An equivalent condition to Frankl's Conjecture

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Frankl's conjecture

For a family $\mathcal{F} \subseteq P(T)$, $\mathcal{F}_a := \{X \in \mathcal{F} : a \in X\}$,

$f_{\mathcal{F}}(a) := \frac{|\mathcal{F}_a|}{|\mathcal{F}|}$ - the frequency of a [in $\mathcal{F}$].

Conjecture (P. Frankl, 1979)

Let $\mathcal{F}$ be a finite family of finite sets closed under unions. If $\mathcal{F} \neq \{\emptyset\}$, then there exists $a \in \bigcup \mathcal{F}$ such that $f_{\mathcal{F}}(a) \geq \frac{1}{2}$.

The version we'll work with today

Definition

Let T be a nonvoid finite set. We say $\mathcal{F} \subseteq P(T)$ is an IC family [on T] if $T, \emptyset \in \mathcal{F}$ and $\mathcal{F}$ is closed under intersection.

Conjecture

Frankl's conjecture says that for every IC family $\mathcal{F}$ on T there exists $a \in T$ such that $f_{\mathcal{F}}(a) \leq \frac{1}{2}$.

Notation

Note that each IC family $\mathcal{F}$, together with inclusion, forms the lattice $\mathbf{L}(\mathcal{F})$. $[N] := \{1, 2, \ldots, N\}$.

When certain parameters get large, $a \ll b$ means that the ratio $\frac{a}{b}$ tends to infinity, and $a \approx b$ means that the same ratio tends to 1.

For an IC family $\mathcal{A} \subseteq P(T)$,

$\pi_X(\mathcal{A}) := \{X \cap Y : Y \in \mathcal{A}\}$,

Probability measure is $p : \mathcal{A} \to [0, 1]$ so that $\sum_{X \in \mathcal{A}} p(X) = 1$, and

$$P(a \in X) := \sum_{X \in \mathcal{A}_a} p(X);$$

$$E(f(X)) := \sum_{X \in \mathcal{A}} p(X) f(X);$$

$$\mathcal{A}^p := \{X \in \mathcal{A} : p(X) > 0\}.$$

The Equivalence Theorem

Theorem (The Equivalence Theorem, V. Božin, 2004)

Frankl's Conjecture is true iff for every IC family $\mathcal{A} \subseteq P(T)$, and every probability measure $p : \mathcal{A} \to [0, 1]$ which satisfies $P(a \in X) \geq 1 - f_{\mathcal{A}}(a)$ for all $a \in T$,

$$\frac{E(\log|\pi_X(\mathcal{A})|)}{\log|\mathcal{A}|} \geq \frac{1}{2}. \qquad (1)$$

The main difference compared to weights and threshold functions: The probability measure assigns weights to sets, and this is more general than each element having its weight.

The proof of $(\Leftarrow)$

Assume that Frankl's Conjecture is false for some IC family $\mathcal{A} \subseteq P(T)$.

Then there exists some $q > \frac{1}{2}$ such that for all $a \in T$, $f_{\mathcal{A}}(a) \geq q$.

Hence for all $a \in T$, $1 - f_{\mathcal{A}}(a) \leq 1 - q$.

Select $p(T) = 1 - q$, and $p(\emptyset) = q$.

Then for all $a \in T$, $P(a \in X) = 1 - q \geq 1 - f_{\mathcal{A}}(a)$ and

$$\frac{E(\log|\pi_X(\mathcal{A})|)}{\log|\mathcal{A}|} = \frac{(1-q)\log|\mathcal{A}| + q\log 1}{\log|\mathcal{A}|} = 1 - q < \frac{1}{2}.$$

The proof of (⇒): What we want to construct

Start from a counterexample to the Equivalence Theorem, want a counterexample to Frankl's Conjecture.

We have $f_{\mathcal{A}}(a) + P(a \in X) > 1$ (made strict using continuity).

We want families $\mathcal{A}_3$ and $\mathcal{C}$ on $\{a\text{-elements}\} \cup \{c\text{-elements}\}$ such that

- $\{X \cap Y : X \in \mathcal{A}_3, Y \in \mathcal{C}\} \subseteq \mathcal{C}$,
- $f_{\mathcal{A}_3}(a\text{-element}) \approx f_{\mathcal{A}}(a)$
- $f_{\mathcal{C}}(a\text{-element}) \approx P(a \in X)$, so $f_{\mathcal{A}_3}(a\text{-element}) + f_{\mathcal{C}}(a\text{-element}) > 1$,
- $f_{\mathcal{A}_3}(c\text{-element}) - \frac{1}{2} > \frac{1}{2} - f_{\mathcal{C}}(c\text{-element}) > 0$ and
- $|\mathcal{A}_3| \gg |\mathcal{C}|$.

