 \leq i \leq n\}| = |\mathcal{T}/\rho|$ and the just proved fact that any class i/ρ contains at most one element of the form $\varphi_{\mathcal{T}}(j)$ follows that the restriction of $\varphi_{\mathcal{T}}$ to $S = \varphi_{\mathcal{T}}(\{1, 2, \ldots, n\})$ is a permutation of the set S. In particular, the tournament $\mathcal{T}/\rho$ is isomorphic to the subgraph induced by $\mathcal{T}$ on S.

Now, let $A_i = [a_i, b_i]$, that is, $A_i \cap S = \{b_i\}$ and $S' = \{b_1, b_2, \ldots, b_w\}$. We want to prove that the subtournament $\mathcal{T}'$ induced by $\mathcal{T}$ on S is a locally transitive tournament which satisfies the conclusion of Lemma 4.10 with respect to the Hamiltonian cycle $b_1 \to b_2 \to \ldots \to b_w \to b_1$. Clearly, b_i^+ and b_i^- are transitive in $\mathcal{T}$, and are thus transitive in any subtournament. If $\varphi_{\mathcal{T}}(b_i) = b_j$, we get that $b_i^+ \cap S = (b_i, b_j] \cap S$ which equals the interval $(b_i, b_j]$ in $\mathcal{T}'$. Thus b_{i+1} is the least element of the chain b_i^+ in $\mathcal{T}'$, in particular $b_i \to b_{i+1}$ for all $1 \leq i \leq w$. Hence $\mathcal{T}'$ is strongly connected, and thus intransitive, so $\mathcal{T}'$ is locally transitive. From the proof of Lemma 4.10, we get that Lemma 4.10 holds with respect to the Hamiltonian cycle $b_1 \to b_2 \to \ldots \to b_w \to b_1$. Finally, if $i \neq j$, then $b_i/\rho \neq b_j/\rho$ and $\varphi_{\mathcal{T}'}(b_i) = \varphi_{\mathcal{T}}(b_i) \neq \varphi_{\mathcal{T}}(b_j) = \varphi_{\mathcal{T}'}(b_j)$. Therefore, $\varphi_{\mathcal{T}'}$ is injective, so it is a permutation, and $\mathcal{T}'$ is regular by Lemma 4.12. Finally, if $\mathcal{G}$ is a P-graph parametrized by $\mathcal{T}$ which is in turn a P-graph parametrized by $\mathcal{T}'$, then "compose" the expansion of vertices of $\mathcal{T}'$ into $\mathcal{T}$ with the expansion of the vertices of $\mathcal{T}$ into $\mathcal{G}$ to prove that $\mathcal{G}$ is a P-graph parametrized by $\mathcal{T}'$. $\square$ $\square$

For proofs of the lemmas and theorem that follow till the end of the subsection, we introduce the following convention: all additions and subtractions are taken modulo $n = 2k + 1$, so whenever the result of an arithmetic operation is outside $[1, n]$, just add the appropriate integer multiple of n to put it back into that interval. The regular locally transitive tournament $\mathcal{T} = (\{1, 2, \ldots, 2k + 1\}, \to)$ is also assumed to have the edge relation $i \to j$ iff $0 < j - i \leq k$ (and $j \to i$ otherwise).

We recall a definition from [18] and a most useful theorem from the paper [2]. A sequence of directed edges in a digraph is an *oriented path* when the undirected graph obtained from it by disregarding orientation is a path. For any oriented path α we define the algebraic length $al(\alpha)$ to be $|\{$edges going forward in $\alpha\}| - |\{$edges going backward in $\alpha\}|$. For a digraph $\mathcal{G} = (V,)$ we put

$$al(\mathcal{G}) = \min\{i > 0 : \ (\exists \text{ a closed path } \alpha) \ al(\alpha) = i\},$$

whenever the set on the right hand side is non-empty and ∞ otherwise. Theorem 8.1 of [2] (sometimes dubbed the 'Loop Lemma') states: *If a smooth digraph has algebraic length one and admits a weak near unanimity polymorphism then it contains a loop.*

Lemma 4.15. *Every idempotent polymorphism f of a regular locally transitive tournament $\mathcal{T} = (\{1, 2, \ldots, 2k+1\}, \to)$, where $k > 1$, is a projection.*

Proof. Let f be a binary idempotent polymorphism of $\mathcal{T}$. $\{i, i+1\}^{\forall +} = \{i+2, \ldots, i+k\}$ and $\{i, i+1\}^{\forall -} = \{i-k+1, \ldots, i-1\}$ for all i, so $\{i, i+1\} \subseteq \{i+2, \ldots, i+k\}^{\forall -} \cap \{i-k+1, \ldots, i-1\}^{\forall +}$. The only elements outside $\{i-k+1, \ldots, i-1\} \cup \{i+2, \ldots, i+k\}$ are i, $i+1$ and $i+k+1$, and as $i+k+1 \notin \{i+2, \ldots, i+k\}^{\forall -}$, hence $\{i-k+1, \ldots, i-1\}^{\forall +} \cap \{i+2, \ldots, i+k\}^{\forall -} = \{i, i+1\}$. Hence, $\{i, i+1\}$ is pp-definable in $\mathcal{T}^c$, so $f(\{i, i+1\} \times \{i, i+1\}) \subseteq \{i, i+1\}$. Here the sets $\{i, i+1\}^{\forall +}$ and $\{i, i+1\}^{\forall -}$ are nonempty since $k > 1$.

If we assume that $f(1, j+1) = i$ for some $1 < j \leq 2k$, then we get that $f(k+2, j-k+1) \to f(k+3, j-k+2) \to \ldots \to f(2k, j-1) \to f(2k+1, j)$ and $f(k+2, j-k+1), f(k+3, j-k+2), \ldots, f(2k+1, j) \in f(1, j+1)^- = i^-$. Since the induced subgraph on i^- is the strict linear order with only one directed path of length k, this implies that $f(m, m+j) = m+i-1$ for all m such that $k+2 \leq m \leq 2k+1$. An analogous argument on i^+ proves that $f(m, m+j) = m+i-1$ for all m such that $2 \leq m \leq k+1$. So it remains to find only $f(1, j+1)$ for all $0 < j \leq 2k+1$.

We assume first that $f(1,2) = 1$. As proved above, $f(1,2) = 1$ implies that $f(i, i+1) = i$ for all i. Assume now that for some $2 \leq j \leq 2k$, $f(1, j+1) = i \neq 1$ and that j is the least such. Then $2k+1 = f(2k+1, j-1) \to f(1, j+1) \to f(3, j+2) = 3$, so $f(1, j+1) \in (2k+1, 3)$ and we obtain $f(1, j+1) = 2$ since we assumed that it is not equal to 1. But, then $2 = f(1, j+1) \to f(2, j+k+1) \to f(3, j+2k+1) = f(3, j)$. If $j = 2$, then we know $f(3,2) \in \{2, 3\}$, while if $j \geq 3$, then $f(3, j) = 3$ by the inductive assumption. Either way, from the fact that $2^- \cap 2^+ = \emptyset = 3^- \cap 2^+$ we derive a contradiction. Thus all binary idempotent polymorphisms are projections. If $f(1,2) = 2$, an analogous proof as above works for $g(x,y) := f(y,x)$, just starting from $j = 2k$ and inductively decreasing j. By proving that g is the first projection, we prove f is the second one.

Now let f be an m-ary polymorphism, $m \geq 3$ and inductively assume that all polymorphisms of smaller arity are projections. We know that $al(\mathcal{T}) = 1$ and $\mathcal{T}$ has no loops, so $\mathcal{T}$ has no weak near-unanimity polymorphisms. By Proposition 2.1, $\mathcal{T}$ has no near unanimity polymorphisms (therefore at least one $f_i(x,y) = y$) and $\mathcal{T}$ has no Mal'cev polymorphisms (therefore at most one $f_i(x,y) = y$, or if $f_i(x,y) = f_j(x,y) = y$, we would be able to make a derived Mal'cev polymorphism from f by treating the ith and jth variable as x and z, respectively, and identifying all others as y). Without loss of generality, assume $f_1(x,y) = y$ and $f_i(x,y) = x$ for all $i \neq 1$. For any evaluation $(a_1, a_2, \ldots, a_m)$ where there is any identification $a_i = a_j$ for some $2 \leq i < j \leq m$, from the inductive assumption we know that $f(a_1, a_2, \ldots, a_m) = a_1$.

Now take any tuple $(a_1, a_2, \ldots, a_m)$ and assume that there exist i, j such that $2 \leq i < j \leq m$ and $a_j \notin \{a_i + k, a_i + k + 1\}$. Then there exists $b_i \in \{a_i, a_j\}^{\forall+}$. Select $b_2, \ldots, b_m$ such that $b_j = b_i \in \{a_i, a_j\}^{\forall+}$ and $b_l = a_l + 1$ for all integers $l \in [2, m] \setminus \{i, j\}$. Then for any $x \in a_1^+$, $f(a_1, a_2, \ldots, a_m) \to f(x, b_2, \ldots, b_m) = x$. We get that $a_1^+ \subseteq f(a_1, a_2, \ldots, a_m)^+$, and since $\mathcal{T}$ is regular, this means that $f(a_1, a_2, \ldots, a_m) = a_1$.

Finally, assume that for all integers i, j such that $2 \leq i < j \leq m$, $a_j \in \{a_i + k, a_i + k + 1\}$. This implies that $m = 3$, and that $a_3 \in \{a_2 + k, a_2 + k + 1\}$. Assume that $a_3 = a_2 + k$. But then for any $x \in a_1^+$, $f(a_1, a_2, a_3) \to f(x, a_3, a_3 + 1) = x$, where the equality follows from the previous case, so $a_1^+ \subseteq f(a_1, a_2, a_3)^+$ and this means that $f(a_1, a_2, a_3) = a_1$. In the case when $a_3 = a_2 + k + 1$, the proof goes the same, except that we use $f(a_1, a_2, a_3) \to f(x, a_2 + 1, a_2)$. $\square$

Lemma 4.16. *Every automorphism f of a regular locally transitive tournament $\mathcal{T} = (\{1, 2, \ldots, 2k+1\}, \to)$ is of the form $f(x) = x + t$ for some fixed t.*

Proof. Clearly all such maps are automorphisms of $\mathcal{T}$. On the other hand, if f is an automorphism of $\mathcal{T}$, then select t so that $f(1) = t + 1$. $f(\{2, \ldots, k+1\}) = f(1^+) = (t+1)^+ = \{t+2, \ldots, t+k+1\}$ and $f(\{k+2, \ldots, 2k+1\}) = f(1^-) = (t+1)^- = \{t+k+2, \ldots, t+2k+1\}$. Since the induced subgraphs on the sets $\{2, \ldots, k+1\}$, $\{t+2, \ldots, t+k+1\}$, $\{k+2, \ldots, 2k+1\}$ and $\{t+k+2, \ldots, t+2k+1\}$ are all transitive touraments with k elements, clearly there can be only one map which isomorphically maps the first onto the second and the third onto the fourth subgraph, and that is $f(x) = x + t$. $\square$

Definition 4.17. Let $\mathcal{G} = (V, \to)$ be a digraph. The subset $I \subseteq V$ is called a *triangular ideal* if for all $a, b \in I$ such that $a \to b$, if $c \in V$ is such that $b \to c \to a$, then $c \in I$. I is trivial if $|I| = 1$ or $I = V$.

Lemma 4.18. *Any digraph $\mathcal{G} = (V, E)$ which contains a regular locally transitive tournament on V as an edge-subgraph has only trivial triangular ideals.*

Proof. Assume that $V = \{1, 2, \ldots, 2k + 1\}$, $\mathcal{T} = (V, \to)$, $\to \subseteq E$, I is a triangular ideal of $\mathcal{G}$ and $|I| \geq 2$. We will prove that for any $a \in I$, $a+k \in I$, too. This will imply that $I = \{1, 2, \ldots, 2k+1\}$, since the element k generates the additive cyclic group Z_{2k+1}.

So, let $b \in I$, $b \neq a$. If $b \in [a+k+1, a+2k]$, then $a \to a+k \to b \to a$, so $a+k \in I$. On the other hand, if $b \in [a+1, a+k-1]$, then first $a+k+1 \in I$ since $b \to a+k+1 \to a \to b$, and then from $a \to a+k \to a+k+1 \to a$ we get $a+k \in I$. □

From now until Theorem 4.25, we fix a finite P-graph $\mathcal{G}_\mathcal{T} = (V, E)$ parametrized by a locally transitive tournament $\mathcal{T}$ and assume that $\mathcal{G}_\mathcal{T}$ is not a 3-cycle. Our goal is Theorem 4.25 which says that $\mathcal{G}_\mathcal{T}$ has only trivial idempotent polymorphisms. According to Lemma 4.14, we may assume that $\mathcal{T}$ is the regular locally transitive tournament $(\{1, 2, \ldots, 2k+1\}, \to)$. The partition from Definition 4.13 is denoted by ρ and we denote by A_i the ρ-class corresponding to i in the parametrization of $\mathcal{G}_\mathcal{T}$ by $\mathcal{T}$.

Lemma 4.19. *Let f be an m-ary idempotent polymorphism of $\mathcal{G}_\mathcal{T}$, let $j_1, \ldots, j_m \in \{1, \ldots, m\}$, let $a_{i,t} \in A_{j_i+t}$ for $i = 1, \ldots, m$ and $t = 0, \ldots, 2k$ and denote $a_t := f(a_{1,t}, a_{2,t}, \ldots, a_{m,t})$. Then there exists $j \in \{1, \ldots, n\}$ such that either $a_t \in A_j$ for all $t = 0, \ldots, 2k$, or $a_t \in A_{j+t}$ for all $t = 0, \ldots, 2k$. (All additions are modulo $2k + 1$).*

Proof. Let $V_1 := \{a_0, \ldots, a_{2k}\}$. $\mathcal{G}_\mathcal{T}$ is parametrized by $\mathcal{T}$, so $s \to t$ in $\mathcal{T}$ iff $(a_{i,s}, a_{i,t}) \in E$ for all i. Thus, $s \to t$ implies $(a_s, a_t) = (f(a_{1,s}, \ldots, a_{m,s}), f(a_{1,t}, \ldots, a_{m,t})) \in E$. Hence, $|V_1| = 2k + 1$ by irreflexivity of E, and the mapping $\varphi : \{1, \ldots, n\} \to V_1$ given by $f(t) = a_t$ ($a_n := a_0$, as usually) is a bijective homomorphism from $\mathcal{T}$ to the induced subgraph $\mathcal{V}_1 := (V_1, E \restriction_{V_1})$. Therefore, $\mathcal{V}_1$ contains an isomorphic copy of $\mathcal{T}$ as an edge-subgraph. We observe that if $x, y \in A_j$ and $z \in V \setminus A_j$, then either $(x, z), (y, z) \in E$, or $(z, x), (z, y) \in E$ and no double edges exist between z and $\{x, y\}$. So, if the restriction of E to $\{x, y, z\}$ contains the edges of a 3-cycle, then from $\{x, y\} \subseteq A_j$ follows that $z \in A_j$, too. Therefore, the intersection of any A_j with the induced subgraph $\mathcal{V}_1$ is a triangular ideal of the latter. Since $\mathcal{V}_1$ contains as an edge-subgraph a tournament isomorphic to $\mathcal{T}$, according to Lemma 4.18, either $V_1 \subseteq A_j$ for some j, or no two elements of V_1 are in the same ρ-class. The first possibility is one of the desired outcomes, so we assume that V_1 is a set of representatives for $\{A_1, \ldots, A_n\}$. In this case, if $a_s \in A_l$ and $a_t \in A_m$, since $s \neq t$ iff $l \neq m$, then $(a_s, a_t) \in E$ iff $l \to m$. Thus the induced subgraph $\mathcal{V}_1$ is a tournament, and the mapping $\psi : V_1 \to \{1, \ldots, n\}$ given by $\psi(a_i) = j$ iff $a_i \in A_j$ is an isomorphism from $\mathcal{V}_1$ to $\mathcal{T}$. Both $\mathcal{V}_1$ and $\mathcal{T}$ have $\binom{n}{2}$ edges, so φ is also an isomorphism, not just a bijective homomorphism. Hence, the composition $\psi \circ \varphi$ is an automorphism of $\mathcal{T}$. The only such, according to Lemma 4.16, are mappings of the sort $f(x) = x + j$ for some fixed $j \in \{0, 1, \ldots, 2k\}$, so if $a_0 \in A_j$, then $a_t \in A_{j+t}$, as desired. ☐

Lemma 4.20. *The equivalence relation ρ is compatible with all idempotent polymorphisms (i. e. it is a congruence of the algebra of polymorphisms).*

Proof. For any ρ-classes $A_{j_1}, \ldots, A_{j_m}$ we want to prove that there exists a ρ-class A_j such that $f(A_{j_1}, \ldots, A_{j_m}) \subseteq A_j$. Let $a_{i,0}, a'_{i,0} \in A_{j_i}$ for $i = 1, \ldots, m$. Select $a_{i,t} \in A_{j_i+t}$ for $i = 1, \ldots, m$ and $t = 1, \ldots, 2k$. Let $a_t = f(a_{1,t}, \ldots, a_{m,t})$, $t = 0, 1, \ldots, 2k$ and $a'_0 = f(a'_{1,0}, \ldots, a'_{m,0})$. Let $a_0 \in A_j$. The sets $V_1 = \{a_0, a_1, \ldots, a_{2k}\}$ and $V'_1 = \{a'_0, a_1, \ldots, a_{2k}\}$ are, according to Lemma 4.19 either both subsets of A_j, or both are sets of representatives for $\{A_1, \ldots, A_n\}$. Either way, we obtain that $a'_0 \in A_j$, as desired. ☐

We know from Lemma 4.20 that any idempotent polymorphism f induces an operation $\hat{f}$ on $\mathcal{T}$ given by $\hat{f}(j_1, j_2, \ldots, j_m) = j$ iff $f(A_{j_1}, \ldots, A_{j_m}) \subseteq A_j$. In other words, $f(A_{j_1}, \ldots, A_{j_m}) \subseteq A_{\hat{f}(j_1, j_2, \ldots, j_m)}$. Given any $j_1, \ldots, j_m \in \{1, \ldots, n\}$, from Lemma 4.19 follows that either $\hat{f}(j_1 + t, j_2 + t, \ldots, j_m + t) = \hat{f}(j_1, j_2, \ldots, j_m)$ for all $0 \leq t < n$, or $\hat{f}(j_1 + t, j_2 + t, \ldots, j_m + t) = \hat{f}(j_1, j_2, \ldots, j_m) + t$ for all $0 \leq t < n$.

Lemma 4.21. *Let $f \in Pol_{id}(\mathcal{G}_\mathcal{T})$ be m-ary. For all $1 \leq t \leq 2k$, if $f(A_{j_1}, \ldots, A_{j_m}) \subseteq A_j$, then $f(A_{j_1+t}, \ldots, A_{j_m+t}) \subseteq A_{j+t}$.*

Proof. From the above considerations, we know that the alternative is that there exist $A_{j_1}, A_{j_2}, \ldots, A_{j_m}$ and A_i such that $f(A_{j_1+t}, A_{j_2+t}, \ldots, A_{j_m+t}) \subseteq A_i$ for all $t = 0, 1, \ldots, 2k$, so we assume that. We select and fix representatives $a_r \in A_r$ for $r = 1, 2, \ldots, 2k+1$. We desire that $f(a_{j_1}, a_{j_2}, \ldots, a_{j_m}) \in A_i \setminus \{a_i\}$. If this is the case we ignore the rest of the paragraph, otherwise assume that $f(a_{j_1}, \ldots, a_{j_m}) = a_i$. Consider $a'_i := f(a_{j_1+1}, a_{j_2+1}, \ldots, a_{j_m+1})$. By assumption $a'_i \in A_i$ and $(f(a_{j_1}, a_{j_2}, \ldots, a_{j_m}), a'_i) \in E$, so $a'_i \neq a_i$. By substituting $j_l + 1$ for j_l, we get an idempotent polymorphism and ρ-classes such that $f(a_{j_1}, a_{j_2}, \ldots, a_{j_m}) = a'_i \in A_i \setminus \{a_i\}$.

Without loss of generality, assume that $(a_i, a'_i) \in E$. a'_i is in a directed cycle within A_i which consists of elements of the form $f(a_{j_1+t}, a_{j_2+t}, \ldots, a_{j_m+t})$ for $0 \leq t \leq 2k$. So, there exists $a''_i \in A_i$ such that $(a'_i, a''_i) \in E$.

Let $b_{j_l} = a'_i$ if $j_l \in [i+k+1, i]$ and $b_{j_l} = a_{i+k+1}$ if $j_l \in [i+1, i+k]$. Define as $b'_i := f(b_{j_1}, b_{j_2}, \ldots, b_{j_m})$. From $(a_{j_l}, b_{j_l}) \in E$ for all $1 \leq l \leq m$ follows that $(a'_i, b'_i) \in E$, while from $(a'_i, a''_i), (a_{k+1}, a''_i) \in E$ follows that $(b'_i, a''_i) \in E$. Therefore, $b'_i \in a'^{+}_i \cap a''^{-}_i$, so it must be that $b'_i \in A_i$. Finally, from $(a'_i, b'_i) \in E$ and $(a_{k+1}, b'_i) \in E$ follows that $(b_{j_l}, b'_i) \in E$ for all $1 \leq l \leq m$, and therefore $(f(b_{j_1}, b_{j_2}, \ldots, b_{j_m}), f(b'_i, b'_i, \ldots, b'_i)) \in E$. But this is the same as saying that $(b'_i, b'_i) \in E$, a contradiction. □

Lemma 4.22. *$\hat{f}$ is a projection operation.*

Proof. We prove first that $\hat{f}$ is a polymorphism of $\mathcal{T}$. Suppose $j_1 \to i_1, \ldots, j_m \to i_m$. Hence, $i_1 \to (j_1 + k + 1), \ldots, i_m \to (j_m + k + 1)$. Let us pick elements $a_1 \in A_{j_1}, \ldots, a_m \in A_{j_m}, b_1 \in A_{i_1}, \ldots, b_m \in A_{i_m}, c_1 \in A_{j_1+k+1}, \ldots, c_m \in A_{j_m+k+1}$. Also, let $\hat{f}(j_1, \ldots, j_m) = j$ and $\hat{f}(i_1, \ldots, i_m) = i$. We know from Definition 4.13 and $(f(a_1, \ldots, a_m), f(b_1, \ldots, b_m)) \in E$ that $i = j$ or $j \to i$, that is, $i \in [j, j + k]$. Also, by Lemma 4.21, we know that $f(c_1, \ldots, c_m) \in A_{j+k+1}$, so from Definition 4.13 and $(f(b_1, \ldots, b_m), f(c_1, \ldots, c_m)) \in E$ that $i = j + k + 1$ or $i \to (j + k + 1)$, that is, $i \in [j + 1, j + k + 1]$. Therefore, $i \in [j + 1, j + k]$, so $j \to i$ and $\hat{f}$ is a polymorphism of $\mathcal{T}$, as desired.

