On Vladimir Božin's equivalent to Frankl's union-closed sets conjecture

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Arbeitstagung Allgemeine Algebra 99,
Siena, February 2020

Frankl's conjecture

It is a simple enough statement:

Conjecture (P. Frankl, 1979)

Let $\mathcal{F}$ be a finite family of finite sets closed under unions. If $\mathcal{F} \neq \{\emptyset\}$, then there exists $a \in \bigcup \mathcal{F}$ which is an element of at least one-half of the sets in $\mathcal{F}$.

Equivalently,

Conjecture

Let $\mathbf{L}$ be a finite lattice. Then there exists a join-irreducible element $a \in L$ such that $|a\uparrow| \leq \frac{1}{2}|L|$.

Probably around 100 research papers and surveys written on it, some by famous people (Poonen, Stanley, Bollobas, ...)

Weights

Poonen [1992] introduced weights on elements. The idea was used for several things:

- Families on a small enough set satisfy the conjecture,
- Families with few enough sets satisfy the conjecture,
- Finding FC families - when an FC family $\mathcal{F}$ is a subfamily of an union-closed family $\mathcal{G}$, then the conjecture is true with one of the elements of $\bigcup \mathcal{F}$ being in $\geq$ half of the sets.

Ultimately, these weights just allow for quick checking of large, but finite sets of cases - useless for the whole problem.

The version we'll work with today and a bit of notation

Definition

Let T be a nonvoid finite set. We say $\mathcal{F} \subseteq P(T)$ is an IC family [on T] if $T, \emptyset \in \mathcal{F}$ and $\mathcal{F}$ is closed under intersection.

For an IC family $\mathcal{F} \subseteq P(T)$, $\mathcal{F}_a := \{X \in \mathcal{F} : a \in X\}$,

$f_{\mathcal{F}}(a) := \frac{|\mathcal{F}_a|}{|\mathcal{F}|}$ - the frequency of a [in $\mathcal{F}$].

Conjecture

Frankl's conjecture says that for every IC family $\mathcal{F}$ on T there exists $a \in T$ such that $f_{\mathcal{F}}(a) \leq \frac{1}{2}$.

Note that each IC family $\mathcal{F}$, together with inclusion, forms the lattice $\mathbf{L}(\mathcal{F})$.

More notation

$[N] := \{1, 2, \ldots, N\}$.

When certain parameters get large, $a \ll b$ means that the ratio $\frac{a}{b}$ tends to infinity, and $a \approx b$ means that the same ratio tends to 1.

For an IC family $\mathcal{A} \subseteq P(T)$,

$\pi_X(\mathcal{A}) := \{X \cap Y : Y \in \mathcal{A}\}$,

Probability measure is $p : \mathcal{A} \to [0, 1]$ so that $\sum_{X \in \mathcal{A}} p(X) = 1$, and

$$P(a \in X) := \sum_{X \in \mathcal{A}_a} p(X);$$

$$E(f(X)) := \sum_{X \in \mathcal{A}} p(X) f(X);$$

$$\mathcal{A}^p := \{X \in \mathcal{A} : p(X) > 0\}.$$

The Equivalence Theorem

Theorem (The Equivalence Theorem, V. Božin, 2004)

Frankl's Conjecture is true iff for every IC family $\mathcal{A} \subseteq P(T)$, and every probability measure $p : \mathcal{A} \to [0,1]$ which satisfies $P(a \in X) \geq 1 - f_{\mathcal{A}}(a)$ for all $a \in T$,

$$\frac{E(\log |\pi_X(\mathcal{A})|)}{\log |\mathcal{A}|} \geq \frac{1}{2}.$$

The main difference compared to old weights: The probability measure assigns weights to sets, and this is more general than each element having its weight.

The proof of ($\Leftarrow$)

Assume that Frankl's Conjecture is false for some IC family $\mathcal{A} \subseteq P(T)$.

