QCSP on semicomplete digraphs

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We fix an only relational language when speaking logic here.

A positive Horn formula is a formula in the prenex form (quantifiers are preceded only by quantifiers and their scope is until the end of the formula), such that the unquantified part is just a conjunction of atomic formulas.

The graph of a positive Horn formula $\varphi$ is the relational structure $\mathcal{V}$ on the set of variables $V$ used in the formula, where $R^\mathcal{V}$ is the set of all tuples $\overline{x}$ such that $R(\overline{x})$ occurs as part of the formula $\varphi$. 
Definition

Let $\mathbb{A}$ be a finite relational structure. The decision problem $QCSP(\mathbb{A})$ takes as input any first order sentence in prenex form such that the unquantified part is a conjunction of atoms. The accepted sentences are the ones which hold true in $\mathbb{A}$.

Definition

The problem $CSP(\mathbb{A})$ additionally stipulates that sentences are using only existential quantifiers.

$CSP(\mathbb{A})$ is at worst NP-complete, while $QCSP(\mathbb{A})$ is at worst Pspace-complete (for polynomial time many-one reductions).

Applications in non-monotonic reasoning and planning, yadda, yadda.
A Galois connection

An operation compatible with all relations of the relational structure $\mathbb{A}$ is a polymorphism of $\mathbb{A}$. All polymorphisms of $\mathbb{A}$ form the clone $Pol(\mathbb{A})$. The subset of surjective polymorphisms is denoted as $s – Pol(\mathbb{A})$.

A relation compatible with all operations of an algebra $\mathbb{A}$ is an invariant relation (or subalgebra of power, subpower) of $\mathbb{A}$. The set of all subpowers of $\mathbb{A}$ is the relational clone $Inv(\mathbb{A})$. 
Chen’s collapsibility and switchability

The collapsibility given below is not the original definition, but an equivalent over idempotent finite algebras:

**Definition (Collapsibility)**
An algebra $\mathbf{A}$ is $k$-collapsible if for all $n > k$, $\mathbf{A}^n$ is generated by the set of all $n$-tuples in which there are at least $n - k$ coordinates which are equal.

**Definition (Switch)**
A tuple $(a_1, \ldots, a_n) \in \mathbf{A}^n$ has a switch at $i$ if $a_{i-1} \neq a_i$.

**Definition (Switchability)**
An algebra $\mathbf{A}$ is $k$-switchable if for all $n > k$, $\mathbf{A}^n$ is generated by the set of all $n$-tuples with at most $k$ switches. $\mathbf{A}$ is switchable if it is $k$-switchable for some $k$. 
The central notions to Chen’s approach to QCSP are PGP and EGP.

**Definition (PGP and EGP)**

A finite algebra $A$ has *polynomially generated powers* (PGP) if there exists a polynomial $p(x)$ such that for all $n$, $A^n$ is generated by some set of tuples with at most $p(n)$ elements. $A$ has *exponentially generated powers* (EGP) if there exists a constant $c > 0$ such that for almost all $n$, any generating set of $A^n$ has more than $2^{cn}$ elements.

**Theorem (Zhuk 2015)**

Let $A$ be a finite algebra. Then either $A$ is switchable, or $A$ has EGP.

It is easy to show that collapsibility implies switchability, which implies PGP. So the above theorem gives a dichotomy between PGP and EGP for finite algebras.
Applications to QCSP

Theorem (Chen)

Let $A$ be a finite idempotent algebra. Switchability of $A$ implies that $QCSP(A)$ reduces to $CSP(A)$ for any relational structure $A$ which consists of relations in $Inv(A)$.

HOWEVER:

Example [Zhuk]

There exists a finite relational structure $A$ such that $QCSP(A)$ is in $P$, while the algebra $Pol(A)$ has EGP.

Also, Martin and Zhuk (2015) proved that on three-element idempotent algebras which have no type 1 covers, finite relatedness and switchability imply collapsibility. They conjecture that the same holds for all idempotent algebras.
Theorem (Jeavons, 1998)

If $\text{Pol}(\Gamma_1) \subseteq \text{Pol}(\Gamma_2)$ and $\Gamma_2$ is finite, then $\text{CSP}(\Gamma_2)$ logspace-reduces to $\text{CSP}(\Gamma_1)$.

Theorem (BBCJK, 2009)

If $s - \text{Pol}(\Gamma_1) \subseteq s - \text{Pol}(\Gamma_2)$ and $\Gamma_2$ is finite, then $\text{QCSP}(\Gamma_2)$ logspace-reduces to $\text{QCSP}(\Gamma_1)$.
Algebraic approach - practicalities

Problems with the analogy:

- $s - Pol(A)$ is not a clone;
- no reduction to cores (important!);
- we do not have a Mal’cev characterization of the 'nice' class,
- or expect one.