Since f is an idempotent operation, thus $\hat{f}$ is also an idempotent operation, so by Lemma 4.15, either $k = 1$ or $\hat{f}$ is a projection. So assume that $k = 1$ and $\mathcal{T} = (\{1, 2, 3\}, \to)$. Note first that since $\mathcal{G}_{\mathcal{T}}$ is not a 3-cycle, at least one of the ρ-classes is not a singleton. Without loss of generality, assume that $a_1, a'_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$ and $(a_1, a'_1) \in E$. Since $a'_1, a_2 \in a_1^+$ for any idempotent polymorphism $g(x_1, \ldots, x_s)$ we have that $g(b_1, \ldots, b_s) \in a_1^+ \subseteq A_1 \cup A_2$ whenever for all i, $b_i \in \{a'_1, a_2\}$. By Lemma 4.20, this implies that $A_1 \cup A_2$ is closed under all idempotent polymorphisms of $\mathcal{G}_{\mathcal{T}}$. From this and Lemma 4.21, we conclude that $A_2 \cup A_3$ and $A_3 \cup A_1$ are also closed under all idempotent polymorphisms of $\mathcal{G}_{\mathcal{T}}$.

Claim. If $f(A_1, A_2, A_2, \ldots, A_2) \subseteq A_1$, then for all $A_{j_1}, A_{j_2}, \ldots, A_{j_m} \in \{A_1, A_2, A_3\}$, $f(A_{j_1}, A_{j_2}, \ldots, A_{j_m}) \subseteq A_{j_1}$.

We prove the claim by treating separately the cases $m = 2$, $m = 3$ and $m > 3$ (there in nothing to prove if $m = 1$ since then $f(x) = x$ follows by idempotence). If $m = 2$ then from idempotence and Lemma 4.21 follows that all that we have to prove is $f(A_2, A_1) \subseteq A_2$. From Lemma 4.21 and $f(A_1, A_2) \subseteq A_1$ follows that $b_3 := f(a_3, a'_1) \in A_3$. Thus $(f(a_2, a_1),$

$f(a_3, a_1')) \in E$ and from the discussion preceding the Claim we infer $f(a_2, a_1) \in b_3^- \cap (A_1 \cup A_2) = A_2$. By Lemma 4.20, $f(A_2, A_1) \subseteq A_2$, as desired.

In the case $m = 3$, from $f(A_1, A_2, A_2) \subseteq A_1$ and the binary case follow all cases where $A_{j_2} = A_{j_3}$ (using $g(x, y) := f(x, y, y)$). Since $b_1 := f(a_3, a_1, a_1) \in A_3$ and $f(a_1', a_1', a_2) \in b_1^+ \cap (A_1 \cup A_2) \subseteq A_1$, we infer by Lemmas 4.20 and 4.21 that $f(A_i, A_i, A_{i+1}) \subseteq A_i$. Moreover, since $c_1 := f(a_1, a_1, a_2) \in A_1$ and $f(a_1', a_1', a_3) \in c_1^+ \cap (A_1 \cup A_3) \subseteq A_1$, from Lemmas 4.20 and 4.21 we also infer that $f(A_i, A_i, A_{i+2}) \subseteq A_i$. By transposing the last two coordinates of f in the previous arguments we also conclude that $f(A_i, A_{i+1}, A_i) \subseteq A_i$ and $f(A_i, A_{i+2}, A_i) \subseteq A_i$. Finally, since $c_1 = f(a_1, a_1, a_2) \in A_1$ and $f(a_1', a_2, a_3) \in c_1^+$ we conclude $f(a_1', a_2, a_3) \notin A_3$, while from $d_1 := f(a_1', a_3, a_1') \in A_1$ and $f(a_1, a_2, a_3) \in d^-$ we conclude $f(a_1, a_2, a_3) \notin A_2$. These two by Lemma 4.20 imply that $f(A_1, A_2, A_3) \subseteq A_1$ and thus by Lemma 4.21 follows $f(A_i, A_{i+1}, A_{i+2}) \subseteq A_i$. The proof that $f(A_i, A_{i+2}, A_{i+1}) \subseteq A_i$ is analogous, just transpose the last two coordinates of f.

Finally, let $m > 3$ and $f(A_1, A_2, A_2, \ldots, A_2) \subseteq A_1$. Fix a sequence $A_{j_1}, A_{j_2}, \ldots, A_{j_m}$ such that all $A_{j_i} \in \{A_1, A_2, A_3\}$. Let $u_2, \ldots, u_m, v_2, \ldots, v_m \in \{x, y, z\}$ be such that $u_i = y$ if $A_{j_i} = A_{j_1}$ and $u_i = z$ if $A_{j_i} \neq A_{j_1}$, while $v_i = x$ if $A_{j_i} = A_{j_1}$, $v_i = y$ if $A_{j_i} = A_{j_1+1}$ and $v_i = z$ if $A_{j_i} = A_{j_1+2}$. Let $g(x, y, z) = f(x, u_2, u_3, \ldots, u_m)$ and $h(x, y, z) = f(x, v_2, v_3, \ldots, v_m)$. From $f(A_1, A_2, A_2, \ldots, A_2) \subseteq A_1$ follows that $g(A_1, A_2, A_2) \subseteq A_1$. By the case $m = 3$, $h(A_1, A_2, A_2) = g(A_1, A_1, A_2) \subseteq A_1$. Again by the case $m = 3$ follows that $f(A_{j_1}, A_{j_2}, \ldots, A_{j_m}) = h(A_{j_1}, A_{j_1+1}, A_{j_1+2}) \subseteq A_{j_1}$, so the Claim is proved. □

The Claim (with an appropriate permutation of variables) implies the Lemma is true if there exists a position i is such that $f(A_2, A_2, \ldots, A_2, A_1, A_2, A_2, \ldots, A_2) \subseteq A_1$, where A_1 is in ith position (in particular, the Lemma is proved for $m = 2$, so we assume $m > 2$). Let $f_i(x, y, z)$ be $f(x, x, \ldots, x, y, z, z, \ldots, z)$, where the first i variables are evaluated as x. Moreover, let i be maximal among those that satisfy $f_i(A_1, A_2, A_2) \subseteq A_2$ (by our assumptions, $1 \leq i < m - 1$). Thus $f_i(A_1, A_1, A_2) = f_{i+1}(A_1, A_2, A_2) \subseteq A_1$. By applying the Claim to $f_i(x, y, y)$ and $f_i(x, x, y)$ we deduce that $f_i(A_l, A_j, A_j) \subseteq A_j$ and $f_i(A_l, A_l, A_j) \subseteq A_l$ for all $j, l \in \{1, 2, 3\}$. From $f_i(a_1, a_1, a_2) \in A_1$ and $f_i(a_1', a_2, a_3) \in f_i(a_1, a_1, a_2)^+$ follows $f_i(a_1', a_2, a_3) \notin A_3$. By Lemmas 4.20 and 4.21 we obtain that $\neg\, f_i(A_1, A_2, A_3) \subseteq A_3$ and hence that $\neg\, f_i(A_3, A_1, A_2) \subseteq A_2$. On the other hand, from $f_i(a_2, a_1, a_1) \in A_1$ and $f_i(a_3, a_1', a_2) \in f_i(a_2, a_1, a_1)^+$ follows that $f_i(a_3, a_1', a_2) \notin A_3$, and by Lemma 4.20 this implies $\neg\, f_i(A_3, A_1, A_2) \subseteq A_3$. The remaining possibility allowed by Lemma 4.20 is $f_i(A_3, A_1, A_2) \subseteq A_1$. Therefore by Lemma 4.21

we obtain $f_i(A_1, A_2, A_3) \subseteq A_2$, so $f_i(a_1, a_2, a_3) \in A_2$. Since $f_i(a_1', a_3, a_1') \in f_i(a_1, a_2, a_3)^+ \cap (A_1 \cup A_3) \subseteq A_3$ we get that $f_i(A_1, A_3, A_1) \subseteq A_3$. By the Claim applied to $f_i(x, y, x)$, this implies that $f_i(A_2, A_1, A_2) \subseteq A_1$, so the Claim may be applied to prove $f(A_{j_1}, A_{j_2}, \dots, A_{j_m}) \subseteq A_{j_{i+1}}$. □ □

Without loss of generality, we may assume that $\hat{f}$ is the first projection, so now we know that $f(A_{j_1}, \dots, A_{j_m}) \subseteq A_{j_1}$ for any m-tuple of ρ-classes $(A_{j_1}, \dots, A_{j_m})$. Since we will not use the tournament $\mathcal{T}$ in the remainder of the proof, from this point onwards we change the notation to write $u \to v$ instead of $(u, v) \in E$.

Lemma 4.23. *Let the induced subgraph on A_i contain a nontrivial strong component C. Then $f(C, V, \dots, V) = C$.*

Proof. We may consider the case when $i = k+1$ to make the notation easier (we use the isomorphic copy $\mathcal{T}'$ of $\mathcal{T}$ obtained by the cyclic automorphism which maps i into $k+1$ and parametrize $\mathcal{G}$ by $\mathcal{T}'$ instead of $\mathcal{T}$). Let $c \in C$ and $v_2, v_3, \dots, v_m \in V$.

- If $v_2, \dots, v_m \in \bigcup_{j=1}^{k} A_j$, then from $f(c, v_2, \dots, v_m)^+ \supseteq A_{k+1} \cap c^+$ follows that $f(c, v_2, \dots, v_m) \in A_{k+1} \cap C^-$. If it were $f(c, v_2, \dots, v_m) \in A_{k+1} \cap C^{\forall -} = (A_{k+1} \cap C^-) \setminus C$, then $A_{k+1} \cap C^{\forall -} \ni f(c, v_2, \dots, v_n) \leftarrow f(c', a_{2k+1}, \dots, a_{2k+1}) \leftarrow f(c'', c'', \dots, c'') = c'' \in C$ where $c \leftarrow c' \leftarrow c''$ are vertices in C and $a_{2k+1} \in A_{2k+1}$. However, $(c''^+)^+ \cap C^{\forall -} \cap A_{k+1} = \emptyset$, which is a contradiction. Therefore,

$$f(C, (\bigcup_{j=1}^{k} A_j)^{m-1}) \subseteq C.$$

- We can prove dually that

$$f(C, (\bigcup_{j=k+2}^{2k+1} A_j)^{m-1}) \subseteq C.$$

- Now let $v_2, \dots, v_m \in V \setminus A_{k+1}$. Let $c' \to c \to c''$ in C, $a_1 \in A_1$ and $a_{2k+1} \in A_{2k+1}$. Since the first two cases hold also for a polymorphism obtained from f by identifying some variables with x_1, from $a_{2k+1} \to \bigcup_{j=1}^{k} A_j$ and $c' \to \bigcup_{j=k+2}^{2k+1} A_j$ follows that $C \ni f(c', u_2, \dots, u_m) \to f(i, v_2, \dots, v_m)$, where $u_j = a_{2k+1}$ if $v_j \in \bigcup_{j=1}^{k} A_j$, while $u_j = c'$ if $v_j \in \bigcup_{j=k+2}^{2k+1} A_j$. Similarly, from $\bigcup_{j=k+2}^{2k+1} A_j \to a_1$ and $\bigcup_{j=1}^{k} A_j \to c''$ follows that $f(c, v_2, \dots, v_m) \to$

$f(c'', w_2, \dots, w_m) \in C$, where $w_j = a_1$ if $v_j \in \bigcup_{j=k+2}^{2k+1} A_j$, while $w_j = c''$ if $v_j \in \bigcup_{j=1}^{k} A_j$. So,

$$f(C, (V \setminus A_{k+1})^{m-1}) \subseteq C.$$

- From the previous inclusion follows that $f(C, V^{m-1}) \subseteq C$ similarly as in the proof of the previous inclusion. $\square$

Lemma 4.24. *Let the induced subgraph on A_i contain a nontrivial strong component C and let $c \in C$. Then $f(\{c\}, V, \dots, V) = \{c\}$.*

Proof. Again as in the previous proof we assume that $j = k+1$. Also, let us assume $v_2, \dots, v_m \in A_{k+1}^{\forall -} = \bigcup_{j=1}^{k} A_j$ and denote $c' := f(c, v_2, \dots, v_m)$. Then for any $d \in c^+ \cap C$, $c' = f(c, v_2, \dots, v_m) \to f(d, d, \dots, d) = d$, so $c^+ \subseteq c'^+$. By semicompleteness, either $c' = c$, or $c' \to c$. Since we wish to prove $c' = c$, assume instead that $c' \to c$. But then for any $a_{2k+1} \in A_{2k+1}$ follows that $c'' = f(c', a_{2k+1}, \dots, a_{2k+1}) \to f(c, v_2, \dots, v_m) = c'$. Finally, $c'' = f(c'', c'', \dots, c'') \to f(c', a_{2k+1}, \dots, a_{2k+1}) = c''$ contradicting the irreflexivity of $\to$. Moreover, a dual argument proves that $f(c, v_2, \dots, v_m) = c$ when $v_2, \dots, v_m \in A_{k+1}^{\forall +}$.

Next assume that $v_2, \dots, v_m \in V \setminus A_{k+1}$. Then for any $a_1 \in A_1$, $a_{2k+1} \in A_{2k+1}$ and $c', c'' \in C$ such that $c' \to c \to c''$, there exist $u_2, \dots, u_m \in \{a_{2k+1}, c'\}$ and $w_2, \dots, w_m \in \{a_1, c''\}$ such that $u_i \to v_i \to w_i$ for all $2 \le i \le m$. To see this, denote first $J_1 = \{j : 1 < j \le m \text{ and } v_j \in C^-\}$, while $J_2 = \{2, 3, \dots, m\} \setminus J_1 = \{j : 1 < j \le m \text{ and } v_j \in C^+\}$. Now just take $u_j = a_{2k+1}$ and $w_j = c''$ when $v_j \in J_1$, while $u_j = c'$ and $w_j = a_1$ when $v_j \in J_2$. From the previous case applied to the binary polymorphisms g and h which are obtained from f by identification of the variables in $\{1\} \cup J_1$ and in J_2, respectively in $\{1\} \cup J_2$ and in J_1, we get $c' = g(c', a_{2k+1}) = f(c', u_2, \dots, u_m) \to f(c, v_2, \dots, v_m) \to f(c'', w_2, \dots, w_m) = h(c'', a_1) = c''$. Since c' and c'' were arbitrarily chosen, this implies that $(c^- \cap C) \subseteq f(c, v_2, \dots, v_m)^-$ and $(c^+ \cap C) \subseteq f(c, v_2, \dots, v_m)^+$, while from Lemma 4.23 follows that $f(c, v_2, \dots, v_m) \in C$, so together this implies that $f(c, v_2, \dots, v_m) = c$. We proved now that $f(\{c\}, (V \setminus A_{k+1})^{m-1}) = \{c\}$.

The general case of the lemma follows now easily by the same argument as the case above, as for any $v_i \in V$ it is easy to find u_i and w_i such that $u_i \to v_i \to w_i$ and that $u_i \in \{c'\} \cup (V \setminus A_{k+1})$ and $w_i \in \{c''\} \cup (V \setminus A_{k+1})$. $\square$

Theorem 4.25. *Every idempotent polymorphism f of a P-graph $\mathcal{G}_\mathcal{T}$ parametrized by the locally transitive tournament $\mathcal{T}$ is a projection, except when $\mathcal{G}_\mathcal{T}$ is the 3-cycle.*

Proof. We may assume that $f(A_{j_1}, \dots, A_{j_m}) \subseteq A_{j_1}$ and after Lemma 4.24 is applied, we are left with proving that $f(v_1, v_2, \dots, v_m) = v_1$ when $v_1 \in A_i$ is the only element of a trivial strong component of the induced subgraph on A_i. First let us denote by S the union of the maximal sequence of consecutive trivial strong components of the induced subgraph on A_i which contains $\{v_1\}$. Here the use of term 'consecutive' is with respect to the linear order of strong components induced by $\to$. S is bounded from above either by the start of A_i, or by a nontrivial strong component I of A_i, and from below either by the end of A_i or by the nontrivial strong component J of A_i. We know that $f(v_1, v_2, \dots, v_m) \in A_i$, so if $S = A_i$, then $f(v_1, v_2, \dots, v_m) \in S$. Otherwise, we select any $u_j \to v_j \to w_j$ for $2 \leq j \leq m$. If a nontrivial strong component I of the subgraph on A_i such that $S \subseteq I^{\forall +}$, then for all $x \in I$, from Lemma 4.24 we get $x = f(x, u_2, \dots, u_m) \to f(v_1, v_2, \dots, v_m)$, so for all $x \in I$, $f(v_1, v_2, \dots, v_m) \in x^+$, i. e. $f(v_1, v_2, \dots, v_m) \in I^{\forall +}$. Dually, for any nontrivial strong component J of the subgraph on A_i such that $S \subseteq J^{\forall -}$, then for all $y \in J$, we get $f(v_1, v_2, \dots, v_m) \to f(y, w_2, \dots, w_m) = y$, so for all $y \in J$, $f(v_1, v_2, \dots, v_m) \in y^-$, i. e. $f(v_1, v_2, \dots, v_m) \in J^{\forall -}$. We conclude that $f(v_1, v_2, \dots, v_m) \in S$, and by extension that $f(S, V, V, \dots, V) \subseteq S$.

Now let $S = \{a_1^1, a_1^2, \dots, a_1^{|S|}\}$, where $a_1^1 \to a_1^2 \to \dots \to a_1^{|S|}$ and $v_1 = a_1^j$. Moreover, select any vertices $a_r^t \in V$ for $2 \leq r \leq m$ and $1 \leq t \leq |S|$ which satisfy that $a_r^t \to a_r^{t+1}$ for all r, t such that $2 \leq r \leq m$ and $1 \leq t < |S|$, and which also have the property that $a_r^j = v_r$ for all $2 \leq r \leq m$. Denote by $b_t = f(a_1^t, a_2^t, \dots, a_m^t)$. We get from compatibility that $b_1 \to b_2 \to \dots \to b_{|S|}$, while from the previous paragraph follows that $\{b_1, b_2, \dots, b_{|S|}\} \subseteq S$. However, the induced subgraph on S is the strict linear order (i. e. transitive tournament) in which the only directed path of length $|S|$ is $a_1^1 \to a_1^2 \to \dots \to a_1^{|S|}$, so it must be that $a_1^t = b_t$ for all $1 \leq t \leq |S|$. In particular, for $t = j$ we get $f(v_1, v_2, \dots, v_m) = b_j = a_1^j = v_1$, as desired. □

4.2 All strongly connected semicomplete digraphs

Lemma 4.26. *Let $\mathcal{G} = (V, \to)$ be a strongly connected semicomplete digraph which contains at least one 2-cycle. Then for each 2-cycle $a \leftrightarrow b$ in $\mathcal{G}$, the set $\{a, b\}$ is closed with respect to all idempotent polymorphisms of $\mathcal{G}$ and each binary idempotent polymorphism of $\mathcal{G}$ restricted to $\{a, b\}$ is a projection.*

Proof. First, note that the 2-cycle has no idempotent binary polymorphisms other than projections (the only other options are $\wedge$ and $\vee$, and those two are clearly not polymorphisms of the 2-cycle). So the second statement follows from the first one.

Given $f \in Pol_{id}(\mathcal{G})$ of arity $n > 1$ and $\overline{c} \in \{a,b\}^n$, there is a binary $g \in Pol_{id}(\mathcal{G})$ (obtained from f by identification of variables) and $\overline{d} \in \{a,b\}^2$ such that $f(\overline{c}) = g(\overline{d})$. So, it suffices to prove that any 2-cycle $a \leftrightarrow b$ is closed under all binary $f \in Pol_{id}(\mathcal{G})$. There is nothing to prove for $|V| = 2$. Assume that it holds for all strongly connected semicomplete digraphs with fewer than $|V|$ vertices. In this proof we will call the 2-cycles $\{a,b\} \subseteq V$ which are not closed under all idempotent polymorphisms of $\mathcal{G}$ the *bad pairs* of $\mathcal{G}$. We are trying to prove no bad pairs exist and assume the opposite.

Claim 1: For a bad pair $\{a,b\}$ of $\mathcal{G}$, $\{a,b\}^{\forall+} = \{a,b\}^{\forall-} = \emptyset$.

Assume not, and without loss of generality, let $x \in \{a,b\}^{\forall-}$. Then $\{a,b\} \subseteq x^+$ and x^+ is closed under all idempotent polymorphisms of $\mathcal{G}$, since it is pp-definable with $\to$ and the constant x. Let $\mathcal{G}_1 = (x^+, \to)$ be the induced subgraph on x^+. The strong component S of $\mathcal{G}_1$ which contains $\{a,b\}$ is pp-definable within $\mathcal{G}_1$ using all constants from $x^+ \setminus S$ and $\to$, so it is also pp-definable within $\mathcal{G}$ using $\to$ and constants. Therefore, S is closed under all idempotent polymorphisms of $\mathcal{G}$. The assumption that $\{a,b\}$ is a bad pair of $\mathcal{G}$ implies that $\{a,b\}$ is a bad pair of the induced subgraph of $\mathcal{G}$ on the set S. Since $x \notin S$, thus $|S| \leq |V| - 1$ and the induced subgraph on S is a strongly connected semicomplete digraph, contradicting the inductive assumption and proving Claim 1. $\square$

Another way to write Claim 1 is to say that for all bad pairs $\{a,b\}$ of $\mathcal{G}$ and $x \in V \setminus \{a,b\}$, $|x^+ \cap \{a,b\}| = |x^- \cap \{a,b\}| = 1$.

Claim 2: Let $\{a,b\}$ be a bad pair of $\mathcal{G}$ which is not closed under the idempotent binary polymorphism f. Then $\{f(a,b), f(b,a)\}$ is also a bad pair of $\mathcal{G}$.