Then there exists some $q > \frac{1}{2}$ such that for all $a \in T$, $f_{\mathcal{A}}(a) \geq q$.

Hence for all $a \in T$, $1 - f_{\mathcal{A}}(a) \leq 1 - q$.

Select $p(T) = 1 - q$, and $p(\emptyset) = q$.

Then for all $a \in T$, $P(a \in X) = 1 - q \geq 1 - f_{\mathcal{A}}(a)$ and

$$\frac{E(\log|\pi_X(\mathcal{A})|)}{\log|\mathcal{A}|} = \frac{(1-q)\log|\mathcal{A}| + q\log 1}{\log|\mathcal{A}|} = 1 - q < \frac{1}{2}.$$

The proof of ($\Rightarrow$): Synopsis

We aim toward constructing two IC families of sets, $\mathcal{A}_3$ and $\mathcal{C}$, both on a universe consisting of two types of elements, the a-elements and the c-elements. $\mathcal{A}_3$ and $\mathcal{C}$ will have the following properties:

- $\{X \cap Y : X \in \mathcal{A}_3, Y \in \mathcal{C}\} \subseteq \mathcal{C}$,
- $f_{\mathcal{A}_3}(a\text{-element}) + f_{\mathcal{C}}(a\text{-element}) > 1$,
- $f_{\mathcal{A}_3}(c\text{-element}) - \frac{1}{2} > \frac{1}{2} - f_{\mathcal{C}}(c\text{-element}) > 0$ and
- $|\mathcal{A}_3| \gg |\mathcal{C}|$.

Then we make a *convex combination* of one copy of $\mathcal{A}_3$ and below it many copies of $\mathcal{C}$ so that the number of sets in the copies of $\mathcal{C}$ is almost equal to $|\mathcal{A}_3|$. Thus the frequencies of all elements in the combination are $\approx$ the arithmetic mean of the frequencies in $\mathcal{A}_3$ and $\mathcal{C}$, hence greater than $\frac{1}{2}$.

The construction $\mathcal{A} \otimes I$

Let $\mathcal{A} \subseteq P(T)$ be an IC family and I a nonvoid set. The IC family $\mathcal{A} \otimes I \subseteq P(T \times I)$ is given by:

For any $X \subseteq T \times I$, $X \in \mathcal{A} \otimes I$ if for all $i \in I$, $X \cap (T \times \{i\}) \in \mathcal{A}$.

In other words,

- The universe of $\mathcal{A} \otimes I$ can be seen as a disjoint union of $|I|$ copies of the universes of $\mathcal{A}$,
- and the sets in $\mathcal{A} \otimes I$ are represented by sequences of sets in $\mathcal{A}$ of length $|I|$, where $X_i := \{a \in T : (a, i) \in X\}$.

Properties:

- The lattice $\mathbf{L}(\mathcal{A} \otimes I) \cong (\mathbf{L}(\mathcal{A}))^{|I|}$.
- $f_{\mathcal{A} \otimes I}(a, i) = f_{\mathcal{A}}(a)$ for all $a \in T$ and $i \in I$.
- $(\mathcal{A} \otimes I) \otimes J = \mathcal{A} \otimes (I \times J)$, up to $((a, i), j) \leftrightarrow (a, (i, j))$.

For $X \in \mathcal{A}$, X^I denotes the "constant tuple" in $\mathcal{A} \otimes I$, i.e. $Y = X^I$ iff for all $i \in I$, $Y_i = X$.

The proof of ($\Rightarrow$): On the constants

We will have four natural parameters K_1, K_2, K_3 and K_4.

The first two are related by $K_1 > 2K_2$ and both are chosen simultaneously as large enough that several conditions are fulfilled (the conditions will be specified by-and-by).

The frequencies of c-elements in $\mathcal{A}_3$ and $\mathcal{C}$ depend on these two parameters.