Good thing that we know complete graphs with $\geq 3$ vertices are Pspace-complete for QCSP (a reduction from Q-NAE-3-SAT in BBCJK), which implies:

**Theorem (BBCJK, 2009)**

*If $s - Pol(A)$ consists only of essentially unary operations and $A$ is finite, then QCSP($A$) is Pspace-complete.*

Idea of the proof: all polymorphisms of a complete graph are essentially unary and they are the clone generated by all permutations of the universe.
When the structure $\mathbb{A}$ is a core, then $s - Pol(\mathbb{A}) = Pol(\mathbb{A})$ (for $f \in Pol(\mathbb{A})$, just look at $f(x, x, \ldots, x)$ and it better be surjective), so $s - Pol(\mathbb{A})$ is a clone.

Essentially unary polymorphisms are permutations of a variable.

So, for a core structure $\mathbb{A}$, all surjective polymorphisms are essentially unary iff all polymorphisms are essentially unary iff the only idempotent polymorphism is the identity.

That last bit: use $g(x) := f(x, x, \ldots, x)$ and then $f^* := g^{n!-1} \circ f$ is an idempotent operation of the same essential arity as $f$ (where $|A| = n$).
Semicomplete digraphs

Definition

A digraph is semicomplete if it is irreflexive and any two distinct vertices are connected by at least one directed edge.

Why semicomplete?

- You gotta start someplace (Bang-Jensen, Hell and MacGillivray, 1988);
- Barny Martin posed the problem in Dagstuhl 2012;

Why really semicomplete?

- They are easy (= cores), and have no weak nus (except for the tractable few),
- They are even easier: sinks are hit by everybody and sources hit everybody, which helps with dealing with $\forall$,
- They are EVEN easier: when in trouble deduce that there’s a loop and you’ve got a contradiction.
Theorem

Let $G$ be a semicomplete digraph. Then

- Either $G$ has at most one cycle, in which case $\text{QCSP}(G)$ is tractable, or
- $G$ has at least two cycles, a source and a sink, in which case $\text{QCSP}(G)$ is NP-complete, or
- $G$ has at least two cycles, but not both a source and a sink, in which case $\text{QCSP}(G)$ is Pspace-complete.

Proof took 50+ pages (and I said it was easy). We are not very smart. That is as it should be at this stage of the game.
Tractable and NP-complete cases

When there exists both a source and a sink, universal quantification (except formal) implies failure of the formula, so QCSP reduces to CSP.

\(K_2\) and \(C_3\) have majority polymorphisms, and thus are tractable by BBCJK.

Then we prove that \(QCSP(K_2^{\rightarrow j+1})\) reduces to \(QCSP(K_2^{\rightarrow j})\) and similarly for \(QCSP(C_3^{\rightarrow j+1})\) and \(QCSP(C_3^{\rightarrow j})\).
The Good
The Good

This is the case when we reduce from smooth but not strongly connected to strongly connected case.

Take a polymorphism which acts on each strong component which is not a cycle as a projection and prove it is a projection on the smooth digraph with more than one strong component.

Not much to say, mostly straightforward.
The Ugly
It is the strongly connected case.

Here we prove that the only strongly connected semicomplete digraphs with nontrivial idempotent polymorphisms are $K_2$ and $C_3$.

We first prove it for the locally transitive (strongly connected) tournaments.

Next we deal with the semicomplete digraphs which are obtained from strongly connected locally transitive tournaments by blowing up each vertex into a semicomplete graph. We call these P-graphs. (This bit we can shorten and will before the result is published).

Finally, we deal with the remaining cases, which is painfully long and requires several ideas.
The Bad
Assume that a semicomplete digraph has more than one cycle and a sink, but no sources. Here we don’t have only essentially unary polymorphisms, though we don’t have a weak nu, either.

We reduce to a single-sink extension of a smooth semicomplete digraph. Then we find a semicomplete digraph $G'$ which is also a single-sink extension of a smooth semicomplete digraph, and such that $s - pol(G) \subseteq s - Pol(G)'$, but this one has either

- a pp-definable subgraph $\overrightarrow{K_n}$ where $n > 2$, or
- a pp-definable subgraph $\overrightarrow{K_{2\rightarrow2}}$, or
- a pp-definable subgraph $\overrightarrow{T_n}$, where $n > 2$.

Here $\overrightarrow{K_{2\rightarrow2}} :=$ a copy of $\overrightarrow{K_2}$ beats another copy of $\overrightarrow{K_2}$, while $\overrightarrow{T_n} := \langle n; < \cup \{(n - 1, 0)\} \rangle$. 
So we are forced to prove that $QCSP(\overrightarrow{K_n})$, where $n > 2$, $QCSP(\overrightarrow{K_{2\rightarrow2}})$ and $QCSP(\overrightarrow{T_n})$, where $n > 2$, are all Pspace-complete.