From the fact that f is a polymorphism, it follows that $f(a,b) \leftrightarrow f(b,a)$. Moreover, assuming that $f(a,b) = c \notin \{a,b\}$, then from Claim 1 follows that $\{a,c\}$ and $\{b,c\}$ are not 2-cycles, and so $f(b,a) = d \notin \{a,b\}$. If $\{c,d\}$ is not a bad pair, then it is closed under all idempotent polymorphisms. Assume that $c \to a \to d$ or $d \to a \to c$. Then by Lemma 4.8, $\{a,c,d\}$ is nice. Moreover, since the induced subgraph on $\{a,c,d\}$ is strongly connected and $b^+ \cap \{a,c,d\} \neq \emptyset \neq b^- \cap \{a,c,d\}$, then $\{a,b,c,d\}$ is also nice by Lemma 4.7, which is a contradiction with the assumption that $\{a,b\}$ is a bad pair. So, $\{c,d\} \subseteq a^-$ or $\{c,d\} \subseteq a^+$. We may assume without loss of generality that $\{c,d\} \subseteq a^+$, that $\{c,d\} \cap a^- = \emptyset$, and also that f restricts to $\{c,d\}$ as the first projection. Now we get that $d = f(b,a) \to f(a,c) \to f(c,d) = c$ and from Lemma 4.8 follows that the subset $\{c,d,f(a,c)\}$ is nice and that f restricts to it as the first projection. Moreover, if it were $a \to f(a,c)$, then we would get that $f(a,c) \to f(f(a,c),d) = f(a,c)$ (the equality follows since f is the first projection on $\{c,d,f(a,c)\}$), which is impossible. From $d \to f(a,c)$

follows that $f(a,c) \neq a$. The only remaining possibility is that $f(a,c) \to a$. But together with Lemma 4.7, this implies that $\{a, c, d, f(a,c)\}$ is nice and then again from Lemma 4.7 and $b \leftrightarrow a$ we get that $\{a, b, c, d, f(a,c)\}$ is nice, which contradicts the assumption that $\{a,b\}$ is a bad pair. This final contradiction proves that $\{c,d\} = \{f(a,b), f(b,a)\}$ must be a bad pair. $\square$

Claim 3: The set $B := \bigcup\{\{a,b\} : \{a,b\} \text{ is a bad pair of } \mathcal{G}\}$ is closed under all binary idempotent polymorphisms of V.

By Claim 1, if $a \in V$ is a member of the bad pair $\{a,b\}$, then the only 2-cycle containing a is $a \leftrightarrow b$. Let $a_1, a_2 \in B$ and let f be an idempotent polymorphism of $\mathcal{G}$. We aim to prove that $f(a_1,a_2) \in B$. This follows from idempotence if $a_1 = a_2$. If $a_1 \leftrightarrow a_2$, then $\{a_1, a_2\}$ is a bad pair, so $f(a_1,a_2) \in B$ by Claim 2. So, we may assume without loss of generality that $a_1 \to a_2$, $a_1 \leftrightarrow b_1$ and $a_2 \leftrightarrow b_2$, where $\{a_1,b_1\}$ and $\{a_2,b_2\}$ are bad pairs, and that $|\{a_1,a_2,b_1,b_2\}| = 4$. From $a_1 \to a_2$ and Claim 1 applied to $a_1 \leftrightarrow b_1$, resp. to $a_2 \leftrightarrow b_2$, we get $a_2 \to b_1$, resp. $b_2 \to a_1$, while Claim 1 applied to $a_2 \leftrightarrow b_2$ and $a_2 \to b_1$ imply $b_1 \to b_2$. The induced subgraph on $\{a_1,a_2,b_1,b_2\}$ is given in Figure 2:

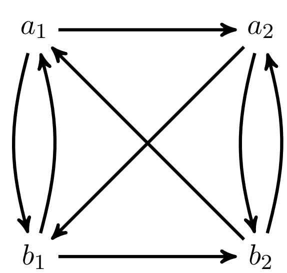

Figure 2: The induced subgraph on $\{a_1, a_2, b_1, b_2\}$

Assume that $c := f(a_1,a_2) \notin B$. Therefore, $f(a_1,a_2) \leftrightarrow f(b_1,b_2) =: d$ and the pair $\{c,d\}$ is not a bad pair, and hence $\{c,d\}$ is closed under all idempotent polymorphisms of $\mathcal{G}$. Now $c = f(a_1,a_2) \to f(a_2,b_1) \to f(b_1,b_2) = d$, so the subset $\{c,d,f(a_2,b_1)\}$ is nice by Lemma 4.8. Moreover, since $f(a_1,b_2) \leftrightarrow f(a_2,b_1)$, thus $\{c,d,f(a_2,b_1),f(a_1,b_2)\}$ is also nice by Lemma 4.7. Since $f(a_2,b_1) \leftrightarrow f(a_1,b_2)$, $\{f(a_2,b_1), f(a_1,b_2)\}$ is not a bad pair and elements of B are in precisely one 2-cycle, thus $f(a_2,b_1) \notin B$ and $f(a_1,b_2) \notin B$. Now, $a_1 = f(a_1,a_1) \to f(a_2,b_1)$, and since $f(a_2,b_1) \notin B \ni b_1$, from Claim 1 follows that $f(a_2,b_1) \to b_1$. Also, we know that $b_1 = f(b_1,b_1) \to f(a_1,b_2)$, so from Lemma 4.7 follows that $\{c,d,f(a_2,b_1), f(a_1,b_2), b_1\}$ is nice. Finally, from this, Lemma 4.7 and $a_1 \leftrightarrow b_1$ follows that $\{c, d, f(a_2, b_1), f(a_1, b_2), a_1, b_1\}$ is nice, which contradicts the assumption that the pair $\{a_1, b_1\}$ is bad. This finishes the proof of Claim 3. $\square$

Now we consider the case when B contains at least three distinct bad pairs. If this is the case, we claim that there exist three distinct elements $a_1, a_2, a_3 \in B$ such that $\{a_1, a_2, a_3\}$ contains no bad pairs and it is closed under all binary idempotent polymorphisms. We will obtain this set as an intersection of a pp-definable subset of V and B, and therefore all binary polymorphisms will be compatible with it by Claim 3. Note that for any $b \in B$, $|b^+ \cap B| = \frac{|B|}{2}$, since $b^+ \cap B$ contains the other half of the bad pair which contains b and exactly one element of each other bad pair, according to Claim 1. So, $b^+ \cap B = \{a_1, a_2, \ldots, a_n\}$, this is a set which contains no bad pairs, it is closed under all idempotent binary polymorphisms of $\mathcal{G}$ and we assumed that $n \geq 3$. Now we inductively intersect this set with another pp-definable subset to make it smaller, but still no less than 3. To do this, assume that the subset $S = \{a_1, a_2, \ldots, a_m\} \subseteq B$ is closed under all idempotent binary polymorphisms and contains no bad pairs and let $m > 3$. If $m \in \{2k, 2k+1\}$ for some integer k, then we know that $k \geq 2$. Now the induced subgraph on S is a tournament and since $|S| \in \{2k, 2k+1\}$, then either $|a_m^+ \cap S| \geq k$ or $|a_m^- \cap S| \geq k$. Without loss of generality, let $|a_m^+ \cap S| \geq k$. The set $T := a_m^+ \cap S$ contains no bad pairs since S doesn't contain them, it is closed under all idempotent binary polymorphisms and since $T \subseteq S \setminus \{a_m\}$, thus $|T| < m$. If $|T| \geq 3$, then this is our desired set T. If $|T| = 2$, then for $b_m \in B$ such that $a_m \leftrightarrow b_m$, i. e. that $\{a_m, b_m\}$ is a bad pair, Claim 1 implies that $b_m^- \cap S = T \cup \{a_m\}$ and so $T' = |b_m^- \cap S| = 3$. As above, this set T' is also closed under all idempotent binary polymorphisms and contains no bad pairs.

So we have proved that there exist three distinct elements $a_1, a_2, a_3 \in B$ such that $\{a_1, a_2, a_3\}$ is closed under all idempotent binary polymorphisms of $\mathcal{G}$ and contains no bad pairs. In fact, from the proof in previous paragraph we know that there exists a pp-formula $\varphi(x)$ in the language of the pointed digraph $\mathcal{G}^c$ with one free variable x such that $\{x : \varphi^{\mathcal{G}}(x)\} \cap B = \{a_1, a_2, a_3\}$. Without loss of generality, we may assume that $a_1 \to a_2 \to a_3$ (no assumption is made on the edge between a_1 and a_3) and let $a_i \leftrightarrow b_i$, i. e. $\{a_i, b_i\}$ are bad pairs. This and Claim 1 force the situation depicted on Figure 3:

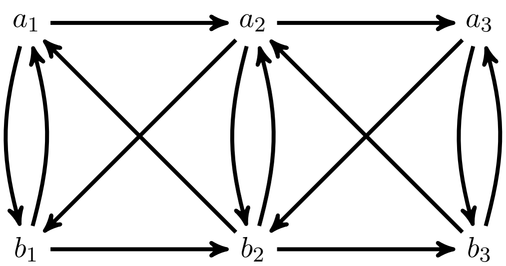

Figure 3: The induced subgraph on $\{a_1, a_2, a_3, b_1, b_2, b_3\}$

$\{b_1, b_2, b_3\} = \{x : (\exists y)(\varphi^{\mathcal{G}}(y) \& x \leftrightarrow y)\} \cap B$, so $\{b_1, b_2, b_3\}$ is also closed under all idempotent binary polymorphisms of $\mathcal{G}$. Also since $\{a_1, a_2\} = b_2^+ \cap \{a_1, a_2, a_3\}$, $\{a_2, a_3\} = b_2^- \cap \{a_1, a_2, a_3\}$, $\{b_1, b_2\} = a_2^+ \cap \{b_1, b_2, b_3\}$, $\{b_2, b_3\} = a_2^- \cap \{b_1, b_2, b_3\}$, then the sets $\{a_1, a_2\}$, $\{a_2, a_3\}$, $\{b_1, b_2\}$ and $\{b_2, b_3\}$ are all closed under idempotent binary polymorphisms of $\mathcal{G}$. Let f be an idempotent binary polymorphism of $\mathcal{G}$ such that $\{a_2, b_2\}$ is not closed under f. We have the following cases:

Case 1: $f(a_1, a_2) = a_2$. Since $a_2 = f(a_1, a_2) \to f(a_2, a_3)$ and $\{a_2, a_3\}$ is closed under f, we get $f(a_2, a_3) = a_3$. Then we have these two subcases:

Case 1a: $f(a_3, a_2) = a_2$. Then $a_2 = f(a_1, a_2) \to f(a_2, b_2) \to f(a_3, a_2) = a_2$ and the only element $x \in V$ such that $a_2 \leftrightarrow x$ is b_2, it follows that $f(a_2, b_2) = b_2$. By Claim 2, $\{f(a_2, b_2), f(b_2, a_2)\} = \{b_2, f(b_2, a_2)\}$ is a bad pair, so $f(b_2, a_2) = a_2$, and from these and idempotence of f follows that $\{a_2, b_2\}$ is closed under f, contradicting the choice of f.

Case 1b: $f(a_3, a_2) = a_3$. Now $f(a_2, b_2) \to f(a_3, a_2) = a_3$ and $f(b_2, a_2) \to f(a_2, a_3) = a_3$, so $a_3 \in f(a_2, b_2)^+ \cap f(b_2, a_2)^+$. Since $\{f(a_2, b_2), f(b_2, a_2)\}$ is a bad pair by Claim 2, we get a contradiction with Claim 1.

Case 2: $f(a_2, a_1) = a_2$. This case is analogous to Case 1.

Case 3: $f(a_1, a_2) = a_1$ and $f(a_2, a_1) = a_1$. This case is analogous to Case 1b, with all edges and the roles of a_1 and a_3 reversed, since the contradiction there was derived only from $f(a_2, a_3) = a_3 = f(a_3, a_2)$, without using $f(a_1, a_2) = a_2$ at all.

Finally, we deal with the case when B contains at most two bad pairs. By Claim 2 it cannot contain exactly one, so $|B| = 4$. Without loss of generality, $B = \{a_1, a_2, b_1, b_2\}$ and the induced subgraph on B is isomorphic to the one in Figure 2. Also, since $b_1^+ \cap B = \{a_1, b_2\}$, the subset $\{a_1, b_2\}$ is invariant under all idempotent binary polymorphisms of $\mathcal{G}$. Let f be an idempotent binary polymorphism such that $\{a_1, b_1\}$ is not closed with respect to f.

Then by Claim 2 $\{f(a_1, b_1), f(b_1, a_1)\}$ is a bad pair distinct from $\{a_1, b_1\}$, so $\{f(a_1, b_1), f(b_1, a_1)\} = \{a_2, b_2\}$ and without loss of generality we may assume that $f(a_1, b_1) = a_2$ and $f(b_1, a_1) = b_2$ (if not, just replace f with $g(x, y) := f(y, x)$). Now $a_2 = f(a_1, b_1) \to f(a_2, b_2) \to f(b_1, a_1) = b_2$, so $f(a_2, b_2) = b_1$ and from $f(a_2, b_2) \leftrightarrow f(b_2, a_2)$ follows that $f(b_2, a_2) = a_1$ (see Figure 2). Since $a_1 = f(b_2, a_2) \to f(a_1, b_2)$, then $f(a_1, b_2) \in a_1^+ \cap B = \{a_2, b_1\}$, so $\{a_1, b_2\}$ is not closed under f. This contradiction establishes that B must be empty, as desired. □ □

Definition 4.27. Let $\mathcal{G} = (V, \to)$ be a strongly connected semicomplete digraph. We say that L *splits* $\mathcal{G}$ if $\emptyset \neq L \subsetneq V$ is a subset with the following properties:

1. $\{L, L^{\forall +}, L^{\forall -}\}$ is a partition of V and
2. for any 2-cycle $a \leftrightarrow b$ in $\mathcal{G}$, $\{a, b\}$ is contained in one of L, $L^{\forall +}$, or $L^{\forall -}$.

Lemma 4.28. *Let $\mathcal{G} = (V, \to)$ be a strongly connected semicomplete digraph which is not a cycle. Let L_0 be either a 2-cycle or a nice subset of V. Then either all idempotent polymorphisms of $\mathcal{G}$ are projections, or there exists an L such that $L_0 \subseteq L \subseteq V$ and that*

- *L splits $\mathcal{G}$ and*
- *either the induced subgraph on L is a 2-cycle, or L is nice.*

Proof. We inductively construct a sequence of subsets such that for all i, $L_i \subseteq L_{i+1}$, and also such that L_i are nice for all $i > 0$. We terminate our inductive construction if L_i splits $\mathcal{G}$ and make $L := L_i$. We have two possibilities:

Case 1: If there exists an element $v \in V \setminus L_i$ such that $v^+ \cap L_i \neq \emptyset$ and $v^- \cap L_i \neq \emptyset$, then select $L_{i+1} := L_i \cup \{v\}$. If $i = 0$ and the induced subgraph on L_0 is a 2-cycle, then from Lemmas 4.26 and 4.8 follows that L_{i+1} is nice. Otherwise, the same conclusion follows from Lemma 4.7 since L_i is nice.

Case 2: Assume that for all $v \in V \setminus L_i$, either $v^+ \cap L_i = \emptyset$, or $v^- \cap L_i = \emptyset$. Thus either $v \in L_i^{\forall -}$ or $v \in L_i^{\forall +}$, but not both, and so $\{L, L^{\forall +}, L^{\forall -}\}$ is a partition of V. Now either L_i splits $\mathcal{G}$, in which case we put $L := L_i$ and terminate the sequence, or there exists a 2-cycle $c \leftrightarrow d$ such that $c \in L^{\forall -}$ and $d \in L^{\forall +}$. We put $L_{i+1} := L_i \cup \{c, d\}$. The induced subgraph on any subset of L_{i+1} which contains $\{c, d\}$ is strongly connected as any element is in a 3-cycle with c and d. Moreover if $L_i = \{v_1, v_2, \ldots, v_k\}$, starting from $\{c, d\}$ by Lemmas 4.26 and 4.8 we get that $\{c, d, v_1\}$ is nice. From the assumption that $\{c, d, v_1, \ldots, v_j\}$ is nice and Lemma 4.7 we get that $\{c, d, v_1, \ldots, v_j, v_{j+1}\}$ is nice, hence inductively L_{i+1} is nice. $\square$

Lemma 4.29. *Let $\mathcal{G} = (V, \to)$ be a strongly connected semicomplete digraph which is not a P-graph and let L split $\mathcal{G}$. Then there exist vertices $a_0, a_1, b_0 \in V$ such that $a_1 \leftarrow a_0 \to b_0 \to a_1$ and that either*

- *(a) $b_0 \in L^{\forall -}$ and a_0, a_1 are in the same strong component, or two consecutive strong components, of the induced subgraph on $L^{\forall +}$, or*
- *(b) $b_0 \in L^{\forall +}$ and a_0, a_1 are in the same strong component, or two consecutive strong components, of the induced subgraph on $L^{\forall -}$.*

Proof. Firstly, if there were no elements $a \in L^{\forall -}$ and $b \in L^{\forall +}$ such that $b \to a$, then $\mathcal{G}$ would not be strongly connected. On the other hand, if there were no elements $a \in L^{\forall -}$ and $b \in L^{\forall +}$ such that $a \to b$, then $\mathcal{G}$ would be a P-graph parametrized by the 3-cycle into components $L^{\forall -}$, L and $L^{\forall +}$.

Let the strong components of the induced subgraphs on $L^{\forall -}$ and $L^{\forall +}$ be, respectively, $A_1 \Rightarrow A_2 \Rightarrow \ldots \Rightarrow A_{k_1}$ and $B_1 \Rightarrow B_2 \Rightarrow \ldots \Rightarrow B_{k_2}$ (here $\Rightarrow$ indicates that all edges between those subsets are in that direction and none in the other). If there is an element $c \in L^{\forall -}$ and a strong component B_j of the induced subgraph on $L^{\forall +}$ such that $c^+ \cap B_j \neq \emptyset \neq c^- \cap B_j$, then consider the Hamiltonian cycle $d_0 \to d_1 \to \ldots \to d_{k-1} \to d_0$ of B_j. There must be consecutive elements $d_i \to d_{i+1}$ in this cycle (here we use addition modulo k) such that $d_i \to c \to d_{i+1}$. Then take $b_0 := c$, $a_0 := d_i$ and $a_1 := d_{i+1}$ and the case (a) is satisfied. Dually, if there exist any $c \in L^{\forall +}$ and a strong component A_j of the induced subgraph on $L^{\forall -}$, such that $c^+ \cap B_j \neq \emptyset \neq c^- \cap B_j$, then item (b) of the Lemma would be true.

Now for any $a \in A_i$ and $b \in B_j$, if $a \to b$, then $B_j \subseteq a^+$. Thus for any $c \in B_j$, also $a \to c$, so $A_i \subseteq c^-$, and therefore $A_i \Rightarrow B_j$. Dually, if $b \to a$ then $B_j \Rightarrow A_i$. We proved that $\mathcal{G}$ is parametrized by a tournament $\mathcal{T}$ into $A_1, A_2, \ldots, A_{k_1}, L, B_1, B_2, \ldots, B_{k_2}$. Since $\mathcal{G}$ is not a P-graph, we know that $\mathcal{T}$ is not locally transitive. We know that $A_1 \Rightarrow A_2 \Rightarrow \ldots \Rightarrow A_{k_1} \Rightarrow L \Rightarrow B_1 \Rightarrow B_2 \Rightarrow \ldots \Rightarrow B_{k_2} \Rightarrow A_1$, so $\mathcal{T}$ is strongly connected and by definition, there exists a vertex $v \in V(\mathcal{T})$ such that either v^- or v^+ is not transitive.

Assume that v^+ is not transitive. v is not the vertex which gets expanded into L, since the subtournaments which expand into $L^{\forall -}$ and $L^{\forall +}$ are both transitive (as $A_i \Rightarrow A_j$ and $B_i \Rightarrow B_j$ whenever $i < j$). If v expands into A_i, from the assumption that v^+ is not transitive follows that there must be some j and $k > i$ such that $B_j \subseteq A_i^+$ and $B_j \Rightarrow A_k$. Let k be the least integer greater than i such that $B_j \Rightarrow A_k$. From $A_i \Rightarrow B_j$ follows that $A_{k-1} \Rightarrow B_j$, so select any $a_0 \in A_{k-1}$, $a_1 \in A_k$ and $b_0 \in B_j$ to fulfil the requirements of (b). On the other hand, if v gets expanded into B_i from intransitivity of v^+ follows that there must be some j and $k > i$ such that $A_j \subseteq B_i^+$ and $A_j \Rightarrow B_k$. The proof of this case follows by permuting the letters A and B in the previous proof and leads to fulfilment of (a). The case when v^- is intransitive is dual. $\square$

The next lemma will be used only in the case when there are no 2-cycles in $\mathcal{G}$, so we assume that $\mathcal{G}$ is a tournament. It will help with the inductive base of the main proof.

Lemma 4.30. *If a strongly connected tournament $\mathcal{G} = (V, \to)$ is not a P-graph and for all $v \in V$, all strong components of the induced subgraphs on v^+ and on v^- are of sizes 1 or 3, then there is a 3-cycle $a \to b \to c \to a$ in $\mathcal{G}$ such that all idempotent polymorphisms of $\mathcal{G}$ restrict to $\{a, b, c\}$ as projections.*

Proof. **Claim 1**: There exist five distinct vertices such that the induced subgraph on them contains all edges of the partial tournament depicted on Figure 4, or of its dual (the edge between a_2 and b_0 is missing since it doesn't matter which way it goes).

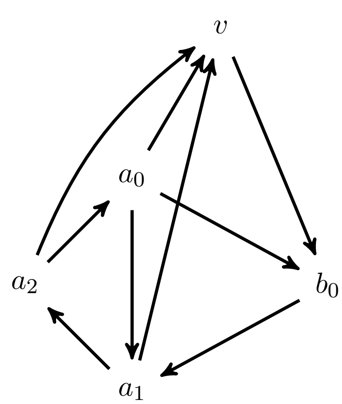

Figure 4: The partial tournament from Claim 1

Since $\mathcal{G}$ is a tournament, each singleton $\{v\}$ splits $\mathcal{G}$, so $V = \{v\} \cup v^- \cup v^+$. First assume that for some vertex v there exist strong components A of v^+ and B of v^- such that neither $A \Rightarrow B$ nor $B \Rightarrow A$. At least one of these components is of size 3, say A. From the proof of Lemma 4.29 we get statement (a) or statement (b) of that Lemma to hold, with a_0, a_1 both in the same component (the consecutive case arises when always $A \Rightarrow B$ or $B \Rightarrow A$). If (b) holds then the situation is like in Figure 4, while if (a) holds, then it is like in its dual.

Now assume that for all vertices v and all strong components A of v^+ and B of v^-, either $A \Rightarrow B$ or $B \Rightarrow A$. Select $L \subsetneq V$ to be maximal so that L is singleton or a strongly connected proper subtournament, and that L splits $\mathcal{G}$. Denote the strong components of $L^{\forall -}$ and $L^{\forall +}$, respectively, by $A_1 \Rightarrow A_2 \Rightarrow \ldots \Rightarrow A_{k_1}$ and $B_1 \Rightarrow B_2 \Rightarrow \ldots \Rightarrow B_{k_2}$. For any $v \in L$, A_i and B_j are strong components of the induced graph on v^- and v^+, respectively, since $A_i \Rightarrow (L \cap v^-)$ and $(L \cap v^+) \Rightarrow B_j$, so by our assumptions $|A_i|, |B_j| \in \{1, 3\}$, and either $A_i \Rightarrow B_j$ or $B_j \Rightarrow A_i$. Without loss of generality, assume that Lemma 4.29, item (b) holds, i. e. $A_{i+1} \Leftarrow A_i \Rightarrow B_j \Rightarrow A_{i+1}$, (in particular, $k_1 > 1$).