Then the other two parameters are chosen, even larger than the first two, depending on the choice of K_1 and K_2, and such that the ratio $\frac{K_3}{K_4}$ is fixed.

K_3 makes $|\mathcal{A}_3|$ grow but does not affect $|\mathcal{C}|$, while K_4 makes $|\mathcal{C}|$ grow but does not affect $|\mathcal{A}_3|$.

The proof of $(\Rightarrow)$: The initial adjustment

Assume that $\mathcal{A} \subseteq P(T)$ is an IC family, and $p : \mathcal{A} \to [0,1]$ a probability measure so that $P(a \in X) \geq 1 - f_{\mathcal{A}}(a)$ for all $a \in T$, but

$$\frac{E(\log|\pi_X(\mathcal{A})|)}{\log|\mathcal{A}|} < \frac{1}{2}.$$

Due to continuity, we may slightly increase $p(T)$ and decrease all other positive $p(X)$ so that

- for some $t > 0$, $P(a \in X) \geq 1 - f_{\mathcal{A}}(a) + t$ for all $a \in T$,
- $p(X) \in \mathbb{Q}$ for all $X \in \mathcal{A}$ and still
- $\dfrac{E(\log|\pi_X(\mathcal{A})|)}{\log|\mathcal{A}|} = m < \frac{1}{2}$.

The constants $t > 0$ and $0 < m < \frac{1}{2}$ are fixed from now on.

The proof of (⇒): The family $\mathcal{A}_1$

We introduce the parameters $K_1, K_2 \in \mathbb{N}$. Let $K_2 > \frac{1}{\frac{1}{2}-m}$, $K_1 > 2K_2$ and K_1 is a common multiple of all denominators of $p(X)$, $X \in \mathcal{A}$. Moreover, both K_1 and K_2 must be large enough (denoted: $K_1 \gg 1$ and $K_2 \gg 1$) to make two further estimates close enough for our purpose.

Let $I := [K_1]$.

Let $\mathcal{A}_1 := \mathcal{A} \otimes I$.

The proof of ($\Rightarrow$): The probability measure p_1

Fix $Z := \{Z_i : i \in [K_1]\}$, such that for all $i \in [K_1]$, $Z_i \in \mathcal{A}$ and for all $X \in \mathcal{A}$, if $p(X) = \frac{k}{K_1}$, then $\{i \in [K_1] : Z_i = X\}$ is a set of consecutive indices of size k.

Define the probability measure $p_1 : \mathcal{A}_1 \to [0,1]$ by

$$p_1(X) = \begin{cases} \frac{1}{K_1} & \text{if } (\exists k \leq K_1)(\forall i \in I) X \cap (T \times \{i\}) = Z_{i+k}; \\ 0 & \text{otherwise.} \end{cases}$$

(Addition in indices is mod K_1, so $\mathcal{A}_1^{p_1}$ consists of Z and its cyclic permutations, each with the probability $\frac{1}{K_1}$).

We see that $P_1((a,i) \in X) = P(a \in X)$ for all $a \in T$.

Denote $\mathcal{A}_1^{p_1} := \{Y \in \mathcal{A}_1 : p_1(Y) \neq 0\}$. We have

$$|\mathcal{A}_1^{p_1}| = \{Z \text{ and all its cyclic permutations}\} = K_1.$$

(Unless $|\mathcal{A}^p| = 1$, which easily leads to a contradiction.)