The proof of (⇒): Convex combination setup

$\mathcal{A}_3$

$$f_{\mathcal{A}_3}(a - \text{element}) \approx f_{\mathcal{A}}(a)$$
$$f_{\mathcal{A}_3}(c - \text{element}) > \tfrac{1}{2}$$

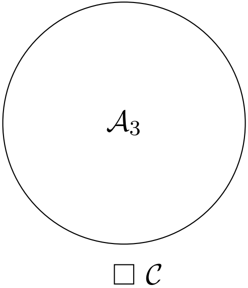
$$|\mathcal{A}_3| \gg |\mathcal{C}|$$

$\boxed{\mathcal{C}}$

$$f_{\mathcal{C}}(a - \text{element}) \approx P(a \in X)$$
$$f_{\mathcal{C}}(c - \text{element}) < \tfrac{1}{2}$$

The proof of (⇒): Convex combination blowup

Letting the parameters get large, we have



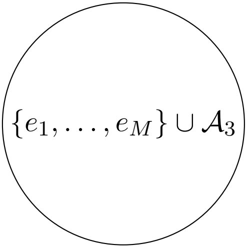
$$f_{\mathcal{A}_3}(a - \text{element}) \approx f_{\mathcal{A}}(a)$$
$$f_{\mathcal{A}_3}(c - \text{element}) > \tfrac{1}{2}$$

$$|\mathcal{A}_3| \gg |\mathcal{C}|$$

$$f_{\mathcal{C}}(a - \text{element}) \approx P(a \in X)$$
$$f_{\mathcal{C}}(c - \text{element}) < \tfrac{1}{2}$$

The proof of ($\Rightarrow$): Convex combination

Let $M|\mathcal{C}| < |\mathcal{A}_3| \leq (M+1)|\mathcal{C}|$. Add new e-elements:



$$f_{\mathcal{A}_3}(a - \text{element}) \approx f_{\mathcal{A}}(a)$$
$$f_{\mathcal{A}_3}(c - \text{element}) > \tfrac{1}{2}$$

$$|\mathcal{A}_3| \gg |\mathcal{C}|$$

- $\square$ $\{e_1, \ldots, e_{M-1}\} \cup \mathcal{C}$
- $\vdots$
- $\square$ $\{e_1\} \cup \mathcal{C}$
- $\square$ $\mathcal{C}$

$$f_{\mathcal{C}}(a - \text{element}) \approx P(a \in X)$$
$$f_{\mathcal{C}}(c - \text{element}) < \tfrac{1}{2}$$

Recall: $\mathcal{C}$ is closed under intersections with sets from $\mathcal{A}_3$

How to blow up IC families: The construction $\mathcal{A} \otimes I$

Let $\mathcal{A} \subseteq P(T)$ be an IC family and I a nonvoid set. The IC family $\mathcal{A} \otimes I \subseteq P(T \times I)$ is given by:

For any $X \subseteq T \times I$, $X \in \mathcal{A} \otimes I$ if for all $i \in I$, $X \cap (T \times \{i\}) \in \mathcal{A} \times \{i\}$.

In other words,

- The universe of $\mathcal{A} \otimes I$ can be seen as a disjoint union of $|I|$ copies of the universe of $\mathcal{A}$,
- and the sets in $\mathcal{A} \otimes I$ are represented by sequences of sets in $\mathcal{A}$ of length $|I|$, where $X_i := \{a \in T : (a, i) \in X\}$.

Properties:

- The lattice $\mathbf{L}(\mathcal{A} \otimes I) \cong (\mathbf{L}(\mathcal{A}))^{|I|}$.
- $f_{\mathcal{A} \otimes I}(a, i) = f_{\mathcal{A}}(a)$ for all $a \in T$ and $i \in I$.
- $(\mathcal{A} \otimes I) \otimes J = \mathcal{A} \otimes (I \times J)$, up to $((a, i), j) \leftrightarrow (a, (i, j))$.

The proof of (⇒): The initial adjustment

Let $\mathcal{A} \subseteq P(T)$ be an IC family, and $p : \mathcal{A} \to [0, 1]$ a probability measure which fail condition (1) of the Equivalence Theorem.

- $P(a \in X) \geq 1 - f_{\mathcal{A}}(a)$ for all $a \in T$, but
- $\dfrac{E(\log|\pi_X(\mathcal{A})|)}{\log|\mathcal{A}|} < \dfrac{1}{2}$.

Due to continuity, we may slightly increase $p(T)$ and decrease all other positive $p(X)$. Thus both inequalities are strict and all probabilities are rational.

Next we apply the construction $\mathcal{A} \otimes I$ and cleverly choose probabilities to obtain an IC family $\mathcal{A}_1 \subseteq T \times I$ so that

- $P((a, i) \in X) > 1 - f_{\mathcal{A}_1}(a, i)$ for all $(a, i) \in T \times I$,
- $\dfrac{E(\log|\pi_X(\mathcal{A}_1)|)}{\log|\mathcal{A}_1|} < \dfrac{1}{2}$ and
- $p(X) \in \{0, \dfrac{1}{K_1}\}$ for all $X \in \mathcal{A}_1$ (where $I = [K_1]$).