Let us show that we may assume that $i = k_1 - 1$ and $j = 1$. If $i < k_1 - 1$ then for any $a_i \in A_i$, a_i^+ contains a cycle of length at least 4, as $A_{i+1} \Rightarrow A_{k_1} \Rightarrow L \Rightarrow B_j \Rightarrow A_{i+1}$. This is a contradiction with the conditions of the Lemma. Let $A_i = k_i - 1$ and $j > 1$. If $A_i \Rightarrow B_1$, then for any $a_i \in A_i$, a_i^+ contains a cycle of length 4 or more since $A_{k_1} \Rightarrow L \Rightarrow B_1 \Rightarrow B_j \Rightarrow A_{k_1}$, and this again contradicts the conditions of the Lemma. On the other hand, if $B_1 \Rightarrow A_i$ then select $a_0 \in A_i$, $a_1 \in L$, $a_2 \in B_1$, $v \in B_j$ and $b_0 \in A_{i+1}$. We get the following edges: $a_0 \to a_1 \to a_2 \to a_0$, then $\{a_0, a_1, a_2\} \in v^-$, $v \to b_0$ and finally $a_0 \to b_0 \to a_1$. In other words, the induced subgraph on these five vertices contains all edges of the partial tournament depicted on Figure 4.

Now let $i = k_1 - 1$ and $j = 1$. If there exists A_l such that $l < i$, $B_1 \Rightarrow A_l$, then select $a_0 \in A_{k_1}$, $a_1 \in B_1$, $a_2 \in L$, $v \in A_{k_1 - 1} = A_i$ and $b_0 \in A_l$. Now we get that $a_0 \leftarrow a_1 \leftarrow a_2 \leftarrow a_0$, that $\{a_0, a_1, a_2\} \subseteq v^+$, $b_0 \to v$ and that $a_0 \leftarrow b_0 \leftarrow a_1$. In other words, the induced subgraph on these five vertices contains all edges of dual of the partial tournament depicted on Figure 4.

The remaining case is when for all $l < i$, the strong components $A_l \Rightarrow B_1$. Also, $|A_{k_1}| = |L| = |B_1| = 1$, otherwise from $A_{k_1} \cup L \cup B_1 \subseteq A_{k_1 - 1}^{\forall +}$ and $A_{k_1} \Rightarrow L \Rightarrow B_1 \Rightarrow A_{k_1}$ would follow that for any $v \in A_{k_1 - 1}$, v^+ contains a cycle of length greater than 3. Moreover, we know from strong connectedness of $\mathcal{G}$ that $k_2 > 1$, otherwise there would be no edge from $A_{k_1} \cup L \cup B_1$ into the rest of V (which is nonempty since $k_1 > 1$).

Now we prove that also for all $l > 1$, $A_{k_1} \Rightarrow B_l$. Assume not, then select B_l such that $l > 1$ and $B_l \Rightarrow A_{k_1}$. Now, if $A_{k_1 - 1} \Rightarrow B_l$, we would get that $A_{k_1} \Rightarrow L \Rightarrow B_1 \Rightarrow B_l \Rightarrow A_{k_1}$ and for any $v \in A_{k_1 - 1}$, $A_{k_1} \cup L \cup B_1 \cup B_l \subseteq v^+$.

This would imply that v^+ contains a 4-cycle, a contradiction. On the other hand, if $B_l \Rightarrow A_{k_1-1}$, then select $a_2 \in L$, $a_0 \in A_{k_1}$, $a_1 \in B_1$, $v \in A_{k_1-1}$ and $b_0 \in B_l$. We get that $a_0 \leftarrow a_1 \leftarrow a_2 \leftarrow a_0$, that $\{a_0, a_1, a_2\} \subseteq v^+$, $b_0 \rightarrow v$ and that $a_0 \leftarrow b_0 \leftarrow a_1$. As before, the induced subgraph on these five vertices contains all edges of dual of the partial tournament depicted on Figure 4.

Finally, if for all $l > 1$, $A_{k_1} \Rightarrow B_l$, then we get that $A_{k_1} \cup L \cup B_1$ also splits $\mathcal{G}$ and it is a proper subset of V. This contradicts the choice of L. Claim 1 is thus proved. $\square$

Note that $\{a_0, a_1, a_2\} \subseteq v^-$ and $\{a_1, b_0, v\} \subseteq a_0^+$, hence each of those three-element sets is closed under all idempotent polymorphisms of $\mathcal{G}$ (since they are 3-cycles, they must be strong components of v^- and a_0^+, respectively, so they are pp-definable in $\mathcal{G}^c$).

Claim 2: Every idempotent binary polymorphism $f \in Pol_{id}(\mathcal{G})$ is the same projection on the sets $\{a_0, a_1, a_2\}$ and $\{a_1, v, b_0\}$.

Since $a_0 = f(a_0, a_0) \rightarrow f(b_0, a_1) \rightarrow f(a_1, a_2) \rightarrow f(v, v) = v$, it follows that $f(a_1, a_2) \neq a_0$ since there are no 2-cycles in $\mathcal{G}$. From $f(a_0, a_1) \rightarrow f(a_1, a_2) \neq a_0$ and $f(a_0, a_1), f(a_1, a_2) \in \{a_0, a_1, a_2\}$ follows $f(a_0, a_1) \in \{a_0, a_1\}$. By switching the coordinates of f in the previous sentence we also get that $f(a_1, a_0) \in \{a_0, a_1\}$. By the same argument, since v is not in a 2-cycle, we get that $f(b_0, a_1) \neq v \neq f(a_1, b_0)$. Therefore, also $f(b_0, a_1)$, $f(a_1, b_0) \in \{a_1, b_0\}$. Let us assume that $f(a_0, a_1) = a_0$. This implies that $a_0 = f(a_0, a_1) \rightarrow f(a_1, a_2) \rightarrow f(a_2, a_0) \rightarrow f(a_0, a_1) = a_0$, so $f(a_1, a_2) = a_1$ and $f(a_2, a_0) = a_2$. Since $f(a_0, a_1) \rightarrow f(b_0, a_2) \rightarrow f(a_1, a_0)$ and there are no 2-cycles in $\mathcal{G}$, therefore $f(a_1, a_0) \neq a_0$ and so $f(a_1, a_0) = a_1$. Similarly as before, $a_1 = f(a_1, a_0) \rightarrow f(a_2, a_1) \rightarrow f(a_0, a_2) \rightarrow f(a_1, a_0) = a_1$ implies $f(a_2, a_1) = a_2$ and $f(a_2, a_0) = a_2$, so f is the first projection on $\{a_0, a_1, a_2\}$. Moreover, from $\{a_1, b_0\} \ni f(b_0, a_1) \rightarrow f(a_1, a_2) = a_1$ implies that $f(b_0, a_1) = b_0$. Thus from $b_0 = f(b_0, a_1) \rightarrow f(a_1, v) \rightarrow f(v, b_0) \rightarrow f(b_0, a_1) = b_0$ we get $f(a_1, v) = a_1$ and $f(v, b_0) = v$. Finally, from $a_1 = f(a_1, v) \rightarrow f(a_2, b_0) \rightarrow f(v, a_1)$ follows that $f(v, a_1) \neq a_1$ since there are no 2-cycles in $\mathcal{G}$, while from $a_1 = f(a_1, a_0) \rightarrow f(v, a_1)$ follows that $f(v, a_1) \neq b_0$ since $b_0 \rightarrow a_1$ and not the other way round. So the remaining possibility is that $f(v, a_1) = v$ and then from $v = f(v, a_1) \rightarrow f(b_0, v) \rightarrow f(a_1, b_0) \rightarrow f(v, a_1) = v$ we get $f(a_1, b_0) = a_1$ and $f(b_0, v) = b_0$. So, f is the first projection both on $\{a_0, a_1, a_2\}$ and on $\{a_1, v, b_0\}$. The proof that if $f(a_0, a_1) = a_1$ then f is the second projection both on $\{a_0, a_1, a_2\}$ and on $\{a_1, v, b_0\}$ is analogous. $\square$

Claim 3: No idempotent polymorphism of $\mathcal{G}$ restricts to $\{a_1, v, b_0\}$ as a Mal'cev or as a near-unanimity operation.

Assume first that $d(x, y, z)$ is a ternary idempotent polymorphism of

$\mathcal{G}$ which restricts to $\{a_1, v, b_0\}$ as a Mal'cev operation, i. e. $d(x, x, y) = y = d(y, x, x)$ holds identically on $\{a_1, v, b_0\}$. Applying Claim 2 we get that $d(x, x, y) = y = d(y, x, x)$ holds identically on $\{a_0, a_1, a_2\}$, as well. Thus, we get $a_1 = d(a_1, a_0, a_0) \to d(v, v, a_1) = a_1$, a contradiction.

On the other hand, assume that $f(x_1, x_2, \ldots, x_n)$ is an idempotent polymorphism of $\mathcal{G}$ which restricts to $\{a_1, v, b_0\}$ as a near-unanimity operation, i. e. $f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = \ldots = f(x, x, \ldots, x, y) = x$ is true for all $x, y \in \{a_1, v, b_0\}$. Applying Claim 2 we get that $f(y, x, \ldots, x) = f(x, y, x, \ldots, x) = f(x, x, \ldots, x, y) = x$ holds identically on $\{a_0, a_1, a_2\}$, as well. Let i be such that $f(x^i y^{n-i}) = x$, while $f(x^{i-1} y^{n-i+1}) = y$ for all $x, y \in \{a_1, v, b_0\}$ (and consequently, for all $x, y \in \{a_0, a_1, a_2\}$). From compatibility of f with $\{a_0, a_1, a_2\}$ and

$$f(a_0^{i-1} a_1 a_2^{n-i}) \to f(a_1^{i-1} a_2 a_0^{n-i}) \to f(a_2^{i-1} a_0 a_1^{n-i}) \to f(a_0^{i-1} a_1 a_2^{n-i})$$

we conclude that the triples

$$\big(f(a_0^{i-1} a_1 a_2^{n-i}), f(a_1^{i-1} a_2 a_0^{n-i}), f(a_2^{i-1} a_0 a_1^{n-i})\big) \text{ and } (a_0, a_1, a_2)$$

are cyclic permutations of each other. We have two cases:

Case 1: $a_2 \to b_0$. Then $f(a_0^{i-1} a_1 a_2^{n-i}) = a_0$ implies that $f(a_1^{i-1} a_2 a_0^{n-i}) = a_1$, so $a_1 = f(a_1^{i-1} a_2 a_0^{n-i}) \to f(v^{i-1} b_0^{n-i+1}) = b_0 \to a_1$, which would be a double-edge, a contradiction. On the other hand, if $f(a_0^{i-1} a_1 a_2^{n-i}) = a_2$, then $f(a_2^{i-1} a_0 a_1^{n-i}) = a_1$ and $a_1 = f(a_2^{i-1} a_0 a_1^{n-i}) \to f(b_0^i v^{n-i}) = b_0 \to a_1$, a contradiction. Finally, if $f(a_0^{i-1} a_1 a_2^{n-i}) = a_1$, then $a_1 = f(a_0^{i-1} a_1 a_2^{n-i}) \to f(b_0^{i-1} v b_0^{n-i}) = b_0 \to a_1$, again a contradiction.

Case 2: $b_0 \to a_2$. If $f(a_0^{i-1} a_1 a_2^{n-i}) = a_0$, then $a_0 = f(a_1^{i-1} a_0^{n-i+1}) \to f(a_2^{i-1} b_0^{n-i+1}) \to f(a_0^{i-1} a_1 a_2^{n-i}) = a_0$, a contradiction. On the other hand, if $f(a_0^{i-1} a_1 a_2^{n-i}) = a_2$, then $f(a_1^{i-1} a_2 a_0^{n-i}) = a_0$ and $a_0 = f(a_0^i a_1^{n-i}) \to f(b_0^i a_2^{n-i}) \to f(a_1^{i-1} a_2 a_0^{n-i}) = a_0$, again the same contradiction. Finally, if $f(a_0^{i-1} a_1 a_2^{n-i}) = a_1$, then $f(a_2^{i-1} a_0 a_1^{n-i}) = a_0$ and hence $a_0 = f(a_2^{i-1} a_0 a_1^{n-i}) \leftarrow f(b_0^{i-1} a_2 b_0^{n-i}) \leftarrow f(v^{i-1} a_1 v^{n-i}) = v$, again a contradiction since $(v^+)^+ = b_0^+ = \{a_1, a_2\}$, so $a_0 \notin (v^+)^+$. Thus Claim 3 is proved. $\square$

Let f be an idempotent polymorphism of $\mathcal{G}$. We deduce from Lemma 4.1 that there exists a unique k such that $f_k(b_0, v) = v$. Without loss of generality, assume that $k = 1$, so let $f(v, b_0^{n-1}) = v$. By the binary case we proved in Claim 2, it follows that $f(a_1, a_0^{n-1}) = a_1$. However, since $f(\{v\}, \{a_1, v, b_0\}^{n-1}) \subseteq f(a_1, a_0^{n-1})^+ = a_1^+$, we obtain $f(\{v\}, \{a_1, v, b_0\}^{n-1}) = \{v\}$. It follows that f is the first projection on $\{a_1, v, b_0\}$ by the same "going around the 3-cycle" argument we used several times in this proof. $\square$ $\square$

Theorem 4.31. *A strongly connected semicomplete digraph which is not a cycle has only trivial idempotent polymorphisms.*

Proof. We prove it by an induction on $|V| = n$. By Theorem 4.25, if $\mathcal{G}$ is a P-graph, we are done, so we assume that $\mathcal{G}$ is not a P-graph. For $n = 2$ the only strongly connected semicomplete digraph must be a cycle. If $n = 3$ and $\mathcal{G}$ is not a cycle, then there is a 2-cycle $a \leftrightarrow b$ in $\mathcal{G}$, and the third vertex c must satisfy either $a \to c \to b$ or $b \to c \to a$ (possibly even both!), so by Lemma 4.26 and Lemma 4.8 all idempotent polymorphisms are projections. Also, if $n = 4$, then $\mathcal{G}$ is a P-graph parametrized by the 3-cycle if $\mathcal{G}$ is the only 4-element strongly connected tournament or in the case when $V = \{a, b, c, d\}$ has exactly one 2-cycle $a \leftrightarrow b$, $c \in \{a, b\}^{\forall +}$ and $d \in \{a, b\}^{\forall -}$. Otherwise, from Lemmas 4.26, 4.8 and 4.7 follows that all idempotent polymorphisms of $\mathcal{G}$ are projections.

Now assume that $n > 4$ and that the Theorem holds in all strongly connected semicomplete digraphs with fewer than n vertices. We are going to prove that the maximal nice subset of V is V itself. We would like to prove this by finding first a nice subset, then going to the maximal nice subset which contains it and finally proving that this maximal nice set is V. However, the actual argument we found is slightly murkier, we may not be able to start off from a nice subset. Our starting point might be instead a 2-cycle, which may have a polymorphism (of arity greater than 2) which is nontrivial. However, if not even Lemma 4.28 provides a nice subset which contains the 2-cycle, then we are able to proceed inductively to a nice set which contains the 2-cycle by Lemma 4.29.

First we need to prove that there exists a 2-cycle, or a nice subset with more than one element. If there exists a 2-cycle $a \leftrightarrow b$, then we set $L_0 = \{a, b\}$. Otherwise, $\mathcal{G}$ is a tournament, and if there exists any vertex $v \in V$ and a strong component L_0 of the induced subgraph on v^- or on v^+ such that $|L_0| > 3$, then L_0 is clearly pp-definable with constants in $\mathcal{G}$, so L_0 must be nice by the inductive assumption. Finally, if $\mathcal{G}$ is a tournament and for all $v \in V$ all strong components of the induced subgraphs on v^- and on v^+ have at most three elements, then $\mathcal{G}$ is either a P-graph, in which case we are done by Theorem 4.25, or from Lemma 4.30 follows that there is a three element subset L_0 which is nice.

Let L be a maximal nice subset of V such that $L_0 \subseteq L$. Assume that $L \neq V$. Either L exists, so by Lemma 4.28 L splits $\mathcal{G}$, or L_0 is contained in no nice set, so L_0 is a 2-cycle, so by Lemmas 4.26 and 4.8 L_0 splits $\mathcal{G}$ (in this case we also set $L := L_0$). From Lemma 4.29 follows that either a strong component L' of the induced subgraph on a_0^+ contains $L \cup \{a_1, b_0\}$

(if Lemma 4.29 (b) holds), or that a strong component L' of the induced subgraph on a_1^- contains $L \cup \{a_0, b_0\}$ (Lemma 4.29 (a) holds). Either way, L' is pp-definable in $\mathcal{G}^c$, $L \subsetneq L' \subsetneq V$ and the induced subgraph on L' is strongly connected, so by the inductive assumption L' is nice. This contradicts the assumed maximality of L (or nonexistence of the nice set which contains L_0, as the case may be). The remaining alternative is $L = V$, but then the Theorem holds by niceness of L. $\square$

As we mentioned at the start of this section, by proving Theorem 4.31 and invoking Proposition 4.2 we also proved Theorem 4.3.

5 Smooth semicomplete digraphs with several strong components

In this section we deal with the smooth semicomplete digraphs $\mathcal{G}$ and we will show that QCSP($\mathcal{G}$) is PSPACE-complete whenever $\mathcal{G}$ is not a 2-cycle nor a 3-cycle (i.e. when $\mathcal{G}$ has at least two cycles, cf. the proof of Theorem 3.3). The case when $\mathcal{G}$ has only one strong component was resolved in Theorem 4.3, so we may assume that $\mathcal{G}$ has at least two strong components. Moreover, smoothness implies that the largest and smallest one in the linear order of strong components induced by the edge relation are nontrivial.

5.1 Two strong components

Let us first deal with the case when the digraph $\mathcal{G} = (V, \to)$ consists of exactly two nontrivial strong components, U and L such that $L \Rightarrow U$. Since they are nontrivial, the induced subgraph on each of L and U is either a cycle (a 2-cycle or 3-cycle would be the only semicomplete ones, but our proof would also work if there was a k-cycle instead!) or a strongly connected semicomplete digraph with at least two cycles, for which we know from Theorem 4.31 that the only idempotent polymorphisms of that subgraph are projections. Recall the notation $\preceq_{\mathcal{G}}$ from Section 2. Note that $a^+ \subseteq b^+$ iff $a \preceq_{\mathcal{G}^\partial} b$, where $\mathcal{G}^\partial$ is the dual graph. Denote by $x \to^k y$ the assertion that there exists a directed path of length k from x to y.

Lemma 5.1. *Let $a \in L$ and $b \in U$ such that a is maximal in the poset $(L, \preceq_{\mathcal{G}})$ and b is maximal in $(U, \preceq_{\mathcal{G}^\partial})$. Then $\{a, b\} = (a^-)^{\forall +} \cap (b^+)^{\forall -}$. In particular, $\{a, b\}$ is pp-definable in $\mathcal{G}^c$, and thus compatible with any idempotent polymorphism of $\mathcal{G}$.*

Proof. ($\subseteq$) follows the fact that $L \Rightarrow U$.
($\supseteq$): Let c be in the right hand side, but not in $\{a, b\}$. Assume first that $c \in L$. Then for all $x \in a^-$, $x \to c$, so $a^- \subseteq c^-$. Thus $a \preceq_{\mathcal{G}} c$, and from maximality of a follows $c = a$. The case when $c \in U$ is dual.

The final sentence requires no proof. $\square$

Note that, in the case when L is a cycle, any of its elements satisfies the maximality condition for a, and dually when U is a cycle.

Lemma 5.2. *Let $f(x_1, \ldots, x_n)$ be an idempotent polymorphism of $\mathcal{G}$ and $1 \leq k \leq n$.*

(i) For all $(a, b_2, \ldots, b_n) \in L^k \times U^{n-k}$, if $a^- \subseteq f(a, b_2, \ldots, b_n)^-$ and $U \subseteq f(a, b_2, \ldots, b_n)^+$, then $f(a, b_2, \ldots, b_n) = a$ for all $(a, b_2, \ldots, b_n) \in L^k \times U^{n-k}$.

(ii) For all $(a, b_2, \ldots, b_n) \in L^k \times U^{n-k}$, if $a^+ \subseteq f(a, b_2, \ldots, b_n)^+$ and $U \subseteq f(a, b_2, \ldots, b_n)^+$, then $f(a, b_2, \ldots, b_n) = a$ for all $(a, b_2, \ldots, b_n) \in L^k \times U^{n-k}$.

(iii) Assume also that U is a cycle and that for all $(a', b'_2, \ldots, b'_n) \in L^k \times U^{n-k}$, $(a')^- \subseteq f(a', b'_2, \ldots, b'_n)^-$. If $(a, b_2, \ldots, b_n) \in L^k \times U^{n-k}$ satisfy $n - k \geq 2$, $b_{k+1} \neq b_{k+2}$, $b_{k+1}^+ \subseteq f(a, b_2, \ldots, b_n)^+$ and $b_{k+2}^+ \subseteq f(a, b_2, \ldots, b_n)^+$, then $f(a, b_2, \ldots, b_n) = a$.

Remark: Note that the above Lemma also implies its dual statement for $(a, b_2, \ldots, b_n) \in U^k \times L^{n-k}$ (obtained by transposing U and L, $^+$ and $^-$, etc.). Also, the order of variables may be permuted arbitrarily and the same statements would hold.

Proof. In all three cases we have that $a \preceq_{\mathcal{H}} f(a, b_2, \ldots, b_n)$ with respect to some graph $\mathcal{H} = \mathcal{G}$ in (i) and (iii), while in (ii) we use $\mathcal{H} = \mathcal{G}^{\partial}$. In (iii) from the assumption that U is a cycle follows that $|f(a, b_2, \ldots, b_n)^+ \cap U| \geq |b_{k+1}^+ \cup b_{k+2}^+| = 2$, hence $f(a, b_2, \ldots, b_n) \in L$, as $|x^+| = 1$ for all $x \in U$. The same $f(a, b_2, \ldots, b_n) \in L$ can be concluded from $U \subseteq f(a, b_2, \ldots, b_n)^+$ for the cases (i) and (ii). So let a be maximal in the poset $(L, \preceq_{\mathcal{H}})$ such that there exist $(b_2, \ldots, b_n) \in L^{k-1} \times U^{n-k}$ for which $e := f(a, b_2, \ldots, b_n) \in L$ and $a \neq e$. The relations $a \preceq_{\mathcal{H}} e$ and $a \neq e$ mean that $a \to e$ in cases (i) and (iii), while they mean $a \leftarrow e$ in (ii). Select $(d_2, \ldots, d_n) \in L^{k-1} \times U^{n-k}$ such that $b_i \to d_i$ in cases (i) and (iii), respectively $b_i \leftarrow d_i$ in case (ii). Since b_i and d_i are in the same strong component of $\mathcal{G}$, there exists some m such that, in cases (i) and (iii), $a \to e \to^m a$ and $b_i \to d_i \to^m b_i$, while in (ii) the same holds with $\leftarrow$ in place of $\rightarrow$ (we take $m+1$ to be the least common multiple of several lengths of cycles). Since $f(e, d_2, \ldots, d_n) \rightarrow^m f(a, b_2, \ldots, b_n) = e \in L$ (in (*ii*) replace $\rightarrow^m$ with $\rightarrow$), it follows that $f(e, d_2, \ldots, d_n) \in L$. From maximality of a follows that $f(e, d_2, \ldots, d_n) = e$. But this is a contradiction, as $e = f(a, b_2, \ldots, b_n) \rightarrow f(e, d_2, \ldots, d_n) = e$ (in case (*ii*) use $\leftarrow$) contradicts the irreflexivity of $\rightarrow$. $\square$

Lemma 5.3. *Let $f(x, y)$ be a binary idempotent polymorphism of $\mathcal{G}$. Then f is one of the two projections.*

Proof. Fix a and b which fit the conditions of Lemma 5.1. According to Lemma 5.1, without loss of generality, we may assume that $f(a, b) = a$. We claim first that f is the first projection on L.