The proof of (⇒): The expectation in $\mathcal{A}_1$

For each $Y \in \mathcal{A}_1$ such that $p_1(Y) = \frac{1}{K_1}$ (i.e. for $Y \in \mathcal{A}_1^{p_1}$),

$$|\pi_Y(\mathcal{A}_1)| = \prod_{i=1}^{K_1} \pi_{Y_i}(\mathcal{A}) = \prod_{X \in \mathcal{A}} \pi_X(\mathcal{A})^{p(X)K_1}.$$

$$\frac{E(\log|\pi_Y(\mathcal{A}_1)|)}{\log|\mathcal{A}_1|} = \frac{\sum_{Y \in \mathcal{A}_1^{p_1}} \frac{1}{K_1} \log|\pi_Y(\mathcal{A})|}{\log|\mathcal{A}_1|} =$$

$$\frac{\frac{K_1}{K_1}\log|\prod_{X \in \mathcal{A}} \pi_X(\mathcal{A})^{p(X)K_1}|}{\log|\mathcal{A}_1|} = \frac{K_1 \sum_{X \in \mathcal{A}} p(X)\log|\pi_X(\mathcal{A})|}{K_1 \log|\mathcal{A}|} =$$

$$\frac{E(\log|\pi_X(\mathcal{A}|)}{\log|\mathcal{A}|} = \frac{m\log|\mathcal{A}|}{\log|\mathcal{A}|} = m.$$

To remember: For all $Y \in \mathcal{A}_1^{p_1}$, $|\pi_Y(\mathcal{A}_1)| = |\mathcal{A}_1|^m < \sqrt{|\mathcal{A}_1|}$.

The proof of ($\Rightarrow$): The family $\mathcal{A}_2$

Let $J = [2K_2 - 1]$ and $L = [K_3]$. K_3 is such that $K_3 \gg 2^{K_2}$.

Let $\mathcal{A}_2 \subseteq P(T \times I \times J \times L)$ be given by
$\mathcal{A}_2 := \mathcal{A} \otimes (I \times J \times L) = \mathcal{A}_1 \otimes (J \times L)$.

We know that $f_{\mathcal{A}_2}(a, i, j, l) = f_{\mathcal{A}_1}(a, i) = f_{\mathcal{A}}(a)$ for all $a \in T$.
Define $p_2 : \mathcal{A}_2 \to [0, 1]$ by

$$p_2(X) = \begin{cases} \frac{1}{K_1}, & \text{if } (\exists Y \in \mathcal{A}_1^{p_1})(X = Y^{J \times L}) \\ & (X \text{ is a "constant" tuple in } \mathcal{A}_1 \otimes (J \times L)); \\ 0, & \text{otherwise.} \end{cases}$$

Analogously as with $\mathcal{A}_1$, $P_2((a, i, j, l) \in X) = P(a \in X)$ and for each $Y \in \mathcal{A}_2^{p_2}$, the projection $|\pi_Y(\mathcal{A}_2)| = |\mathcal{A}_2|^m$.

The proof of (⇒): The family $\mathcal{A}_3'$

We make $\mathcal{A}_3' \subseteq P(\{c_1, \ldots, c_{2K_2-1}\} \cup (T \times I \times J \times L))$ in several steps:

First we consider only the sets $Y \in \mathcal{A}_2$ for which there exists $j \in J$ such that $Y \cap (T \times I \times \{j\} \times L) = \emptyset$.

For any $Y \in \mathcal{A}_2$ and $j \in J$ such that $Y \cap (T \times I \times \{j\} \times L) = \emptyset$, put $X = Y \cup \{c_j, c_{j+1}, \ldots, c_{j+K_2-1}\}$ into the family $\mathcal{A}_3^I$. (Addition in indices is modulo $2K_2 - 1$; we add exactly one set into $\mathcal{A}_3^I$ for each suitable choice of the pair Y and j.)

Close $\mathcal{A}_3^I$ under intersections to obtain $\mathcal{A}_3'$. Denote $\mathcal{A}_3^{II} := \mathcal{A}_3' \setminus \mathcal{A}_3^I$.

For any $X \in \mathcal{A}_3^{II}$ we will have $|X \cap \{c_1, \ldots, c_{2K_2-1}\}| < K_2$, while for $X \in \mathcal{A}_3^I$, $|X \cap \{c_1, \ldots, c_{2K_2-1}\}| = K_2$.