The proof of ($\Rightarrow$): a-elements and c-elements

The universe on which $\mathcal{A}_3$ and $\mathcal{C}$ are defined is a disjoint union $A \cup C$.

$A = T \times I \times J \times U$ (a-elements)

$C = \{c_{i,j,k} : i \in I, j \in J, k \in V\}$ (c-elements)

Here $J = [2K_2 - 1]$, $U = [K_3]$ and $V = [K_4]$. Recall that $I = [K_1]$.

The proof of ($\Rightarrow$): On the constants

We have four natural parameters K_1, K_2, K_3 and K_4.

The first two are related by $K_1 > 2K_2$ and both are chosen simultaneously as large enough for several conditions to be fulfilled.

The frequencies of c-elements in $\mathcal{A}_3$ and $\mathcal{C}$, depend on these two parameters.

Then the other two parameters are chosen, even larger than the first two, depending on the choice of K_1 and K_2, and such that the ratio $\frac{K_3}{K_4}$ is fixed.

K_3 makes $|\mathcal{A}_3|$ grow but does not affect $|\mathcal{C}|$, while K_4 makes $|\mathcal{C}|$ grow but does not affect $|\mathcal{A}_3|$.

The proof of (⇒): Blinking family

Start from $\mathcal{F} \subseteq P(C)$ (not closed under intersection)

To each set $S \in \mathcal{F}$ attach an IC family $\mathcal{B}_S \subseteq P(A)$ (the blinking family of S). Properties:

- $|\mathcal{B}_S| = |\mathcal{B}_{S'}|$ for all $S, S' \in \mathcal{F}$, but
- when $S \neq S'$, then $|\{X \cap Y : X \in \mathcal{B}_S, Y \in \mathcal{B}_{S'}\}| \ll |\mathcal{B}_S|$.

Definition of $\mathcal{A}_3$

$\mathcal{A}_3' := \{S \cup X : S \in \mathcal{F}, X \in \mathcal{B}_S\}$ and $\mathcal{A}_3$ is the closure of $\mathcal{A}_3'$ under intersection.

Effect: When we close $\mathcal{A}_3' := \{S \cup X : S \in \mathcal{F}, X \in \mathcal{B}_S\}$ under intersection, the new sets obtained in the intersection are so few, that the frequencies $f_{\mathcal{A}_3'}$ and $f_{\mathcal{A}_3}$ are as close as we want to make them.

$\mathcal{C}$ is constructed using the same basic idea, with a-elements and c-elements switching roles.

The proof of ($\Rightarrow$): Constructing $\mathcal{A}_3$

We start from $\mathcal{F} = \{S_j : 1 \leq j \leq 2K_2 - 1\}$, where

$$S_j := \{c_{i,t,k} : i \in I, j \leq t < j + K_2, k \in V\}$$

(here $j \leq t < j + K_2$ is meant cyclically, so if $K_2 = 3$ and $j = 4$, we get $t \in \{4, 5, 1\}$).
The blinking family associated with S_j is

$$\mathcal{B}_{S_j} := \{(X, i, t, k) \in \mathcal{A} \otimes I \otimes J \otimes U : (t = j \Rightarrow X = \emptyset)\}.$$

First we take $\mathcal{A}_3' := \{S \cup X : S \in \mathcal{F}, X \in \mathcal{B}_S\}$. Then close under intersections to obtain $\mathcal{A}_3$.

The blinking part of families in $\mathcal{A}_3 \setminus \mathcal{A}_3'$ has many more forced empty sets - so $|\mathcal{A}_3 \setminus \mathcal{A}_3'| \ll |\mathcal{A}_3'|$.

The frequencies in $\mathcal{A}_3$ tend to the frequencies in $\mathcal{A}_3'$, which we can compute and control.

The proof of (⇒): The flavor of the construction of $\mathcal{C}$

Much more difficult, but the same main idea.

Now the initial family of sets uses a-elements, while c-elements are used for blinking.

1st issue: Arrange $f_{\mathcal{C}}(a, i, j, k) \approx P(a \in X)$. Using the clever definition of probabilities from the construction of $\mathcal{A}_1$.

2nd issue: we have to make sure that

- there are very few sets obtained as intersections of the generating ones,
- there are very few new sets which are intersections of sets in $\mathcal{A}_3$ and the sets obtained by intersecting generators of $\mathcal{C}$,
- however, there are very few sets in $\mathcal{C}$ compared to $\mathcal{A}_3$.

2nd and 3rd item work against each other. Can both be satisfied when the parameters are large enough and the ratio $\frac{K_3}{K_4}$ is in some interval.

That interval is nonempty provided K_1 and K_2 satisfy some additional condition. This finishes the construction.