If L is not a cycle, then we get for any $x \in a^-$ that $(x, a) \rightarrow (a, b)$ in $\mathcal{G}^2$. We know that $f\restriction_L$ is one of the two projections, according to Theorem 4.31, and $f(x, a)$ can't equal a as we would get $a = f(x, a) \rightarrow f(a, b) = a$, which is impossible. Thus from $\{x, a\} \subseteq L$ and $f(x, a) \neq a$ follows that $f\restriction_L$ must be the first projection. In the case when L is a k-cycle, $a = a_k \leftarrow a_{k-1} \leftarrow \ldots \leftarrow a_1 \leftarrow a_0 = a$, for each l, $0 < l < k$, we get $f(a_{k-1}, a_{k-l-1}) \rightarrow f(a, b) = a = a_k$, so $f(a_{k-1}, a_{k-l-1}) = a_{k-1}$. Continuing like this we inductively get that $f(a_i, a_{i-l}) = a_i$ for all $0 \leq i \leq k-1$, where the subtractions in the last few sentences are modulo k, of course.

Now for some $x \in a^-$ we get $(a, x) \rightarrow (b, a)$ in $\mathcal{G}^2$, so $a = f(a, x) \rightarrow f(b, a)$, so $f(b, a) \neq a$, and therefore $f(b, a) = b$. By the dual argument to that of the last paragraph, we get that f is the first projection in the set U, as well.

It remains to prove that f is the first projection when one of the arguments is in L and the other in U. For any $x \in L$ and $y \in U$ we get that for each $u \in U$ and some $v \in y^+$, $f(x, y) \rightarrow f(u, v) = u$, so $U \subseteq f(x, y)^+$. Also, for each $w \in x^-$, $w = f(w, x) \rightarrow f(x, y)$, so $f(x, y) \in w^+$. In other words, $x^- \subseteq f(x, y)^-$. Now, by Lemma 5.2 (*i*) it follows that $f(x, y) = x$ for all $x \in L$ and $y \in U$. We prove $f(x, y) = x$ for $x \in U$ and $y \in L$ using a dual proof and the dual of Lemma 5.2 (*i*). $\square$

Lemma 5.4. *Let f be an idempotent polymorphism of $\mathcal{G}$. There exists exactly one i such that $f_i(x, y) = y$, while for all other j, $j \neq i$, it is $f_j(x, y) = x$.*

Proof. According to Lemma 5.3, each $f_i(x, y)$ is identically equal to one of x and y. $\mathcal{G}$ is a smooth digraph of algebraic length 1 (witnessed by the induced subgraph on any two elements of L and one element of U) without loops, so it has not even a weak near-unanimity polymorphism, according to the Loop Lemma of [2] (see the remarks preceding Lemma 4.15), and therefore by Proposition 2.1 $\mathcal{G}$ has no Mal'cev nor near-unanimity polymorphisms. Now apply Lemma 4.1 to $Pol_{id}(\mathcal{G})$. $\square$

Theorem 5.5. *Let $\mathcal{G}$ be a smooth semicomplete digraph with precisely two strong components. Then all idempotent polymorphisms of $\mathcal{G}$ are projections.*

Proof. Assume that $f(x_1, x_2, \ldots, x_n)$ is an idempotent polymorphism of $\mathcal{G}$ with $n \geq 2$. Without loss of generality, we can assume from Lemma 5.4 that $f_1(x, y) = y$ for all $x, y \in V$ and that $f_i(x, y) = x$ for all $x, y \in V$ and $i > 1$. This is for easier notation, if the coordinate singled out by Lemma 5.4 is another, we just permute the coordinates of f to reduce to another idempotent polymorphism which fits this case. We are going to prove that f is the first projection.

We prove it by an induction on n. In the base case $n = 2$ there is nothing to prove.

Fix some $a_1, a_2, \ldots, a_n \in V$. Without loss of generality, we assume that $a_1 \in L$. Also, we may assume for i and j all such that $i \neq j$ and $2 \leq i, j \leq n$, that $a_i \neq a_j$, or there would exist some polymorphism g which is the substitution instance of f obtained by identifying the ith and jth variables which satisfies $g(y, x, \ldots, x) = y$ and which has arity $n - 1$. This g would be the first projection by the inductive assumption and hence $f(a_1, a_2, \ldots, a_n) = a_1$. Also, if $a_1 = a_i$ for some $2 \leq i \leq n$, then define the idempotent polymorphism g of $\mathcal{G}$ from f by identifying the first and ith variables. Now from Lemma 5.4 we get $f_l(x, y) = x$ for all $2 \leq l \leq n$, and hence we get $g_j(x, y) = x$ for all $2 \leq j \leq n - 1$. Then by Lemma 5.4 it follows that $g(y, x, \ldots, x) = g_1(x, y) = y$ and the inductive assumption implies that g is the first projection. Thus, again, $f(a_1, \ldots, a_n) = a_1$. So we are left with the case when $|\{a_1, a_2, \ldots, a_n\}| = n$.

Case 1: Let $|\{a_2, a_3, \ldots, a_n\} \cap L| \neq 0$ and $|\{a_2, a_3, \ldots, a_n\} \cap U| \neq 0$. Without loss of generality (by permuting the coordinates), let $a_2, \ldots, a_k \in L$ and $a_{k+1}, \ldots, a_n \in U$ for some $2 < k < n$. Then for any $d \in U$ and $e \in a_n^+$ we get that $f(a_1, a_2, \ldots, a_n) \to f(d, e, a'_3 \ldots, a'_{k-1}, e) = d$, where the elements $a'_i \in a_i^+$ for $3 \leq i \leq n - 1$ and the equality holds by the inductive assumption, thus $U \subseteq f(a_1, a_2, \ldots, a_n)^+$. Also, for any $d \in a_1^-$ and $e \in a_2^-$, $d = f(d, e, a'_3 \ldots, a'_{k-1}, e) \to f(a_1, a_2, \ldots, a_n)$, where $a'_i \in a_i^-$ for $3 \leq i \leq n-1$ and the equality holds by the inductive assumption, so $a_1^- \subseteq f(a_1, a_2, \ldots, a_n)^-$. Now from Lemma 5.2 (i) we get that $f(a_1, a_2, \ldots, a_n) = a_1$ for all $(a_1, a_2, \ldots, a_n) \in L^k \times U^{n-k}$.

Case 2: Let $(a_1, \ldots, a_n) \in L^n$. There we get for any $d \in a_1^+ \cap L$, $a_i' \in a_i^+ \cap L$ for $1 < i < n$ and $a_n' \in U$ that $f(a_1, a_2, \ldots, a_n) \to f(d, a_2', \ldots, a_n') = d$ (the last equality holds either by the Case 1 or by the inductive assumption, since $n > 2$), so $a_1^+ \subseteq f(a_1, a_2, \ldots, a_n)^+$. Also, for all $d \in U$, $f(a_1, a_2, \ldots, a_n) \to f(d, d, \ldots, d) = d$, so $U \subseteq f(a_1, a_2, \ldots, a_n)^+$. Now by Lemma 5.2 (ii) we get $f(a_1, a_2, \ldots, a_n) = a_1$ for all $a_1, a_2, \ldots, a_n \in L$.

Case 3: Let $(a_1, a_2, \ldots, a_n) \in L \times U^{n-1}$. We have for each $d \in a_1^-$ that $d = f(d, d, \ldots, d) \to f(a_1, a_2, \ldots, a_n)$, so $a_1^- \subseteq f(a_1, a_2, \ldots, a_n)^-$.

First assume that the induced subgraph on U is a cycle. Then $a_i^+ = \{a_i'\}$ for all $2 \leq i \leq n$, and we get $f(a_1, a_2, \ldots, a_n) \to f(a_2', a_2', \ldots, a_n') = a_2'$ and $f(a_1, a_2, \ldots, a_n) \to f(a_3', a_2', \ldots, a_n') = a_3'$, where the equalities hold by the inductive assumption, that is, by the observations at the start of this proof. Since $a_2 \neq a_3$ we have all the conditions of Lemma 5.2 (iii) fulfilled, and so $f(a_1, a_2, \ldots, a_n) = a_1$.

Finally, let the induced subgraph on U be strongly connected and semicomplete with at least two cycles. The restriction of f to U^n is a projection, so it can only be the first one, since $f(y, x, \ldots, x) = y$ for all $x, y \in V$. Therefore, for any $d \in U$, and any $a_i' \in a_i^+$ for all $2 \leq i \leq n$, we get $f(a_1, a_2, \ldots, a_n) \to f(d, a_2', \ldots, a_n') = d$, so $U \subseteq f(a_1, a_2, \ldots, a_n)^+$. Thus by Lemma 5.2 (i), $f(a_1, a_2, \ldots, a_n) = a_1$, completing the proof. □

5.2 Several strong components, but just two nontrivial

We first generalize Theorem 5.5 to the case of smooth semicomplete digraphs with precisely two non-singleton strong components. The order of strong components is linear, and if $x, y \in V$ are in distinct strong components, $x \to y$ and $\neg y \to x$, then we say that the component of x is below that of y. Since $\mathcal{G}$ is smooth, then the only two nontrivial strong components must be the top and bottom one in the order of components.

We denote the strong components by L (the bottom one), U (the top one) and $M_i = \{m_i\}$, for $1 \leq i < k$, which are in between, where M_i is below M_j iff $i < j$.

Lemma 5.6. *$U \cup L$ is closed under any polymorphism of $\mathcal{G}$.*

Proof. Both L and U have Hamiltonian cycles, being nontrivial strong components in a semicomplete digraph. Let the lengths of those cycles be ℓ_1 and ℓ_2, respectively, and let $\ell = \text{lcm}\,(\ell_1, \ell_2)$. Let f be a polymorphism of $\mathcal{G}$ of arity n. For any $(a_1, \ldots, a_n) \in (U \cup L)^n$ we know that $(a_1, \ldots, a_n) \to^\ell (a_1, \ldots, a_n)$ in the digraph $\mathcal{G}^n$. Hence, $f(a_1, \ldots, a_n) \to^\ell f(a_1, \ldots, a_n)$, and therefore $f(a_1, \ldots, a_n)$ can't be in any M_i for $1 \leq i < k$, as any element such that a directed path leads from m_i to it must be in M_i^+, therefore in a strong component above M_i, and not equal to m_i. $\square$

Theorem 5.7. *Let $\mathcal{G}$ be a smooth semicomplete digraph with exactly two non-trivial strong components. Then all idempotent polymorphisms of $\mathcal{G}$ are projections.*

Proof. According to Lemma 5.6 and Theorem 5.5, we may assume that f is the ith projection on $U \cup V$, i.e. that $f(a_1, \dots, a_n) = a_i$ for all $(a_1, \dots, a_n) \in (L \cup U)^n$. We will prove that f is the ith projection on all of V^n.

Case 1: Let $a_i \in M_1 \cup \dots \cup M_{k-1}$. Then for each $d \in L$ and $e \in U$ there exist tuples $(b_1, \dots, b_n) \in L^n$ and $(c_1, \dots, c_n) \in U^n$ such that $b_i = d$, $c_i = e$ and for all $a \leq j < n$, $b_j \in a_j^-$ and $c_j \in a_j^+$. Therefore, $d = f(b_1, \dots, b_n) \to f(a_1, \dots, a_n) \to f(c_1, \dots, c_n) = e$, and so $L \subseteq f(a_1, \dots, a_n)^-$ and $U \subseteq f(a_1, \dots, a_n)^+$. This implies that $f(a_1, \dots, a_n) \in M_1 \cup \dots \cup M_{k-1}$.

Let $a_i = m_j$. Define the tuples $(a_1^{(1)}, \dots, a_n^{(1)})$, $\dots$, $(a_1^{(k-1)}, \dots, a_n^{(k-1)})$ so that for all s, t, $a_s^{(t)} \to a_s^{(t+1)}$, that $a_s^{(j)} = a_s$ and that $a_i^{(t)} = m_t$. These tuples exist because of the structure of $\mathcal{G}$ and its smoothness. Now by the previous paragraph, all $f(a_1^{(t)}, \dots, a_n^{(t)})$ are in $M_1 \cup \dots \cup M_{k-1}$ and $f(a_1^{(t)}, \dots, a_n^{(t)}) \to f(a_1^{(t+1)}, \dots, a_n^{(t+1)})$ for all t. Since the relation $\to$ on $M_1 \cup \dots \cup M_{k-1}$ is the strict partial order on a set with $k-1$ elements, the only path on that set of length $k-1$ is $m_1 \to m_2 \to \dots \to m_{k-1}$, Therefore it must be that $f(a_1^{(t)}, \dots, a_n^{(t)}) = m_t = a_i^{(t)}$ and the case is done.

Case 2: Let $a_i \in U \cup L$. Without loss of generality we may assume $a_i \in L$ (or we would just reverse the edges). Now for each $d \in V \setminus L$, there exist some $b_1, \dots, b_n$ such that for all j, $a_j \to b_j$, $b_i = d$ and $b_j \in U$ for all $j \neq i$. If $d \notin U$, then it follows from Case 1 that $f(a_1, \dots, a_n) \to f(b_1, \dots, b_n) = d$. On the other hand, if $d \in U$, from the fact that $f(b_1, \dots, b_n) = b_i$ if all of b_j are in U, again we get $f(a_1, \dots, a_n) \to f(b_1, \dots, b_n) = d$. Thus, $(V \setminus L) \subseteq f(a_1, \dots, a_n)^+$. On the other hand, if $c_i \in a_i^-$, then there exist $c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_n \in L^{n-1}$ such that $c_j \to a_j$ for all $j \leq n$. Hence $c_i = f(c_1, \dots, c_n) \to f(a_1, \dots, a_n)$, so $a_i^- \subseteq f(a_1, \dots, a_n)^-$. The rest of the proof of this case proceeds exactly like in the proof of Lemma 5.2 (i), with $M_1 \cup \dots \cup M_{k-1} \cup U$ playing the role of U this time, and L still being L. $\square$

5.3 More than two nontrivial strong components

Now we deal with the remaining smooth case, namely the case when there exist more than two nontrivial strong components.

Theorem 5.8. *Let $\mathcal{G}$ be a smooth semicomplete digraph with at least two cycles. Then $\mathcal{G}$ has no idempotent polymorphisms other than projections.*

Proof. Let the strong components of $\mathcal{G}$, ordered by $\Rightarrow$, be $B_1 < B_2 < \ldots < B_m$. Let all nontrivial strong components be $B_1, B_{i_1}, B_{i_2}, \ldots, B_{i_{k-1}}, B_m$, where $1 < i_1 < i_2 < \ldots < i_{k-1} < m$. Define subsets $C_1 = \bigcup_{j=1}^{i_1} B_j$, $C_k = \bigcup_{j=i_{k-1}}^{m} B_j$ and $C_s = \bigcup_{j=i_{s-1}}^{i_s} B_j$ for $1 < s < k$. So, each C_i consists of exactly two consecutive nontrivial strong components, one on the top, one on the bottom, and all trivial strong components between these two nontrivial ones (if any). It is easy to show that each C_j is primitively positively definable in $\mathcal{G}^c$, as $C_j = B^{\forall+}_{i_{j-1}-1} \cap B^{\forall-}_{i_j+1}$ (in the case of C_1 and C_k they are just $B^{\forall-}_{i_1+1}$ and $B^{\forall+}_{i_{k-1}-1}$, respectively). Therefore, each C_j, as well as any union $\bigcup_{j=r}^{\ell} C_j$, where $1 \leq r \leq \ell \leq k$, is closed under all idempotent polymorphisms of $\mathcal{G}$.

From Theorem 5.7 we get that the restriction of each idempotent polymorphism f to each C_j is some projection, let us say it is the ith on C_1. Since C_j and C_{j+1} intersect in the nontrivial strong component B_{i_j} on which f is (inductively) the ith projection as this strong component is a part of C_j, then it must be the ith projection on the set C_{j+1}, too, by Theorem 5.7. Thus the restriction of f to each C_j is the ith projection.

We finish the proof by inductively showing for each tuple $(a_1, \ldots, a_n) \in V^n$ that $f(a_1, \ldots, a_n) = a_i$, where we use the induction on the minimal number ℓ such that there exists some s so that $\{a_1, \ldots, a_n\} \subseteq C_s \cup \ldots \cup C_{s+\ell}$. For $\ell = 0$ we have proved it in the previous paragraph.

If $\ell \geq 1$, we first consider the case when $a_i \notin C_s \cup C_{s+\ell}$. Then we will show by the inductive assumption that $a_i^+ \subseteq f(a_1, \ldots, a_n)^+$ and $a_i^- \subseteq f(a_1, \ldots, a_n)^-$, which can only be satisfied in a semicomplete digraph if $f(a_1, \ldots, a_n) = a_i$. More precisely, we will prove that any element $b_i \in a_i^+$ is equal to $f(b_1, b_2, ..., b_n)$, where $(b_1, b_2, ..., b_n)$ is a properly selected tuple such that $a_j \to b_j$ for all $1 \leq j \leq n$.

If a_j is in a strong component which is below the strong component containing a_i, then we select b_j to be equal to b_i, since $b_i \in a_i^+ \subseteq a_j^+$. For all other j we select b_j to be any element of $C_{s+\ell} \cap a_j^+$. Then each $b_j \in C_{s+1} \cup \ldots \cup C_{s+\ell}$, so by the inductive assumption $f(b_1, b_2, ..., b_n) = b_i$, implying that $b_i \in f(a_1, \ldots, a_n)^+$, as desired. The proof of $a_i^- \subseteq f(a_1, \ldots, a_n)^-$ is dual.

The remaining case is if $a_i \in C_s$ (the case $a_i \in C_{s+\ell}$ is dual to it with respect to reversal of edges). If a_i is in the top strong component of C_s, then $a_i \in C_{s+1}$ and the previous case applies. If a_i is in one of the trivial components of C_s, then we imitate the proof of Case 1 of Theorem 5.7, with L replaced by $B_{i_{s-1}}$, $M_l, \dots, M_{k-1}$ replaced by the trivial strong components of C_s (i.e. B_t, where $i_{s-1} < t < i_s$) and U replaced by $C_{s+1} \cup \ldots \cup C_{s+\ell}$. Note that we used just the fact that L has no sources and U has no sink in the proof of Case 1 of Theorem 5.7, together with the provisions that the strong components (here, unions of the strong components in case of U) are ordered by $L \Rightarrow M_1 \Rightarrow \ldots \Rightarrow M_k \Rightarrow U$ and that all M_i are trivial, so with the said replacement of the meaning of L, U and M_i, the proof transfers verbatim.

Finally, let $a_i \in B_{i_{s-1}}$, but now by L we denote the set $B_{i_{s-1}}$, while U is $(C_s \setminus L) \cup C_{s+1} \cup \ldots \cup C_{s+\ell}$. We will just replicate the proof of Case 2 of Theorem 5.7, which is in fact a reduction to the proof of Lemma 5.2 (i). For each $b_i \in U$ there exists a tuple $(b_1, \dots, b_n) \in U^n$, where in fact for all $j \neq i$, $b_j \in C_{s+\ell}$ such that $a_j \to b_j$ for all $1 \leq j \leq n$. If b_i is in one of the trivial strong components of C_s, then $f(b_1, \dots, b_n) = b_i$, using the case which we proved in the last paragraph (this argument replaces our reference to Case 1 in the proof of Case 2 of Theorem 5.7), and otherwise $\{b_1, \dots, b_n\} \subseteq C_{s+1} \cup \ldots \cup C_{s+\ell}$ and by the inductive assumption on ℓ we obtain again that $f(b_1, \dots, b_n) = b_i$. We conclude that $U \subseteq f(a_1, \dots, a_n)^+$, and then we reduce it to a proof analogous to that of Lemma 5.2 (i): Namely, note that for each $c_i \in a_i^-$ there exists a tuple $(c_1, \dots, c_n) \in L^n$ such that for all j, $c_j \to a_j$, and therefore $c_i = f(c_1, \dots, c_n) \to f(a_1, \dots, a_n)$. So, $(a_i^- \cap L) \subseteq (f(a_1, \dots, a_n)^- \cap L)$. The rest of the proof of Lemma 5.2 (i) transfers verbatim. □

By Proposition 4.2, Theorem 5.8 implies that

Theorem 5.9. *If $\mathcal{G}$ is a smooth semicomplete digraph with more than one cycle, then QCSP($\mathcal{G}$) is Pspace-complete.* □

6 Semicomplete graphs with one sink and no sources

The remaining class of semicomplete graphs whose complexity is not known by Theorem 5.9 or Theorem 3.3 is those which have a sink and not a source, or vice versa. As the two are symmetric, we assume that the graph has no sources, but has a sink (which is unique by semicompleteness). The sink is labelled by t.

6.1 Some Pspace-hardness results

We recall some notation and terminology from basic logic. A formula is in prenex normal form, prenex form for short, if it starts with a sequence of quantifiers (the *prefix*), each of which acts the remainder of the formula after it, followed by the quantifier-free part (the *matrix*). The evaluation of variables of some formula into a model $\mathcal{M}$ is a mapping τ of the set of all variables of that formula into M. The truth value of the formula φ under evaluation τ is denoted by $v_\tau(\varphi)$ and is defined in the usual inductive way, starting from atomic formulae. When the formula in prenex form is a positive Horn formula, then the matrix is essentially a model of its language on the set of its variables. If φ is a positive Horn formula on a signature with a single binary relation, then the matrix of φ is the graph $\mathcal{G}_\varphi$ which we defined in Section 2. Note that there is a homomorphism from $\mathcal{G}_\varphi$ to $\mathcal{H}$ iff the existential quantification of φ is true on $\mathcal{H}$.

Let $\mathcal{K}_{2\to 2}$ be the semicomplete graph built from disjoint copies $\mathcal{H}_1$ and $\mathcal{H}_2$ of $\mathcal{K}_2$ with all edges added from $\mathcal{H}_1$ to $\mathcal{H}_2$ and none other.