The proof of ($\Rightarrow$): The frequencies in family $\mathcal{A}_3'$

$|\mathcal{A}_3^I| = (2K_2 - 1)|\mathcal{A}|^{K_1(2K_2-2)K_3}$.

Bounding $|\mathcal{A}_3^{II}|$ from above:

$$|\mathcal{A}_3^{II}| \leq \sum_{t=2}^{2K_2-1} \binom{2K_2-1}{t} |\mathcal{A}|^{K_1(2K_2-1-t)K_3} \leq$$

$$|\mathcal{A}|^{K_1(2K_2-3)K_3} \sum_{t=2}^{2K_2-1} \binom{2K_2-1}{t} \leq |\mathcal{A}|^{K_1(2K_2-3)K_3} 2^{2K_2-1} =$$

$$\frac{2^{2K_2-1}}{(2K_2-1)|\mathcal{A}|^{K_1K_3}} |\mathcal{A}_3^I| \ll |\mathcal{A}_3^I|.$$

(The last inequality used $K_3 \gg K_2$.)

The frequency $f_{\mathcal{A}_3'}(a,i,j,l) \approx \frac{2K_2-2}{2K_2-1} f_{\mathcal{A}}(a) \approx f_{\mathcal{A}}(a)$ and

$f_{\mathcal{A}_3'}(c_j) \approx \frac{K_2}{2K_2-1}$. Using: $|\mathcal{A}_3^I| \gg |\mathcal{A}_3^{II}|$ and $K_2 \gg 1$.

The proof of ($\Rightarrow$): The family $\mathcal{A}_3$

For $j \in [2K_2 - 1]$ we define new elements

$$C_j := \{c_{i,j,k} : 1 \leq i \leq K_1, 1 \leq k \leq K_4\}.$$

K_4 - a new large constant to be specified later.

$\mathcal{A}_3 \subseteq P(C_1 \cup \cdots \cup C_{2K_2-1} \cup (\mathcal{A} \times I \times J \times L))$ is obtained from $\mathcal{A}'_3$ by replacing each occurrence of c_j with the whole subset C_j.

Still, $|\mathcal{A}_3| = |\mathcal{A}_3^I| + |\mathcal{A}_3^{II}| \approx |\mathcal{A}_3^I| = (2K_2 - 1)|\mathcal{A}|^{K_1(2K_2-2)K_3}$.

The frequencies of $f_{\mathcal{A}_3}(c_{i,j,k}) = f_{\mathcal{A}'_3}(c_j)$ while the frequencies of (a, i, j, l) are unchanged.

The proof of ($\Rightarrow$): The family $\mathcal{C}^I$

Let $\mathcal{A}_2^{p_2} = \{X_\ell : \ell \in [K_1]\}$.

For each $\ell \in [K_1]$, let $\mathcal{D}_\ell$ be the Boolean family

$$\mathcal{D}_\ell := P(\{c_{i,j,k} : i \in [K_1] \setminus \{\ell\}, j \in [2K_2 - 1], k \in [K_4]\}).$$

Define $\mathcal{C}^I := \bigcup_{\ell=1}^{K_1} \{X_\ell \cup Y : Y \in \mathcal{D}_\ell\}$.

We assume $K_4 \gg K_2$ and $2^{K_4} \gg K_1$, but the relationship between K_4 and K_3 TBD later.

The proof of ($\Rightarrow$): The family $\mathcal{C}$ - estimating $|\mathcal{C}^I|$ and $|\mathcal{C}^{II}|$

We define $\mathcal{C}'$ to be the closure of $\mathcal{C}^I$ under intersection, $\mathcal{C}^{II} := \mathcal{C}' \setminus \mathcal{C}^I$, $\mathcal{C} := \mathcal{C}' \cup \{X \cap Y : X \in \mathcal{C}', Y \in \mathcal{A}_3\}$, and $\mathcal{C}^{III} := \mathcal{C} \setminus \mathcal{C}'$. We estimate the sizes $|\mathcal{C}^I|$ and $|\mathcal{C}^{II}|$.