And here we ran out of polymorphism ideas. So we proved it directly by reducing to known Pspace-complete problems via gadgets and similar complexity-theoretic ideas. Actually, the first was proved to be Pspace-complete by (BBCJK, 2009), and we did the other two. The first two admit reductions from $QNAE3 \rightarrow SAT$, while the last one admits a reduction from $Q \rightarrow 1 \rightarrow in \rightarrow 3 \rightarrow SAT$. 
Also, we note that the polymorphism algebras of semicomplete digraphs with PSPACE-complete QCSP have EGP, while polymorphism algebras of semicomplete digraphs with QCSP in the class NP have PGP.
We prove a part of the Bad part

We reduce Q-NAE-3-SAT to QCSP($K_{2\rightarrow 2}$).

![Diagram of $K_{2\rightarrow 2}$]

**Figure:** $K_{2\rightarrow 2}$

First we define the edge gadget which combines two copies of $K_{2\rightarrow 2}$:

![Diagram of edge gadget]

**Figure:** Edge gadget
On the edge gadget

Note that none of $x_i, y_j$ is a sink so they can’t evaluate as the sink. Variables arranged in a copy of $K_{2\to 2}$ (e.g. all $x_i$) evaluate into $K_{2\to 2}$ correctly iff the mapping from indices into values represents one of the four automorphisms of $K_{2\to 2}$.

There exists a way to evaluate the two variables in the middle of the edge gadget iff the copies of $K_{2\to 2}$ to the the left and to the right of them evaluate as different automorphisms of $K_{2\to 2}$. 
Figure: Clause gadget
Now it is possible to correctly evaluate the unmarked variables iff the three automorphisms are all distinct.

Put in other way, two of the bottom copies of $K_2$ must evaluate one way, and one the other way, into $\{0, 1\}$, otherwise it is impossible to evaluate the remaining variables in the clause gadget.
Figure: Variable gadget corresponding to $\nu$ connects to a position in the clause.

Whichever way the variable is evaluated, the link ensures that the bottom copies of $K_2$ in $\nu$ and $x$ evaluate the same way, and no more.
Translating the unquantified part of the NAE formula

Let an unquantified (positive Horn) formula \( \varphi \) in the ternary not-all-equal predicate \( R \) be the conjunction of \( n \) predicates (clauses) in which appear variables from the set \( V = \{u, v, w, \ldots\} \). We make a (partially quantified positive Horn) formula in the language of graphs \( \psi \) which corresponds to \( \varphi \). For each variable of \( \varphi \) we add a variable gadget in \( \psi \), and for each predicate occurrence in \( \varphi \) we add a clause gadget in \( \psi \). If a clause \( R(u, v, w) \) occurs in \( \varphi \), we link the variable gadgets for \( u, v \) and \( w \) to three different positions in the corresponding clause gadget in \( \psi \).

Let \( \tau : V \rightarrow \{0, 1\} \) be an evaluation of the NAE formula. We quantify existentially the variables of \( \psi \) which are in the clause gadgets or in variable gadgets with indices 2 and 3. If \( \tau(v) = 0 \), we assign \( \tau'(v_0) = 0 \), \( \tau'(v_1) = 1 \) and \( \tau'(v_\forall) = 2 \), while if \( \tau(v) = 1 \), we assign \( \tau'(v_0) = 1 \), \( \tau'(v_1) = 0 \) and \( \tau'(v_\forall) = 2 \). I.e., \( \tau(v) = 0 \) iff \( \tau'(v_i) = i \) for \( i = 0, 1 \), while \( \tau(v) = 1 \) iff \( \tau'(v_i) = 1 - i \) for \( i = 0, 1 \).

Verify: \( v_\tau(\varphi) = T \) iff \( v_{\tau'}(\psi) = T \).
Now, let $\varphi$ be a positive Horn formula in the language of NAE, while $\psi$ is the associated digraph formula and $\tau$ and $\tau'$ are the evaluations of the free variables of $\varphi$ and $\psi$, respectively, defined in the previous slide.

We associate to $(\exists v)\varphi$ the formula $(\exists v_\forall)(\exists v_0)(\exists v_1)\psi$, while to $(\forall v)\varphi$ we associate the formula $(\forall v_\forall)(\exists v_0)(\exists v_1)\psi$.

The only way $v_\forall$ can have any impact is if it evaluates as 0 or 1. Then it determines:
if $v_\forall$ is evaluated as 0, then $v_i$ must evaluate as $1 - i$ for $i = 0, 1$;
if $v_\forall$ is evaluated as 1 then $v_i$ must evaluate as $i$ for $i = 0, 1$.
So universal quantification of $v_\forall$ stipulates that the formula $\psi$ must be true both in the evaluation which corresponds to $\tau(v/0)$ and in the evaluation corresponding to $\tau(v/1)$. 