Proposition 6.1. *QCSP($\mathcal{K}_{2\to 2}$) and QCSP($\mathcal{K}^{\to}_{2\to 2}$) are Pspace-complete.*

Proof. There is a fairly straightforward reduction from QCSP($\mathcal{K}_4$), i.e *Quantified* 4*-colouring*, to QCSP($\mathcal{K}_{2\to 2}$), but there is a problem translating it to QCSP($\mathcal{K}^{\to}_{2\to 2}$) with the encoding of universal variables. Let $\mathcal{A} = \langle \{0,1\}; R^{\mathcal{A}} \rangle$, where $R^{\mathcal{A}}$ is the not-all-equal predicate. We give a reduction from QCSP($\mathcal{A}$) to our problems QCSP($\mathcal{K}_{2\to 2}$) and QCSP($\mathcal{K}^{\to}_{2\to 2}$) (the same works for both). Our reduction is vaguely based on that for QCSP($\mathcal{A}$) to QCSP($\mathcal{K}_n$) ($n \geq 3$) in [7], Proposition 5.1.

Let φ be a positive Horn formula in the language $\{R\}$. We construct the corresponding positive Horn formula ψ_φ in the language of digraphs (in linear time) so that φ is a sentence iff ψ_φ is a sentence. For any evaluation τ of the variables of φ into $\mathcal{A}$ we define the corresponding evaluation τ' of variables of ψ_φ into $\mathcal{K}^{\to}_{2\to 2}$. Note that the evaluation τ in our setup is not merely a mapping of the set of all variables into the universe of the model. τ also includes the information which model it maps into. We prove that $v_\tau(\varphi) = \top$ iff $v_{\tau'}(\psi_\varphi) = \top$. The case when φ is a sentence is the desired reduction.

We fix the template $\mathcal{B}$ which is a copy of $\mathcal{K}^{\to}_{2\to 2}$ such that the universe of $\mathcal{B}$ is $\{0,1,2,3,t\}$, where t is the sink, there are double-edges on $\{0,1\}$ and $\{2,3\}$ and also there is an edge from any element of $\{0,1\}$ to any element of $\{2,3\}$. For short we write just ψ for ψ_φ when φ is understood. We will define a few auxiliary graphs, beginning with the *edge gadget* which combines two copies of $K_{2\to 2}$:

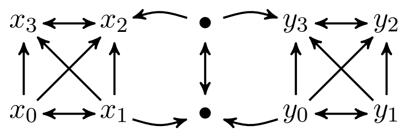

Figure 5: Edge gadget

Each copy of $\mathcal{K}_{2\to 2}$ in the graph of ψ will be denoted by the same letter with indices $0, 1, 2, 3$ which correspond to the same elements of $\mathcal{B}$. The graph of ψ will consist of many such copies with some additional variables. Any evaluation μ of ψ into $\mathcal{B}$ is immediately false (and thus not interesting) unless for all u, the mappings $\mu_u : \{0, 1, 2, 3\} \to \{0, 1, 2, 3\}$ given by $\mu_u(i) = \mu(u_i)$ are automorphism of $K_{2\to 2}$. To any evaluation of ψ we immediately associate all those automorphisms.

Also, the edge gadget depicted in Figure 5 enforces that μ_x and μ_y are distinct, otherwise the middle copy of $\mathcal{K}_2$ ensures that $v_\mu(\psi) = \perp$. On the other hand, if $\mu_x \neq \mu_y$, then they must differ at the upper or at the lower copy of $\mathcal{K}_2$. The connecting copy of $\mathcal{K}_2$ can evaluate at copy of $\mathcal{K}_2$ at which μ_x and μ_y differ and the edge gadget gets the truth value $\top$. The reason we care about this edge gadget is because we chain three of them together to build a triangular *clause gadget* as drawn in Figure 6.

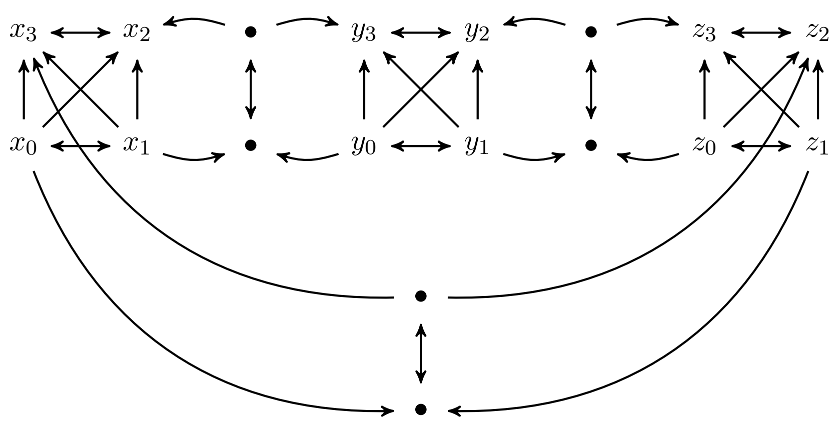

Figure 6: Clause gadget

The salient property of the clause gadget is that the restrictions of μ_x, μ_y and μ_z to $\{0,1\}$ are not all equal, i.e. we can enforce the not-all-equal constraint. This follows since v_μ of the clause gadget is $\top$ iff μ_x, μ_y and μ_z are three distinct automorphisms of $\mathcal{K}_{2\to 2}$, and only two distinct automorphisms of $\mathcal{K}_{2\to 2}$ restrict to $\{0,1\}$ in any fixed way.

Now we define a variable gadget and link variables to clauses. The variable gadget corresponding to s is the subgraph on the vertices $\{s_0, s_1, s_2, s_3, s_\forall\}$ of the graph in Figure 7. The variable gadget links to a copy of $\mathcal{K}_{2\to 2}$ associated to x within some clause gadget iff there is a double edge from s_1 to x_0, as drawn on Figure 7.

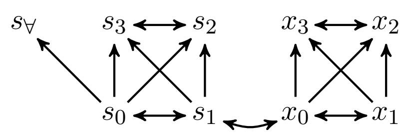

Figure 7: Variable gadget corresponding to s connects to a position in the clause

We first define ψ_φ when φ is quantifier-free, starting with its graph. For each occurrence of the predicate $R(x,y,z)$ in φ we add a clause gadget and for each variable s of φ we add a variable gadget. For any clause $R(u,s,w)$ occurs in φ, we connect $u_1 \leftrightarrow x_0$, $s_1 \leftrightarrow y_0$ and $w_1 \leftrightarrow z_0$. Now ψ_φ is obtained by quantifying existentially all variables with indices 2 and 3 in variable gadgets and also all variables in the clause gadgets.

Assume that τ is an evaluation of φ into $\mathcal{A}$. When $\tau(s) = 0$, we evaluate $\tau'(s_0) = 0$ and $\tau'(s_1) = 1$, while if $\tau(s) = 1$, then $\tau'(s_0) = 1$ and $\tau'(s_1) = 0$. For any s, $\tau'(s_\forall) = 2$.

We claim $v_\tau(\varphi) = v_{\tau'}(\psi_\varphi)$. Assume that $v_\tau(\varphi) = \top$. We select to evaluate the existentially quantified variables in the variable gadgets v_i as i and also for all clause gadgets, we evaluate s_i as $\tau(v_i)$, where $i = 0,1$ and v is the unique variable which is connected to the position s in that clause. As the three variables which appear in some clause are not equally evaluated by τ, in that clause two of the bottom double edges are evaluated equally, while the third one is evaluated differently. The considerations after the definition of the clause gadgets prove that there exists some evaluation of the remaining variables in the clause gadget which has the truth value $\top$.

On the other hand, if τ' is a truthful evaluation of ψ, we choose $\tau(s) = \tau'(s_0)$ for all variables s (of course, the only choices are 0 and 1, as ensured by s_2 and s_3). Whenever there is a clause $R(u,s,w)$, the corresponding clause gadget must have been evaluated as $\tau'(x_0) = \tau'(u_0)$, $\tau'(y_0) = \tau'(s_0)$ and $\tau'(z_0) = \tau'(w_0)$ (where u-gadget is linked to x and so on). The considerations after the definition of clause gadgets showed that $\tau'(x_0) = \tau'(y_0) = \tau'(z_0)$ is impossible since τ' is truthful. Therefore, $\tau'(u_0) = \tau'(s_0) = \tau'(w_0)$ is not true either, and thus $R^{\mathcal{A}}(\tau(u), \tau(s), \tau(w))$ holds.

Now we define the general case of ψ_φ inductively on the number of quantifiers of φ and simultaneously prove our claims about ψ_φ. Assuming that $\varphi = (\exists s)\varphi'$, we define $\psi_\varphi = (\exists s_\forall)(\exists s_0)(\exists s_1)\psi_{\varphi'}$. If $\varphi = (\forall s)\varphi'$, we define $\psi_\varphi = (\forall s_\forall)(\exists s_0)(\exists s_1)\psi_{\varphi'}$.

We are proving $v_\tau(\varphi) = v_{\tau'}(\psi_\varphi)$ by induction on the number of quantifiers of φ. The base case is proved above. Denote by τ_1 the evaluation of the variables of φ which equals τ at all variables except at s, where $\tau(s) \neq \tau_1(s)$ and let the corresponding evaluations of the variables of ψ_φ be τ' and τ'_1.

If $\varphi = (\exists s)\varphi'$, then $v_\tau(\varphi) = \top$ iff $v_\tau(\varphi') = \top$ or $v_{\tau_1}(\varphi') = \top$ iff (by the inductive assumption) $v_{\tau'}(\psi_{\varphi'}) = \top$ or $v_{\tau'_1}(\psi_{\varphi'}) = \top$, which implies $v_{\tau'}(\psi_\varphi) = \top$. We also need the other direction, so assume that $v_{\tau'}(\psi_\varphi) = \top$. Hence there exists an evaluation τ_2 of the variables of ψ_φ which differs from τ' only perhaps at $s_\forall$, s_0 and s_1 such that $v_{\tau_2}(\psi_{\varphi'}) = \top$. Since $\tau_2(s_0) \in \{0,1\}$, if $\tau_3(s_\forall) = 2$ and otherwise τ_3 equals τ_2, then from $v_{\tau_2}(\psi_{\varphi'}) = \top$ follows $v_{\tau_3}(\psi_{\varphi'}) = \top$. But since $\{\tau_3(s_0), \tau_3(s_1)\} = \{0,1\}$, then $\tau_3 = \tau'$ or $\tau_3 = \tau'_1$. By the inductive assumption, $v_\tau(\varphi') = \top$ or $v_{\tau_1}(\varphi') = \top$, so $v_\tau(\varphi) = v_\tau((\exists s)\varphi') = \top$, as desired.

If $\varphi = (\forall s)\varphi'$, then $v_\tau(\varphi) = \top$ iff $v_\tau(\varphi') = v_{\tau_1}(\varphi') = \top$ iff (by the inductive assumption) $v_{\tau'}(\psi_{\varphi'}) = v_{\tau'_1}(\psi_{\varphi'}) = \top$. Now, let τ_2 be any evaluation of the variables of ψ_φ which equals τ' at all variables except possibly $s_\forall$. If $\tau_2(s_\forall) \in \tau'(s_0)^-$, then $v_{\tau_2}(\psi_{\varphi'}) = v_{\tau'}(\psi_{\varphi'}) = \top$, so $v_{\tau_2}((\exists s_0)(\exists s_1)\psi_{\varphi'}) = \top$. Otherwise, $tau_2(s_\forall) = \tau'(s_0) \in \tau'_1(s_0)^-$. Then we select τ_3 to be equal to τ_2, except $\tau_3(s_0) = \tau'(s_1) = \tau'_1(s_0)$ and $\tau_3(s_1) = \tau'(s_0) = \tau'_1(s_1)$. Here $\tau_3(s_\forall) \in \tau_3(s_0)^-$, so $v_{\tau_3}(\psi_{\varphi'}) = v_{\tau'_1}(\psi_{\varphi'}) = \top$, and hence $v_{\tau_2}((\exists s_0)(\exists s_1)\psi_{\varphi'}) = \top$. In all cases we get $v_{\tau_2}((\exists s_0)(\exists s_1)\psi_{\varphi'}) = \top$, hence we proved $v_{\tau'}(\psi_\varphi) = v_{\tau'}((\forall s_\forall)(\exists s_0)(\exists s_1)\psi_{\varphi'}) = \top$. In the other direction, assume $v_{\tau'}(\psi_\varphi) = v_{\tau'}((\forall s_\forall)(\exists s_0)(\exists s_1)\psi_{\varphi'}) = \top$. Let τ_2 and τ_3 be the evaluations of ψ_φ which differ from τ' only at $\tau_2(s_\forall) = 0$ and $\tau_3(s_\forall) = 1$. Our assumption implies $v_{\tau_2}((\exists s_0)(\exists s_1)\psi_{\varphi'}) = v_{\tau_3}((\exists s_0)(\exists s_1)\psi_{\varphi'}) = \top$. Let τ'_2 and τ'_3 be the evaluations which equal τ_2 and τ_3, respectively, on all variables except possibly s_0 and s_1 and such that $v_{\tau_2}(\psi_{\varphi'}) = v_{\tau'_3}(\psi_{\varphi'}) = \top$. We know that $\tau'_2(s_0), \tau'_2(s_1), \tau'_3(s_0), \tau'_3(s_1) \in \{0,1\}$ since they are all in $\{2,3\}^{\forall-}$. From $\tau'_2(s_0) \neq \tau'_2(s_\forall) = 0$ and $\tau'_3(s_0) \neq \tau'_3(s_\forall) = 1$ we get $\tau'_2(s_0) = \tau'_3(s_1) = 1$ and $\tau'_2(s_1) = \tau'_3(s_0) = 0$. But then τ'_2 and τ'_3 are equal to τ' and τ'_1

in some order, except at $s_\forall$. From $v_{\tau_2}(\psi_{\varphi'}) = v_{\tau_3'}(\psi_{\varphi'}) = \top$ we obtain $v_{\tau'}(\psi_{\varphi'}) = v_{\tau_1'}(\psi_{\varphi'}) = \top$, since evaluating $s_\forall$ as 2 can only help, and hence $v_\tau(\varphi) = \top$. $\square$

Corollary 6.2. *Let $\mathcal{G} = (V, \to)$ be a finite loopless digraph. Let $\mathcal{G}$ contain either*

- (i) *a copy of $\mathcal{K}_{2\to 2}$ where $a \leftrightarrow b \to c \leftrightarrow d$ such that any automorphism of this copy extends by the identity map to an automorphism of $\mathcal{G}$ and moreover, $a^+ \cup b^+ = V$, or*
- (ii) *a copy of $\mathcal{K}_3$, $a \leftrightarrow b \leftrightarrow c \leftrightarrow a$ such that any permutation of $\{a, b, c\}$ extends by the identity map to an automorphism of $\mathcal{G}$ and also $a^+ \cup b^+ = a^+ \cup\ c^+ = b^+ \cup\ c^+ = V$,*

then QCSP($\mathcal{G}$) is Pspace-complete.

Proof. We first prove the case (i). Let the subgraph $\mathcal{H}$ be induced by $\mathcal{G}$ on $V \setminus \{a, b, c, d\}$. We modify the proof of Proposition 6.1 by adding variables which are connected in $\mathcal{G}_\psi$ as an isomorphic copy of $\mathcal{H}$. Call these added variables the set H. First we connect the variables in H to the other variables so that for all variable gadgets on variables s_0, s_1, s_2, s_3 and for all clause gadgets which have copies of $\mathcal{K}_{2\to 2}$ induced by $\mathcal{G}_\psi$ on x_0, x_1, x_2, x_3, on y_0, y_1, y_2, y_3 and on z_0, z_1, z_2, z_3, $\mathcal{G}_\psi$ induces on each of $H \cup \{s_0, s_1, s_2, s_3\}$, $H \cup \{x_0, x_1, x_2, x_3\}$, $H \cup \{y_0, y_1, y_2, y_3\}$ and $H \cup \{z_0, z_1, z_2, z_3\}$ an isomorphic copy of $\mathcal{G}$. Basically we make an amalgam of a lot of copies of $\mathcal{G}$ over H. We quantify all variables in H existentially as the outermost quantifiers of the instance.

Whichever evaluation τ of vertices in H is selected, in order to complete τ to a true evaluation of $\mathcal{G}_\psi$, τ together with any one of the variable gadgets must induce an automorphism α of $\mathcal{G}$ which maps $\{a, b, c, d\}$ onto another (possibly different) copy of $\mathcal{K}_{2\to 2}$. Denote by $\mathcal{K}$ this copy of $\mathcal{K}_{2\to 2}$ on $\{\alpha(a), \alpha(b), \alpha(c), \alpha(d)\}$. Any automorphism of $\mathcal{K}$ extends by identity to an automorphism of $\mathcal{G}$ and also $\alpha(a)^+ \cup \alpha(b)^+ = V$ (since (i) is preserved by the automorphism α). The evaluation τ of H is fixed throughout the evaluation of the instance since the variables in H are quantified existentially outermost. Since any automorphism of $\mathcal{K}$ extends by identity to an automorphism of $\mathcal{G}$, this means that the choice of τ does not affect our freedom to evaluate each variable gadget and clause gadget as we will into $\{\alpha(a), \alpha(b), \alpha(c), \alpha(d)\}$, just as if our template was $\mathcal{K}^{\to}_{2\to 2}$. On the other hand, the property $\alpha(a)^+ \cup \alpha(b)^+ = V$ allows us to select any value for the universally quantified variables $s_\forall$ without creating a contradiction. The selection of $s_\forall$ may still limit our choice of the evaluation of the variables s_0 and s_1 to one of the two options, if we evaluate $s_\forall$ as $\alpha(a)$ or as $\alpha(b)$. Now the reduction from QCSP($\mathcal{A}$) follows analogously as in Proposition 6.1.

The case (ii) goes similarly; here we modify in the same way the construction of Proposition 5.1 of [7], proving that QCSP($\mathcal{K}_3$) is Pspace-complete. Let the subgraph $\mathcal{H}$ be induced by $\mathcal{G}$ on $V \setminus \{a, b, c\}$. We modify the proof found in [7] by adding variables which are connected in the graph $\mathcal{G}_\psi$ of the formula as an isomorphic copy of $\mathcal{H}$. Call these added variables the set H. First we connect the variables in H to the other variables so that for all i, $\mathcal{G}_\psi$ induces on $H \cup \{w, x_i, y_i\}$ an isomorphic copy of $\mathcal{G}$. Next, we connect variables in H to each clause gadget used in the proof in [7] to make a copy of $\mathcal{G}$ again. Finally, change the edges between y_i and z_i to $y_i \to z_i$ for all i (in their proof they were undirected). An analogous argument as the one in [7], with modifications just like in the case (i) of this Corollary gives us Pspace-completeness. $\square$

Recall the transitive tournament with an extra edge $\overline{\mathcal{T}_n}$ defined at the end of Section 2. $\overline{\mathcal{T}_n}$ and $\overline{\mathcal{T}_n}^{\rightarrow}$ are depicted in Figure 8.

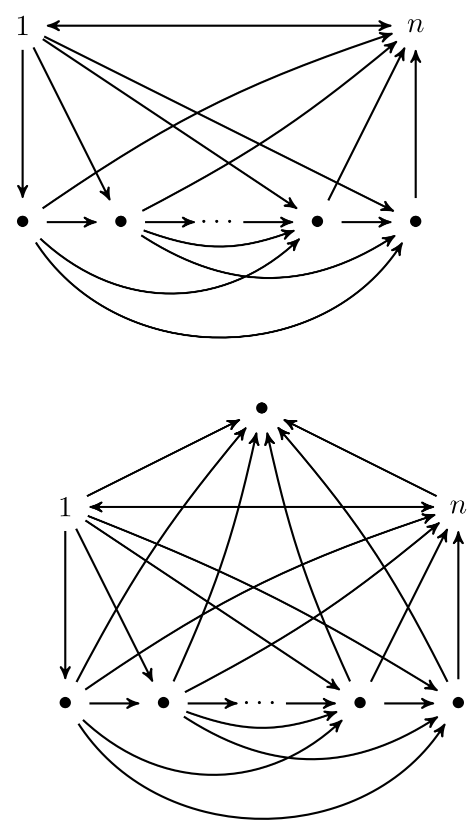

Figure 8: Drawing of $\overline{\mathcal{T}_n}$ with $\overline{\mathcal{T}_n}^{\rightarrow}$

.

Proposition 6.3. *For all $n \geq 3$, QCSP($\overline{\mathcal{T}}_n$) and QCSP($\overline{\mathcal{T}}_n^{\rightarrow}$) are Pspace-complete.*

Proof. The reductions in all cases are exactly the same and so we will prove them as one, referring to $\mathcal{M}_n$ instead of $\overline{\mathcal{T}}_n$ or $\overline{\mathcal{T}}_n^{\rightarrow}$ specifically. The reduction is from *Quantified-1-in-3-Sat* and again owes something in philosophy to the proof of Proposition 5.1 of [7], though they use a reduction from QCSP($\mathcal{A}$), see the proof of Proposition 6.1. Let R be the ternary Boolean operation which is true iff exactly one of its entries is. A literal is a variable or its negation and a clause is R applied to three literals. An instance of

Quantified-1-in-3-Sat (1/3-Q-SAT) is a sentence in prenex form whose matrix is a conjunction of clauses. 1/3-Q-SAT is known to be Pspace-complete after, for example, [31] (see [29]).

Let φ be a formula in prenex form whose matrix is a conjunction of clauses. We construct the corresponding positive Horn formula ψ_φ in the language of graphs so that φ is a sentence iff ψ_φ is a sentence. For any evaluation τ of the propositional variables of φ we define the corresponding evaluation τ' of variables of ψ_φ into $\mathcal{M}_n$. We prove that φ is true in τ iff ψ_φ is true in τ'. The case when φ is a sentence is the desired reduction.

First we define ψ_φ when φ is quantifier-free. For each variable of φ we introduce a variable gadget and for each occurrence of R in φ we introduce a clause gadget. These are depicted in Figure 9, and the third graph corresponds to the clause $R(\neg s_1, s_2, s_3)$.