$$|\mathcal{C}^I| = K_1 2^{(K_1-1)(2K_2-1)K_4},$$

$$|\mathcal{C}^{II}| \leq \sum_{t=2}^{K_1} \binom{K_1}{t} 2^{(K_1-t)(2K_2-1)K_4} <$$

$$\sum_{t=2}^{K_1} K_1^t 2^{(2K_2-1)(K_1-t)K_4} < 2^{(K_1-2)(2K_2-1)K_4} \frac{K_1^2}{1 - \frac{K_1}{2^{(2K_2-1)K_4}}} =$$

$$\frac{K_1}{2^{(2K_2-1)K_4}(1 - \frac{K_1}{2^{(2K_2-1)K_4}})} |\mathcal{C}^I|$$

Since $2^{K_4} \gg K_1$, we get $|\mathcal{C}^I| \gg |\mathcal{C}^{II}|$.

The proof of ($\Rightarrow$): The family $\mathcal{C}$ - estimating $|\mathcal{C}^{III}|$

Any set $X \in \mathcal{C}^{III}$ is a disjoint union of $X \cap (C_1 \cup \cdots \cup C_{2K_2-1})$ and $X \cap (T \times I \times J \times L)$.

$$X \cap (C_1 \cup \cdots \cup C_{2K_2-1}) \subseteq \{c_{i,j,k} : i \in [K_1] \setminus \{\ell\}, \\ j_0 \leq j < j_0 + K_2, k \in [K_4]\},$$

where $(\ell, j_0) \in [K_1] \times [2K_2 - 1]$ must be chosen first. So, $X \cap (C_1 \cup \cdots \cup C_{2K_2-1})$ can be chosen in at most

$$K_1(2K_2 - 1)2^{(K_1-1)K_2K_4}$$

ways. $X \cap (T \times I \times J \times L)$ is an element of $\pi_Y(\mathcal{A}_2)$, where $Y \in \mathcal{A}_2^{p_2}$, so $X \cap (T \times I \times J \times L)$ can be chosen in

$$K_1|\mathcal{A}_2|^m = K_1|\mathcal{A}|^{mK_1(2K_2-1)K_3} = K_1 2^{mK_1(2K_2-1)K_3 \log|\mathcal{A}|}$$

ways. Putting it all together we get

$$|\mathcal{C}^{III}| \leq K_1^2(2K_2 - 1)2^{(K_1-1)K_2K_4 + m\log|\mathcal{A}|K_1(2K_2-1)K_3}.$$

The proof of ($\Rightarrow$): Adjusting K_3 and K_4

In order to obtain $|\mathcal{C}^{III}| \ll |\mathcal{C}^{I}| \ll |\mathcal{A}_3|$, it suffices to have the desired inequalities among the exponents. So, we need

$$m \log|\mathcal{A}| K_1(2K_2-1)K_3 < (K_1-1)(K_2-1)K_4 \text{ and}$$
$$(K_1-1)(2K_2-1)K_4 < \log|\mathcal{A}| K_1(2K_2-1)K_3.$$

These inequalities boil down to

$$\frac{K_1-1}{K_1\log|\mathcal{A}|} < \frac{K_3}{K_4} < \frac{(K_1-1)(K_2-1)}{mK_1(2K_2-1)\log|\mathcal{A}|}.$$

For these to be possible, we need

$$\frac{K_1-1}{K_1\log|\mathcal{A}|} < \frac{(K_1-1)(K_2-1)}{mK_1(2K_2-1)\log|\mathcal{A}|}, \text{ equivalently}$$
$$m < \frac{K_2-1}{2K_2-1}, \text{ i.e. } m < \frac{1}{2} - \frac{1}{2(2K_2-1)},$$
$$4K_2 - 2 > \frac{1}{\frac{1}{2}-m}, \text{ which we know from } K_2 > \frac{1}{\frac{1}{2}-m}.$$