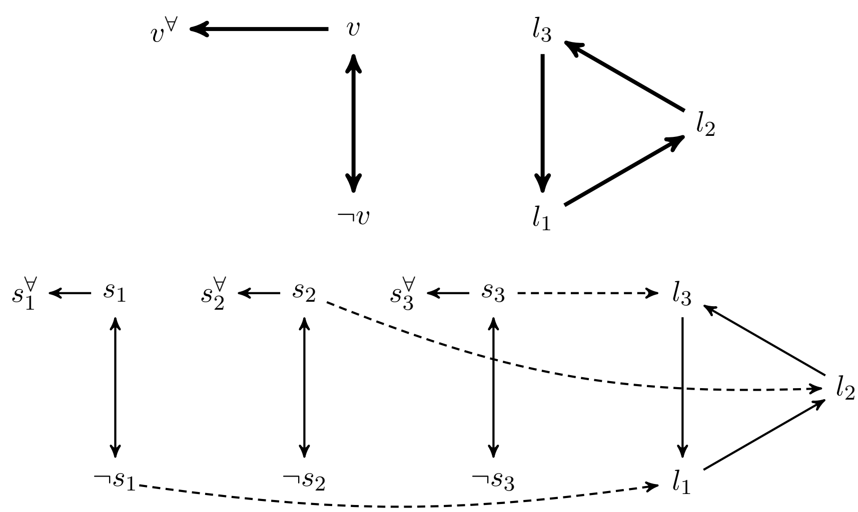

Figure 9: Variable gadget, clause gadget; and their marriage together

Two vertices in the variable gadget correspond to literals of φ with the same names (call them literal vertices), the third vertex (universal vertex) having a special purpose to be explained later. Note that we will use $(\exists \neg s)$ in ψ_φ, which should not create confusion since ψ_φ uses only $\wedge$ of logical connectives. The dashed edges should not be seen as different from the solid edges, they are merely drawn differently to emphasise that they connect the respective gadgets. In particular, the dashed edges are of length 1.

Given a clause C of φ, we draw a directed edge from each of the three literal vertices corresponding to literals of C into a distinct vertex in the clause gadget corresponding to C (each vertex in any clause gadget receives exactly one edge from literal vertices). Now we quantify existentially all variables in the clause gadgets and ψ_φ is defined.

It is clear that the literal vertices in each variable gadget must evaluate to the unique double edge in $\mathcal{M}_n$. The two evaluations it can take will correspond to the literal being evaluated to false (1) or true (n). Thus, if $\tau(s) = \top$, we assign $\tau'(s) = n$, $\tau'(\neg s) = \tau'(s^\forall) = 1$, while if $\tau(s) = \bot$, we assign $\tau'(s) = 1$, $\tau'(\neg s) = \tau'(s^\forall) = n$. A clause is true in τ iff precisely one of the literals is true iff precisely one of the corresponding literal vertices is evaluated as n and the other two as 1 in τ'. Now note that any 3-cycle in $\mathcal{M}_n$ contains no t, and has to contain the edge $n \to 1$. Thus, any 3-cycle in $\mathcal{M}_n$ is of the form $1 \to k \to n \to 1$, for some $1 < k < n$. Since $1^+ = M_n \setminus \{1\}$, and $n^+ \setminus \{t\} = \{1\}$, there exists a way to evaluate correctly the three clause vertices iff one of the corresponding three literal vertices has τ'-value n, and the other two 1. This proves that $v_\tau(\varphi) = v_{\tau'}(\psi_\varphi)$ when φ is quantifier-free.

As in the proof of Proposition 6.1, we proceed by an induction on the number of quantifiers in φ. For the remainder of the proof we fix an evaluation τ of the variables of φ, a variable s, the evaluation τ_1 which differs from τ only at s and we assume that $\varphi = (Qs)\varphi'$, where Q is one of the quantifiers. We insert a table which should help the reader follow the proof below.

	$\varphi = (\exists s)\varphi'$, $(\Leftarrow)$			$\varphi = (\forall s)\varphi'$, $(\Rightarrow)$			$\varphi = (\forall s)\varphi'$, $(\Leftarrow)$		
	(s)	$(\neg s)$	$(s^\forall)$	(s)	$(\neg s)$	$(s^\forall)$	(s)	$(\neg s)$	$(s^\forall)$
τ'	1 or n	$n+1-\tau'(s)$	$\tau'(\neg s)$	same as $\varphi = (\exists s)\varphi'$					
τ'_1	$\tau'(\neg s)$	$\tau'(s)$	$\tau'(s)$	same as $\varphi = (\exists s)\varphi'$					
ρ	1 or n	$n+1-\rho(s)$	?	$\tau'(s)$	$\tau'(\neg s)$	?	$\tau'(s)$	$\tau'(\neg s)$	1
ρ'	$\rho(s)$	$\rho(\neg s)$	$\rho(\neg s)$	1	n	$\rho'(s^\forall)$	$\tau'(s)$	$\tau'(\neg s)$	n
σ	does not apply						n	1	1
σ'	does not apply						1	n	n

If $\varphi = (\exists s)\varphi'$, then we define $\psi_\varphi = (\exists s^\forall)(\exists s)(\exists \neg s)\psi_{\varphi'}$. Assume that $v_\tau(\varphi) = \top$. Thus $v_\tau(\varphi') = \top$ or $v_{\tau_1}(\varphi') = \top$. By the inductive assumption, $v_{\tau'}(\psi_{\varphi'}) = \top$ or $v_{\tau'_1}(\psi_{\varphi'}) = \top$. In the first case, there is nothing to prove, while if $v_{\tau'_1}(\psi_{\varphi'})$, we only need note that τ'_1 and τ' differ exactly at s, $\neg s$ and $s^\forall$. Thus $v_{\tau'}(\psi_\varphi) = v_{\tau'}((\exists s^\forall)(\exists s)(\exists \neg s)\psi_{\varphi'}) = \top$. Now assume that $v_{\tau'}(\psi_\varphi) = v_{\tau'}((\exists s^\forall)(\exists s)(\exists \neg s)\psi_{\varphi'}) = \top$. Let ρ be the evaluation of variables of ψ_φ such that $v_\rho(\psi_{\varphi'}) = \top$ and ρ equals τ' at all variables except possibly $\{s, \neg s, s^\forall\}$. Since $s \leftrightarrow \neg s$, we still know that $\{\rho(s), \rho(\neg s)\} = \{1, n\}$, so

$\rho = \tau'$ or $\rho = \tau_1'$ for all variables except possibly $s^\forall$. But the only relation $s^\forall$ has in the graph G_{ψ_φ} is $s \to s^\forall$. Since we also know that $\rho(s) \to \rho(\neg s)$, we may as well change $\rho(s^\forall)$ to $\rho(\neg s)$, and the new evaluation ρ' will still satisfy $v_{\rho'}(\psi_{\varphi'})$. But, since $\tau'(s^\forall) = \tau'(\neg s)$ and $\tau_1'(s^\forall) = \tau_1'(\neg s)$, we know $\rho' = \tau'$ or $\rho' = \tau_1'$. We obtain $v_{\tau'}(\psi_{\varphi'}) = \top$ or $v_{\tau_1'}(\psi_{\varphi'}) = \top$. By the inductive assumption, $v_\tau(\varphi') = \top$ or $v_{\tau_1}(\varphi') = \top$, which is tantamount to saying $v_\tau(\varphi) = v_\tau((\exists s)\varphi') = \top$.

If $\varphi = (\forall s)\varphi'$, then we define $\psi_\varphi = (\forall s^\forall)(\exists s)(\exists \neg s)\psi_{\varphi'}$. If $v_\tau(\varphi) = \top$, then $v_\tau(\varphi') = \top$, and $v_{\tau_1}(\varphi') = \top$. By the inductive assumption, $v_{\tau'}(\psi_{\varphi'}) = v_{\tau_1'}(\psi_{\varphi'}) = \top$. Choose any evaluation ρ of the variables of ψ_φ which equals τ' everywhere except possibly at $s^\forall$. Assume first that $\rho(s^\forall) = 1$, then $\rho = \tau'$ if $\tau'(s) = n$, or otherwise $\rho = \tau_1'$ for all variables except s and $\neg s$. Since $v_{\tau'}(\psi_{\varphi'}) = v_{\tau_1'}(\psi_{\varphi'}) = \top$, then in either case we obtain that $v_\rho((\exists s)(\exists \neg s)\psi_{\varphi'}) = \top$. Now assume that $\rho(s^\forall) \neq 1$. Let ρ' be the evaluation of the variables of $\psi_{\varphi'}$ such that $\rho'(s) = 1$, $\rho'(\neg s) = n$ and otherwise $\rho' = \rho$. We see that, except at $s^\forall$, either $\rho' = \tau'$, or $\rho' = \tau_1'$ everywhere else. From $1^+ = M_n \setminus \{1\}$ and since the only relation $s^\forall$ has in the graph G_{ψ_φ} is $s \to s^\forall$, follows that if ρ' were changed to $\rho'(s^\forall) = n$, the truth value $v_{\rho'}(\psi_{\varphi'})$ would stay unchanged. But since $v_{\tau'}(\psi_{\varphi'}) = v_{\tau_1'}(\psi_{\varphi'}) = \top$, it follows that $v_{\rho'}(\psi_{\varphi'}) = \top$. Since ρ and ρ' are equal on $M_n \setminus \{s, \neg s\}$, thus $v_\rho((\exists s)(\exists \neg s)\psi_{\varphi'}) = \top$. So for all evaluations ρ which equal τ' on $M_n \setminus \{s^\forall\}$ we have $v_\rho((\exists s)(\exists \neg s)\psi_{\varphi'}) = \top$, so $v_{\tau'}(\psi_\varphi) = v_{\tau'}((\forall s^\forall)(\exists s)(\exists \neg s)\psi_{\varphi'}) = \top$.

Now assume $v_{\tau'}(\psi_\varphi) = \top$. Let the evaluations ρ, ρ' satisfy $\rho(s^\forall) = 1$, $\rho'(s^\forall) = n$, and $\rho = \rho' = \tau'$ on all other variables of $\psi\varphi$. From $v_{\tau'}(\psi_\varphi) = v_{\tau'}((\forall s^\forall)(\exists s)(\exists \neg s)\psi_{\varphi'}) = \top$ follows that $v_\rho((\exists s)(\exists \neg s)\psi_{\varphi'}) = v_{\rho'}((\exists s)(\exists \neg s)\psi_{\varphi'}) = \top$. Therefore, there exist evaluations σ, σ' such that $\sigma = \rho$ and $\sigma' = \rho'$ on all variables of $\psi\varphi$, except possibly for $\{s, \neg s\}$, and such that $v_\sigma(\psi_{\varphi'}) = v_{\sigma'}(\psi_{\varphi'}) = \top$. Since $s \leftrightarrow \neg s$ in G_{ψ_φ}, it follows that $\{\sigma(s), \sigma(\neg s)\} = \{\sigma'(s), \sigma'(\neg s)\} = \{1, n\}$. Also, from $\sigma(s^\forall) = \rho(s^\forall) = 1$ and $s \to s^\forall$ in G_{ψ_φ}, we obtain that $\sigma(s^\forall) = \sigma(\neg s) = 1$ and $\sigma(s) = n$. Analogously we obtain $\sigma'(s^\forall) = \sigma'(\neg s) = n$ and $\sigma'(s) = 1$. Therefore, $\{\sigma, \sigma'\} = \{\tau', \tau_1'\}$, so $v_\sigma(\psi_{\varphi'}) = v_{\sigma'}(\psi_{\varphi'}) = \top$ implies that $v_{\tau'}(\psi_{\varphi'}) = v_{\tau_1'}(\psi_{\varphi'}) = \top$. We have that $v_\tau(\varphi') = v_{\tau_1}(\varphi') = \top$ by the inductive assumption, and thus $v_\tau(\varphi) = v_\tau((\forall s)\varphi') = \top$. This finishes the inductive proof. $\square$

We fix some notation now. For $\mathcal{H}$ a digraph and $H_1, H_2 \subseteq V(H)$, let $\mathrm{QCSP}_{[\exists/H_1]}(\mathcal{H})$ be as $\mathrm{QCSP}(\mathcal{H})$ except all existential variables are relativised to the set H_1. Also, let $\Phi[\exists/H_1, \forall/H_2]$ be Φ with the existential variables relativised to H_1 and the universal variables relativised to H_2. Finally, for a sets of variables X and Y, let $\Phi[X/H_1, Y/H_2]$ be Φ with all variables in the set X relativised to H_1 and all variables in the set Y relativised to H_2 and so on. We will need the following propositions:

Proposition 6.4. *For any digraph $\mathcal{H}$, QCSP($\mathcal{H}^{\rightarrow}$) and QCSP$_{[\exists/H]}$($\mathcal{H}^{\rightarrow}$) are equivalent modulo polynomial-time reductions.*

Proof. Let us first prove that QCSP($\mathcal{H}^{\rightarrow}$) reduces to QCSP$_{[\exists/H]}$($\mathcal{H}^{\rightarrow}$). Let the sink of $\mathcal{H}^{\rightarrow}$ be t. Take an instance Φ (with its unquantified part φ) of QCSP($H^{\rightarrow}$). Call all those existentially quantified variables in Φ which are sinks in $\mathcal{G}_\varphi$ the set X_Φ, the existentially quantified variables in Φ which are not sinks in $\mathcal{G}_\varphi$ the set Y_Φ, and the universally quantified variables of Φ the set Z_Φ (all those better be sinks in $\mathcal{G}_\varphi$, or Φ is a no-instance immediately).

Now, Φ is equivalent to the instance of Φ' of QCSP$_{[X_\Phi/\{t\},Y_\Phi/H]}$($\mathcal{H}^{\rightarrow}$) which is the same instance as Φ, just with restricted universal and existential quantifiers replacing the usual quantifiers at all variables in X_Φ and in Y_Φ, respectively. This follows from the fact that each of these atomic formulae involving variables in X_Φ are in fact of the form $y_j \rightarrow x_i$ for some $x_i \in X_\Phi$ and $y_j \in Y_\Phi$, and they are all true if x_i is evaluated as t and y_j is evaluated as any element of H. Moreover, since all $y \in Y_\Phi$ are not sinks in $\mathcal{G}_\varphi$, they can't be evaluated as the sink.

Next, the instance Φ' is equivalent to the instance of QCSP$_{[\exists/H]}$($\mathcal{H}^{\rightarrow}$) where we delete all the atomic formulae involving variables in X_Φ and the quantifiers involving those variables from Φ'. These atomics are all true no matter what and the instance's truth or falsity is decided on the merits of the rest of the formula. All remaining existentially quantified variables are the ones in Y_Φ which are relativised to H.

On the other hand, any instance Φ with the unquantified part φ of QCSP$_{[\exists/H]}$($\mathcal{H}^{\rightarrow}$) reduces to QCSP($\mathcal{H}^{\rightarrow}$) by just adding a new variable t quantified existentially outermost, and adding $x \rightarrow t$ to φ for any x which is existentially quantified in Φ. $\square$

Proposition 6.5. *Let $\mathcal{H}$ be a digraph. For each $j > 1$ there exists a polytime reduction from QCSP$_{[\exists/H]}$($\mathcal{H}^{\rightarrow}$) to QCSP($\mathcal{H}^{\rightarrow j}$).*

Proof. Let t_1 be the sink added in $\mathcal{H}^{\rightarrow}$ and let $t_1, \ldots, t_j$ be the sinks iteratively added in ($\mathcal{H}^{\rightarrow j}$) (say in the order that makes t_j the true sink).

Let Φ be a positive Horn sentence, φ its unquantified part and $\mathcal{G}_\varphi$ the graph of φ. It is not hard to see that $\mathcal{H}^{\rightarrow} \models \Phi[\exists/H]$ iff $\mathcal{H}^{\rightarrow j} \models \Phi[\exists/H, \forall/H^{\rightarrow}]$ iff $\mathcal{H}^{\rightarrow j} \models \Phi[\exists/H]$. The second equivalence follows since any evaluation which evaluates a universally quantified variable v to one of t_i may be modified by evaluating v to t_1 without changing correctness

(v is a sink in $\mathcal{G}_\varphi$ connected just to some existentially quantified variables, which are restricted to H). Applying Proposition 6.4 to $\mathcal{H}^{\to(j-1)}$ finishes the proof. □

The following two corollaries follow directly.

Corollary 6.6. *For any $j > 1$ and digraph $\mathcal{H}$, QCSP($\mathcal{H}^{\to}$) reduces to QCSP($\mathcal{H}^{\to j}$).*

Proof. This just combines Propositions 6.4 and 6.5. □

Corollary 6.7. *For each $j > 0$, QCSP($\overline{\mathcal{T}}_n^{\to j}$) and QCSP($\mathcal{K}_{2\to 2}^{\to j}$) are both Pspace-complete.*

Proof. This combines Proposition 6.1, Proposition 6.3 and Corollary 6.6. □

6.2 The algebraic part

We will denote the ith projection function on m variables by p_i^m. We may drop the superscript if we deem it unnecessary.

Lemma 6.8. *Let $\mathcal{G} = (V, \to)$ be a semicomplete graph without sources, but with the sink t. Let $f : V^m \to V$ be any idempotent mapping such that $f \restriction_{V\setminus\{t\}}$ is the first projection. f is a polymorphism of $\mathcal{G}$ iff for all $b_1, b_2, \ldots, b_m \in V$, $b_1 \preceq_{\mathcal{G}} f(b_1, b_2, \ldots, b_m)$.*

Proof. Let f be a polymorphism of $\mathcal{G}$. If $f(b_1, b_2, \ldots, b_m) = b_1$, then there is nothing to prove, thus we may assume that $f(b_1, b_2, \ldots, b_m) = a_1 \neq b_1$. Since $\mathcal{G}$ has no sources, we may select $a_2, \ldots, a_m$ such that $a_i \to b_i$ for all $1 < i \leq m$. In particular, this implies that $a_i \neq t$ for all $1 < i \leq m$. If $c \in b_1^-$, i. e. if $c \to b_1$, then from the assumption that f is a polymorphism and $t \notin \{c, a_2, \ldots, a_m\}$ follows that $c = f(c, a_2, \ldots, a_m) \to f(b_1, b_2, \ldots, b_n) = a_1$, which implies that $c \in a_1^-$. (In particular we proved that $\neg a_1 \to b_1$.) By definition, this means $b_1 \preceq_{\mathcal{G}} a_1 = f(b_1, b_2, \ldots, b_m)$, as desired.

Now assume that $f : V^m \to V$ is an idempotent mapping which satisfies $b_1 \preceq_{\mathcal{G}} f(b_1, b_2, \ldots, b_m)$ for all $b_1, b_2, \ldots, b_m \in V$, and that $f(b_1, b_2, \ldots, b_m) = b_1$ if $t \notin \{b_1, b_2, \ldots, b_m\}$. Let $a_i \to b_i$ in $\mathcal{G}$ for all $1 \leq i \leq m$. Then $t \notin \{a_1, a_2, \ldots, a_m\}$ and, consequently, $f(a_1, a_2, \ldots, a_m) = a_1$. Now we know that $a_1 \in b_1^-$ and from $b_1 \preceq_{\mathcal{G}} f(b_1, b_2, \ldots, b_m)$ follows that $a_1 \in f(b_1, b_2, \ldots, b_m)^-$, i. e. $f(a_1, a_2, \ldots, a_m) = a_1 \to f(b_1, b_2, \ldots, b_m)$. Therefore, f is a polymorphism of $\mathcal{G}$. □

Definition 6.9. Let $\mathcal{G} = (V, E)$ be a digraph. We define the partition of the vertex set V into $V^{\mathcal{G}}_{min}$, $V^{\mathcal{G}}_{max}$, $V^{\mathcal{G}}_{both}$ and $V^{\mathcal{G}}_{none}$ so that all vertices in $V^{\mathcal{G}}_{min}$ are minimal, but not maximal, in the order $\preceq_{\mathcal{G}}$, all vertices in $V^{\mathcal{G}}_{max}$ are maximal, but not minimal, in the order $\preceq_{\mathcal{G}}$, all vertices in $V^{\mathcal{G}}_{both}$ are both minimal and maximal in the order $\preceq_{\mathcal{G}}$, while vertices in $V^{\mathcal{G}}_{none}$ are neither minimal nor maximal in the order $\preceq_{\mathcal{G}}$. When the digraph $\mathcal{G}$ is understood, we will omit the superscript $^{\mathcal{G}}$.

Definition 6.10. Given a digraph $\mathcal{G} = (V, E)$, $\mathcal{S}(\mathcal{G}) = (V, \to)$ is a digraph given by:

1. For all $x, y \in V_{max} \cup V_{both}$, $x \leftrightarrow y$,
2. For all $x, y \in V_{min}$, $x \leftrightarrow y$,
3. For all $x, y \in V_{none}$, $x \to y$ iff $E(x, y)$.
4. For all $x \in V_{min}$ and $y \in V_{none} \cup V_{max}$, $x \to y$, but $\neg y \to x$,
5. For all $x \in V_{none}$ and $y \in V_{max}$, $x \to y$, but $\neg y \to x$,
6. For all $x \in V_{both}$ and $y \in V_{none} \cup V_{min}$, $x \to y$, but $\neg y \to x$.

$\mathcal{S}(\mathcal{G})$ is depicted in Figure 10.

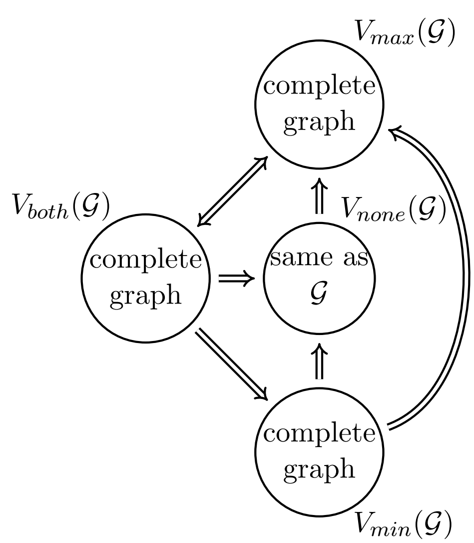

Figure 10: An illustration of $\mathcal{S}(\mathcal{G})$

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Proposition 6.11. $V_{min}^{\mathcal{S}(\mathcal{G})} = V_{min}^{\mathcal{G}}$, $V_{max}^{\mathcal{S}(\mathcal{G})} = V_{max}^{\mathcal{G}}$, $V_{both}^{\mathcal{S}(\mathcal{G})} = V_{both}^{\mathcal{G}}$ *and* $V_{none}^{\mathcal{S}(\mathcal{G})} = V_{none}^{\mathcal{G}}$. *Consequently,* $\mathcal{S}(\mathcal{S}(\mathcal{G})) = \mathcal{S}(\mathcal{G})$.

Proof. Using Definition 6.10, we compute the sets x^- with respect to $\mathcal{S}(\mathcal{G})$ for x in $V_{min}^{\mathcal{G}}$, $V_{max}^{\mathcal{G}}$, $V_{both}^{\mathcal{G}}$ and $V_{none}^{\mathcal{G}}$.

- If $x \in V_{min}^{\mathcal{G}}$, then $x^- = (V_{both}^{\mathcal{G}} \cup V_{min}^{\mathcal{G}}) \setminus \{x\}$.
- If $x \in V_{none}^{\mathcal{G}}$, then $V_{both}^{\mathcal{G}} \cup V_{min}^{\mathcal{G}} \subseteq x^- \subseteq (V_{both}^{\mathcal{G}} \cup V_{min}^{\mathcal{G}} \cup V_{none}^{\mathcal{G}}) \setminus \{x\}$.
- If $x \in V_{max}^{\mathcal{G}}$, then $x^- = V \setminus \{x\}$.
- If $x \in V_{both}$, then $x^- = (V_{both}^{\mathcal{G}} \cup V_{max}^{\mathcal{G}}) \setminus \{x\}$.

Now the statements follow by Definition 6.9. $\square$

We prove the following trivial proposition for the sake of completeness.

Proposition 6.12. *A permutation α of the vertex set V of the digraph $\mathcal{G} = (V, \to)$ (more generally, universe A of a finite model $\mathcal{A}$) is an automorphism iff it is structure-preserving.*

Proof. We need to prove that α^{-1} is also structure preserving. The permutation α applied pointwise induces a permutation $\overline{\alpha}$ of the set V^2 (resp. A^k) which maps injectively the relation $\to$ (resp. each relation R of $\mathcal{A}$) into itself. Thus the restriction of $\overline{\alpha}$ to the set of pairs $\to$ (resp. set of k-tuples R) is a permutation since V is finite, hence α^{-1} must also be structure-preserving. $\square$

Lemma 6.13. *The following statements hold for any digraph $\mathcal{G}$:*

(i) $Aut(\mathcal{G}) \subset Aut(V, \preceq_{\mathcal{G}})$,

(ii) $Aut(\mathcal{G}) \subseteq Aut(\mathcal{S}(\mathcal{G}))$,

(iii) $\preceq_{\mathcal{G}} \subseteq \preceq_{\mathcal{S}(\mathcal{G})}$,

(iv) If $\mathcal{G}$ is semicomplete, then so is $\mathcal{S}(\mathcal{G})$,

(v) If $\mathcal{G}$ is smooth and semicomplete, then so is $\mathcal{S}(\mathcal{G})$ and

(vi) If $\mathcal{G}$ is semicomplete and is not a cycle, then $\mathcal{S}(\mathcal{G})$ is also.

Proof. (i) Let α be an automorphism of $\mathcal{G}$ and let $x \preceq_{\mathcal{G}} y$. This implies $x^- \subseteq y^-$ in $\mathcal{G}$, and since α is an automorphism of $\mathcal{G}$, we get that $\alpha(x)^- = \{\alpha(z) : E(z,x)\} \subseteq \{\alpha(z) : E(z,y)\} = \alpha(y)^-$, so $\alpha(x) \preceq_{\mathcal{G}} \alpha(y)$. According to Proposition 6.12, α is an automorphism of the poset $(V, \preceq_{\mathcal{G}})$. In particular, α restricts to each of the sets V_{min}, V_{max}, V_{both} and V_{none} as a permutation which we will use presently (there is no need to specify the superscript by Proposition 6.11).

(ii) Let α be an automorphism of $\mathcal{G}$ and $x \to y$ in $\mathcal{S}(\mathcal{G})$. If x and y are not both in the same class of the partition $\{V_{min}, V_{max}, V_{both}, V_{none}\}$ of V, then according to Proposition 6.11, the previous paragraph and Definition 6.10, $\alpha(x) \to \alpha(y)$, since all edges between vertices in different classes of that partition are drawn the same way. Similarly, if x and y are both in one of the sets V_{min}, V_{max} and V_{both}, then by the previous paragraph, $\alpha(x)$ and $\alpha(y)$ are also in that set, and the fact that the subgraph induced by $\mathcal{S}(\mathcal{G})$ on each of these sets is the complete graph, while α is bijective, proves $\alpha(x) \to \alpha(y)$. Finally if $x, y \in V_{none}$, since the subgraphs induced on V_{none} by $\mathcal{S}(\mathcal{G})$ and $\mathcal{G}$ are the same graphs, the fact that α is an automorphism of $\mathcal{G}$ implies that $\alpha(x) \to \alpha(y)$. Now Proposition 6.12 proves (ii).

(iii) We assume that $x \preceq_{\mathcal{G}} y$ and we may as well assume that $x \neq y$. This implies that $x \in V_{min} \cup V_{none}$ and $y \in V_{none} \cup V_{max}$. For the rest of this proof, by x^- we will always mean the set of in-neighbours of x with respect to $\mathcal{S}(\mathcal{G})$, rather than $\mathcal{G}$. If $y \in V_{max}$, then $y^- = V \setminus \{y\}$, while $y \notin x^-$ since $x \notin V_{max} \cup V_{none}$, so $x^- \subseteq y^-$. If $x \in V_{min}$ and $y \in V_{none}$, then $x^- = V_{both} \cup V_{min} \setminus \{x\}$, while $y^- \supseteq V_{both} \cup V_{min}$, so again $x^- \subseteq y^-$. Finally, if $x, y \in V_{none}$, then $x^- = V_{both} \cup V_{min} \cup (x^- \cap V_{none}) \subseteq V_{both} \cup V_{min} \cup (y^- \cap V_{none}) = y^-$, where the $\subseteq$ in the middle holds from $x \preceq_{\mathcal{G}} y$ and the fact that on V_{none} both $\mathcal{G}$ and $\mathcal{S}(\mathcal{G})$ restrict the same way.

(iv) If $\mathcal{G}$ is semicomplete, then so is the subgraph induced by $\mathcal{G}$ on V_{none}. Moreover, $\mathcal{S}(\mathcal{G})$ is semicomplete iff the subgraph induced by $\mathcal{S}(\mathcal{G})$ on V_{none} is semicomplete. Thus, the semicompleteness of $\mathcal{S}(\mathcal{G})$ follows from the semicompleteness of $\mathcal{G}$ and Definition 6.10 (3).

(v) We may assume that both $\mathcal{G}$ and $\mathcal{S}(\mathcal{G})$ are semicomplete by (iv). $\mathcal{S}(\mathcal{G})$ has a source iff $V_{both} = \emptyset$ and $|V_{min}| = 1$ iff $\mathcal{G}$ has a source, and analogously for the sinks. (See Proposition 6.11.)

(vi) We may assume that both $\mathcal{G}$ and $\mathcal{S}(\mathcal{G})$ are semicomplete by (iv). By contraposition, if $\mathcal{S}(\mathcal{G})$ is a cycle, then $V = V_{both}^{\mathcal{S}(\mathcal{G})} = V_{both}^{\mathcal{G}}$ (the last equality follows from Proposition 6.11), and so $\mathcal{S}(\mathcal{G})$ is the complete graph. But then $|V| = 2$, otherwise $\mathcal{S}(\mathcal{G})$ could not be at the same time a cycle and a complete graph. Thus $\mathcal{G}$ is a 2-element semicomplete digraph with

$V = V^{\mathcal{G}}_{both}$, so $\mathcal{G}$ is a 2-cycle. $\square$

Corollary 6.14. *Let $\mathcal{G} = (V, E)$ be a smooth semicomplete digraph which is not a cycle. Then $Pol(\mathcal{G}^{\rightarrow}) \subseteq Pol(\mathcal{S}(\mathcal{G})^{\rightarrow})$.*

Proof. Let us assume that $f \in Pol(\mathcal{G}^{\rightarrow})$. Define $g \in Aut(\mathcal{G}^{\rightarrow})$ by $g(x) = f(x, x, \dots, x)$ and $h \in Pol_{id}(\mathcal{G}^{\rightarrow})$ by $h = g^{-1} \circ f$. Now, according to Theorem 5.8, h restricts to V as some projection. Without loss of generality, assume that $h \restriction_V = p_1$. Now, from Lemma 6.13 (v) and (vi) we know that $\mathcal{S}(\mathcal{G})$ is also a semicomplete smooth digraph which is not a cycle. According to Lemma 6.8, for all $b_1, b_2, \dots, b_m \in V \cup \{t\}$, $b_1 \preceq_{\mathcal{G}^{\rightarrow}} f(b_1, b_2, \dots, b_m)$. Since for any digraph $\mathcal{H}$, $\preceq_{\mathcal{H}^{\rightarrow}} = \preceq_{\mathcal{H}} \cup (V(\mathcal{H}^{\rightarrow}) \times \{t\})$ and Lemma 6.13 (iii) guarantees that $\preceq_{\mathcal{G}} \subseteq \preceq_{\mathcal{S}(\mathcal{G})}$, thus for all $b_1, b_2, \dots, b_m \in V \cup \{t\}$, $b_1 \preceq_{\mathcal{S}(\mathcal{G})^{\rightarrow}} f(b_1, b_2, \dots, b_m)$. Again, by Lemma 6.8, $h \in Pol_{id}(\mathcal{S}(\mathcal{G})^{\rightarrow})$. Moreover, $g(t) = t$, and from Lemma 6.13 (ii) we know that the restriction of g to V is in $Aut(\mathcal{S}(\mathcal{G}))$. Therefore, $g \in Aut(\mathcal{S}(\mathcal{G})^{\rightarrow})$, and we get that $f = g \circ h \in Pol(\mathcal{S}(\mathcal{G}))$. $\square$

Definition 6.15. Let $\mathcal{G} = (V, E)$ be a digraph. We define the digraph $\mathcal{L}(\mathcal{G})$ on the set V in the following way:

1. For all $x \in V_{both} \cup V_{min}$ and $y \in V_{none} \cup V_{max}$, $x \rightarrow y$, but $\neg y \rightarrow x$,
2. For all $x \in V_{none}$ and $y \in V_{max}$, $x \rightarrow y$, but $\neg y \rightarrow x$,
3. For all $x, y \in V_{min} \cup V_{both}$, $x \leftrightarrow y$,
4. For all $x, y \in V_{none}$, $x \rightarrow y$ iff $E(x, y)$,
5. For all $x, y \in V_{max}$, $x \leftrightarrow y$.

$\mathcal{L}(\mathcal{G})$ is depicted in Figure 11.

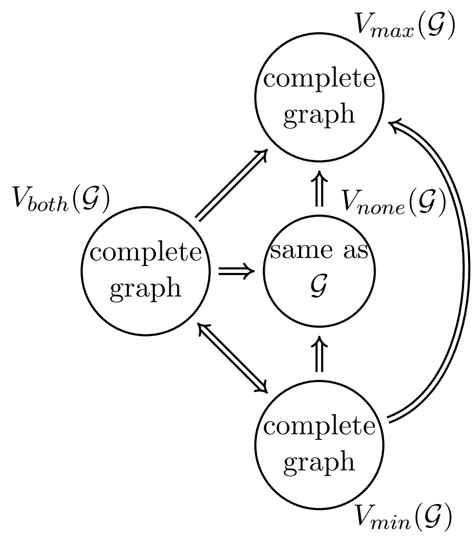

Figure 11: An illustration of $\mathcal{L}(\mathcal{G})$

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Lemma 6.16. *Let $\mathcal{G}$ be a digraph. Either $V = V^{\mathcal{G}}_{both} = V^{\mathcal{L}(\mathcal{G})}_{both}$, or $V^{\mathcal{L}(\mathcal{G})}_{min} = V^{\mathcal{G}}_{both} \cup V^{\mathcal{G}}_{min}$, $V^{\mathcal{L}(\mathcal{G})}_{none} = V^{\mathcal{G}}_{none}$, $V^{\mathcal{L}(\mathcal{G})}_{max} = V^{\mathcal{G}}_{max}$ and $V^{\mathcal{L}(\mathcal{G})}_{both} = \emptyset$.*

Proof. The Lemma follows directly from Definition 6.15. □

Corollary 6.17. *Let $\mathcal{G} = (V, E)$ be a smooth semicomplete digraph which is not a cycle. Then $Pol(\mathcal{S}(\mathcal{G})^{\rightarrow}) \subseteq Pol(\mathcal{L}(\mathcal{G})^{\rightarrow})$.*

Proof. Let us denote the sink of $\mathcal{G}^{\rightarrow}$ by t and assume that $f \in Pol(\mathcal{S}(\mathcal{G})^{\rightarrow})$. Define $g \in Aut(\mathcal{S}(\mathcal{G})^{\rightarrow})$ by $g(x) = f(x, x, \ldots, x)$ and $h \in Pol_{id}(\mathcal{S}(\mathcal{G})^{\rightarrow})$ by $h = g^{-1} \circ f$. By Lemma 6.13 (v) and (vi), our conditions imply that $\mathcal{S}(\mathcal{G})$ is a smooth semicomplete digraph which is not a cycle. According to Theorem 5.8, h restricts to V as some projection. Without loss of generality, assume that $h \restriction_V = p_1$.

Let us prove that $\mathcal{L}(\mathcal{G})$ is also a smooth semicomplete digraph which is not a cycle. From Definition 6.15 follows that $\mathcal{L}(\mathcal{G})$ is semicomplete iff the induced subgraph by $\mathcal{L}(\mathcal{G})$ on $V^{\mathcal{G}}_{none}$ is semicomplete, which is true since it is equal to the induced subgraph by $\mathcal{G}$ on $V^{\mathcal{G}}_{none}$. It has no source since $|V^{\mathcal{L}(\mathcal{G})}_{max}| = |V^{\mathcal{G}}_{max}| \neq 1$ (in both cases of Lemma 6.16) and no sink since either $|V^{\mathcal{L}(\mathcal{G})}_{min}| = |V^{\mathcal{G}}_{min} \cup V^{\mathcal{G}}_{both}| > 1$, or $V = V^{\mathcal{L}(\mathcal{G})}_{both}$ and so $|V^{\mathcal{L}(\mathcal{G})}_{min}| = 0$. Finally, if $\mathcal{L}(\mathcal{G})$ were a cycle, then $V = V^{\mathcal{L}(\mathcal{G})}_{both}$ which would imply that $V = V^{\mathcal{S}(\mathcal{G})}_{both}$ and so that $\mathcal{S}(\mathcal{G}) = \mathcal{L}(\mathcal{G})$, and we know that $\mathcal{S}(\mathcal{G})$ is not a cycle.

From Definition 6.15 follows that the subgraphs induced by $\mathcal{S}(\mathcal{G})$ and $\mathcal{L}(\mathcal{G})$ on the set $V \setminus V^{\mathcal{G}}_{both} = V^{\mathcal{S}(\mathcal{G})}_{min} \cup V^{\mathcal{S}(\mathcal{G})}_{none} \cup V^{\mathcal{S}(\mathcal{G})}_{max}$ are the same. This implies that $\preceq_{\mathcal{S}(\mathcal{G})} \subseteq \preceq_{\mathcal{L}(\mathcal{G})}$, and according to Lemma 6.8, we get that $h \in Pol_{id}(\mathcal{L}(\mathcal{G})^{\rightarrow})$.

Moreover, from $g \in Aut(\mathcal{S}(\mathcal{G})^{\rightarrow})$ follows that $g \in Aut(V \cup \{t\}, \preceq_{\mathcal{S}(\mathcal{G})^{\rightarrow}})$ by Lemma 6.13 (*ii*). Thus, Definition 6.10 implies that g acts independently on $\{t\}$, $V^{\mathcal{G}}_{both} \cup V^{\mathcal{G}}_{min}$, $V^{\mathcal{G}}_{none}$ and $V^{\mathcal{G}}_{max}$ (each is a union of g-orbits). By Definition 6.15, the subgraphs induced by $\mathcal{L}(\mathcal{G})$ on $V^{\mathcal{G}}_{both} \cup V^{\mathcal{G}}_{min}$ and on $V^{\mathcal{G}}_{max}$ are complete, so any permutation is an automorphism. Since $\mathcal{L}(\mathcal{G})$, $\mathcal{S}(\mathcal{G})$ and $\mathcal{G}$ induce on $V^{\mathcal{G}}_{none}$ the same digraph, we get that $g \in Aut(\mathcal{L}(\mathcal{G})^{\rightarrow})$.

Finally, we get that $f = g \circ h \in Pol(\mathcal{L}(\mathcal{G})^{\rightarrow})$. $\square$

Theorem 6.18. *Let $\mathcal{G} = (V, E)$ be a smooth semicomplete digraph which is not a cycle. Then QCSP($\mathcal{G}^{\rightarrow j}$) is Pspace complete for all $j > 0$.*

Proof. According to Corollary 6.6, it suffices to prove this Theorem for $j = 1$, i. e. for QCSP($\mathcal{G}^{\rightarrow}$). By Corollary 6.14, the fact that all polymorphisms of core digraphs (in particular, semicomplete digraphs) are surjective and Theorem 3.16 of [7], we get that we only need to prove that QCSP($\mathcal{S}(\mathcal{G})^{\rightarrow}$) is Pspace complete. According to Corollary 6.17, it suffices to prove that QCSP($\mathcal{L}(\mathcal{G})^{\rightarrow}$) is Pspace complete.

Now if $|V^{\mathcal{G}}_{both} \cup V^{\mathcal{G}}_{min}| \geq 3$, we can use Corollary 6.2 (*ii*) to prove Pspace-completeness of $\mathcal{L}(\mathcal{G})^{\rightarrow}$. $V^{\mathcal{G}}_{both} = \emptyset$ implies that $|V^{\mathcal{G}}_{max}| \geq 2$ and $|V^{\mathcal{G}}_{min}| \geq 2$, or $\mathcal{G}$ would have a source or a sink. However, if $|V^{\mathcal{G}}_{both} \cup V^{\mathcal{G}}_{min}| = 2 \leq |V^{\mathcal{G}}_{max}|$, then we can use Corollary 6.2 (*i*) on $\mathcal{L}(\mathcal{G})^{\rightarrow}$. Moreover, if $V^{\mathcal{G}}_{max} = \emptyset$ or $V^{\mathcal{G}}_{min} = \emptyset$, this implies that $V^{\mathcal{G}}_{max} = V^{\mathcal{G}}_{min} = V^{\mathcal{G}}_{none} = \emptyset$ and $V = V^{\mathcal{G}}_{both}$. Since $|V^{\mathcal{G}}_{both}| = 2$ would imply that $\mathcal{G}$ is the 2-cycle, it must be that $|V^{\mathcal{G}}_{both}| = n \geq 3$, and therefore $\mathcal{S}(\mathcal{G}) = \mathcal{K}_n$ and QCSP($\mathcal{S}(\mathcal{G})^{\rightarrow}$) is Pspace complete.

So we are down to the case when $|V^{\mathcal{G}}_{both}| = |V^{\mathcal{G}}_{min}| = |V^{\mathcal{G}}_{max}| = 1$. Denote $V^{\mathcal{G}}_{both} = \{b\}$ and $V^{\mathcal{G}}_{min} = \{m\}$. In this case, the subgraphs induced by $\mathcal{S}(\mathcal{G})$ and by $\mathcal{G}$ on the set $V \setminus V_{both}$ are the same, and the only elements of V_{min} and V_{max} are the source and the sink of the induced subgraph on the set $V \setminus V_{both}$, respectively.

Let us denote by $\mathcal{G}'$ the subgraph induced by $\mathcal{G}$ on $V^{\mathcal{G}}_{none}$. We have two subcases: if each strong component of $\mathcal{G}'$ is a one-element strong component (i. e. if $\mathcal{G}'$ is a transitive tournament), then $\mathcal{S}(\mathcal{G})$ is isomorphic to the graph $\overline{\mathcal{T}_n}$ (where $|V| = n$) and we can apply Proposition 6.3 to prove that QCSP($\mathcal{S}(\mathcal{G})^{\rightarrow}$) is Pspace complete.

On the other hand, assume that $\mathcal{G}'$ has a nontrivial strong component. All strong components of $\mathcal{G}'$ are at the same time strong components of $\mathcal{L}(\mathcal{G})$, but $\mathcal{L}(\mathcal{G})$ has two more strong components, the singleton $V^{\mathcal{G}}_{max}$ containing its sink, and $\{b,m\}$. In particular, $\mathcal{L}(\mathcal{G})$ has a nontrivial strong component other than $\{b,m\}$. The strict linear order $\Rightarrow_{\mathcal{L}(\mathcal{G})}$ on the set of strong components of $\mathcal{L}(\mathcal{G})$ has the least element $\{b,m\}$, and let C be the maximal nontrivial strong component of $\mathcal{L}(\mathcal{G})$ in this order. Let $W \subseteq V$ be the union of C and all strong components of $\mathcal{L}(\mathcal{G})$ below it. It follows that the subgraph $\mathcal{H}$ induced by $\mathcal{L}(\mathcal{G})$ on W is smooth since the minimal strong component of $\mathcal{H}$ in the order $\sqsubseteq$ is $\{b,m\}$ while the maximal one is C. Furthermore, $\mathcal{L}(\mathcal{G}) = \mathcal{H}^{\rightarrow j}$ for some $j > 0$. Now $\mathcal{L}(\mathcal{G})^{\rightarrow} = \mathcal{H}^{\rightarrow(j+1)}$. We have $|V^{\mathcal{H}}_{min} \cup V^{\mathcal{H}}_{both}| = 2 \leq |V^{\mathcal{H}}_{max}|$, so QCSP($\mathcal{H}^{\rightarrow}$) is Pspace-complete by the earlier case of this proof. By Corollary 6.6, this implies that QCSP($\mathcal{H}^{\rightarrow(j+1)}$) = QCSP($\mathcal{L}(\mathcal{G})^{\rightarrow}$) is Pspace-complete, and the result follows. □

Now we have proved all cases of our main theorem which we restate here:

Theorem 6.19. *If $\mathcal{H}$ is a semicomplete digraph then either*
– $\mathcal{H}$ contains at most one cycle and QCSP($\mathcal{H}$) is in P, or
– $\mathcal{H}$ contains at least two cycles, a source and a sink and QCSP($\mathcal{H}$) is NP-complete, or
– $\mathcal{H}$ contains at least two cycles, but not both a source and a sink, and QCSP($\mathcal{H}$) is Pspace-complete. □

7 Final remarks

Since the conference version of this paper, some companion results, making use of several of our constructions, have appeared in [12]. These new results on algebraic dichotomies, pertaining to growth rates of generating sets of algebra direct powers, are directly motivated by the complexity-theoretic trichotomy we have derived here. Thus the polymorphism classification we give engenders new classifications, both complexity-theoretic and algebraic. Moreover, a good reference on the importance of reflexive digraphs with only projections among their idempotent polymorphisms is [22] and the references found therein (the property is called *idempotent-trivial* there).

We were not able to find any purely algebraic criterion to replace the ad-hoc arguments in Section 6. For a while, there was a conjecture attributed to H. Chen that Pspace completeness of a template $\mathcal{A}$ was equivalent to the algebra $\mathbf{A}$ of polymorphisms of $\mathcal{A}$ having the exponentially generated powers property (EGP property). D. Zhuk was recently [32] able to prove that all finite algebras have either the polynomially generated powers (PGP) or the EGP property, and that PGP of $\mathbf{A}$ implies that QCSP($\mathcal{A}$) reduces to CSP($\mathcal{A}$).

We thank the referees for their many useful remarks. Referee 1's comments helped with presentation and made the paper more palatable, also he/she found a mistake. Referees 2 and 3 have delved very deeply indeed into our arguments, finding several mistakes, some of which were quite serious. Referee 2 also found a way to simplify our definitions by moving a part which was originally in Section 6 into Section 2. All in all, the referees' efforts greatly improved this paper, much more so than usually.

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