The proof of ($\Rightarrow$): The family $\mathcal{C}$ - estimating frequencies

We let K_3 and K_4 simultaneously tend to infinity, their ratio fixed and in the interval $\left(\frac{K_1-1}{K_1 \log|\mathcal{A}|}, \frac{(K_1-1)(K_2-1)}{mK_1(2K_2-1)\log|\mathcal{A}|}\right)$. From $|\mathcal{C}| \approx |\mathcal{C}^I|$ we get

$$f_{\mathcal{C}}(a,i,j,l) \approx f_{\mathcal{C}^I}(a,i,j,l) = \frac{|\{\ell \leq K_1 : (a,i,j,l) \in X_\ell\}|}{K_1} =$$
$$P_2((a,i,j,l) \in X) = P(a \in X) > 1 - f_{\mathcal{A}}(a),$$

and

$$f_{\mathcal{C}}(c_{i,j,k}) \approx f_{\mathcal{C}^I}(c_{i,j,k}) = \frac{K_1 - 1}{2K_1}.$$

The proof of ($\Rightarrow$): Convex combination

We have $|\mathcal{A}_3| \gg |\mathcal{C}|$ and we combine them:
Let MAX be the largest natural number such that $MAX \cdot |\mathcal{C}| < |\mathcal{A}_3|$. We construct the IC family $\mathcal{F}$ of subsets of

$$(T \times I \times J \times L) \cup C_1 \cup C_2 \cup \cdots \cup C_{K_1} \cup \{e_1, e_2, \ldots, e_{MAX}\}$$

as

$$\mathcal{C} \cup \{X \cup \{e_1, \ldots, e_k\} : X \in \mathcal{C}, 1 \leq k < MAX\} \cup$$
$$\{X \cup \{e_1, \ldots, e_{MAX}\} : X \in \mathcal{A}_3\}.$$

The proof of ($\Rightarrow$): The frequencies in $\mathcal{F}$

The frequencies of the new elements are all at least

$$f_{\mathcal{F}}(e_i) \geq s := \frac{|\mathcal{A}_3|}{|\mathcal{A}_3| + MAX|\mathcal{C}|} > \frac{1}{2},$$

while the frequencies of old elements are

$$f_{\mathcal{F}}(x) = sf_{\mathcal{A}_3}(x) + (1-s)f_{\mathcal{C}}(x).$$

Here $s > \frac{1}{2}$ but $s \approx \frac{1}{2}$ as K_3 and K_4 tend to infinity.

Thus

$$f_{\mathcal{F}}(a,i,j,l) \approx sf_{\mathcal{A}}(a) + (1-s)(1 - f_{\mathcal{A}}(a) + t) \approx \frac{1}{2} + (1-s)t > \frac{1}{2}.$$

Here we used $s \approx \frac{1}{2} \approx 1 - s$. Also,

$$f_{\mathcal{F}}(c_{i,j,k}) \approx s\frac{K_2}{2K_2 - 1} + (1-s)\frac{K_1 - 1}{2K_1} > \frac{1}{2}.$$

The last follows from $K_1 > 2K_2$ and $s > 1 - s$. Thus $\mathcal{F}$ contradicts Frankl's conjecture. $\square$

How to (maybe) apply the Equivalence Theorem

There are examples out there of IC families where the frequencies of most elements fail Frankl's Conjecture, but a small number of elements satisfies it.

The idea is to construct a family and select a probability measure such that each set with positive probability contains all elements which satisfy Frankl's conjecture, and the probability of the remaining ones being in a set satisfies the condition of the Equivalence Theorem.

Moreover, the conclusion of the Equivalence Theorem should fail.

We have not been successful in constructing such a family and probability measure yet.

Dedication

This talk is dedicated to the memory of Velibor Tintor and Jarda Ježek.

THANK YOU FOR YOUR ATTENTION!