Doubly biased Maker-Breaker Connectivity game

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Abstract

In this paper we study the $(a:b)$ Maker-Breaker Connectivity game, played on the edge set of the complete graph on n vertices. We determine the winner for almost all values of a and b.

1 Introduction

Let X be a finite set and let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X. In the biased $(a:b)$ *Maker-Breaker game* $(X, \mathcal{F})$, two players, called Maker and Breaker, take turns in claiming previously unclaimed elements of X, with Breaker going first. In each turn, Breaker claims b elements of X, and then Maker claims a elements. The game ends as soon as every element of X is claimed by either player. Maker wins the game $(X, \mathcal{F})$ if, by the end of the game, he is able to fully claim some $F \in \mathcal{F}$; otherwise Breaker wins the game. If Maker has a strategy to win against any strategy of Breaker, then we say that the game is *Maker's win*; otherwise we say it is *Breaker's win*. The set X will be referred to as the *board* of the game, and the elements of $\mathcal{F}$ will be referred to as the *winning sets*. The natural numbers a and b are called the *bias* of Maker and Breaker, respectively.

In this paper, our attention is dedicated to the $(a:b)$ Maker-Breaker *Connectivity game on* $E(K_n)$, that is, the board of this game is the edge set of K_n, the complete graph on n vertices, and the winning sets are all spanning connected subgraphs of K_n. From now on we will denote this game by $\mathcal{T}_n$.

It is easy to see that the $(1:1)$ game $\mathcal{T}_n$ is Maker's win. In fact, the outcome of the $(1:1)$ Connectivity game is known even when the board is the edge set of an arbitrary graph G – it was proved in [9] that this game is Maker's win if and only if G admits two edge disjoint spanning trees. This led Chvátal and Erdős [4] to introduce *biased* games, that is, games for

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which $(a,b) \neq (1,1)$. Since the $(1:1)$ game $\mathcal{T}_n$ is an easy Maker's win, to give Breaker more power they studied the biased $(1:b)$ games for $b>1$.

Chvátal and Erdős [4] have observed that Maker-Breaker games are bias monotone, that is, if some Maker-Breaker $(1:b)$ game $(X,\mathcal{F})$ is Breaker's win, then the $(1:b+1)$ game $(X,\mathcal{F})$ is Breaker's win as well. Since, unless $\emptyset \in \mathcal{F}$, the $(1:|X|)$ game $(X,\mathcal{F})$ is clearly Breaker's win, it follows that, unless $\emptyset \in \mathcal{F}$ or $\mathcal{F}=\emptyset$ (we refer to these cases as *degenerate*), there exists a unique non-negative integer b_0 such that the $(1:b)$ game $(X,\mathcal{F})$ is Maker's win if and only if $b \leq b_0$. This value of b_0 is known as the *threshold bias* of the game $(X,\mathcal{F})$. Chvátal and Erdős [4] proved that the threshold bias of $\mathcal{T}_n$ is between $(1/4-\varepsilon)n/\ln n$ and $(1+\varepsilon)n/\ln n$. They conjectured that the upper bound is in fact asymptotically best possible. This was verified by Gebauer and Szabó [6].

Assume that the $(1:b)$ game $\mathcal{T}_n$ is being played, but instead of playing optimally (as is always assumed in Game Theory), both players play randomly (they will thus be referred to as *RandomMaker* and *RandomBreaker*, and the resulting game will be referred to as the *Random Connectivity game*). It follows that the graph built by RandomMaker by the end of the game is a random graph $G(n, \lfloor \binom{n}{2}/(b+1) \rfloor)$. It is well known that almost surely such a graph is connected if $\lfloor \binom{n}{2}/(b+1) \rfloor \geq (1/2+\varepsilon)n\ln n$ and disconnected if $\lfloor \binom{n}{2}/(b+1) \rfloor \leq (1/2-\varepsilon)n\ln n$. Hence, almost surely RandomBreaker wins the game if $b \geq (1+\tilde{\varepsilon})n/\ln n$ but loses if $b \leq (1-\tilde{\varepsilon})n/\ln n$, just like when both players play optimally. This remarkable relation between positional games and random graphs, first observed in [4], has come to be known as the *probabilistic intuition* or *Erdős paradigm*. Much of the research in the theory of positional games has since been devoted to finding the threshold bias of certain games and investigating the probabilistic intuition. Many of these results can be found in [2].

While $(a:b)$ games, where $a>1$, were studied less than the case $a=1$, they are not without merit. Indeed, the small change of going from $a=1$ to $a=2$ has a considerable impact on the outcome and the course of play of certain positional games (see [2]). Moreover, it was shown in [1] that the *acceleration* of the so-called *diameter*-2 game partly restores the probabilistic intuition. Namely, it was observed that, while $G(n,1/2)$ has diameter 2 almost surely, the $(1:1)$ diameter-2 game (that is, the board is $E(K_n)$ and the winning sets are all spanning subgraphs of K_n with diameter at most 2) is Breaker's win. On the other hand, it was proved in [1] that the seemingly very similar $(2:2)$ game is Maker's win. Further examples of $(a:b)$ games, where $a>1$, can be found in [2, 1, 5].

Similarly to the $(1:b)$ game, one can define the *generalized threshold bias* for the $(a:b)$ game as well. Given a non-degenerate Maker-Breaker game $(X,\mathcal{F})$ and $a \geq 1$, let $b_0(a)$ be the unique non-negative integer such that the $(a:b)$ game $(X,\mathcal{F})$ is Maker's win if and only if $b \leq b_0(a)$. In this paper we wish to estimate $b_0(a)$ for the Connectivity game $\mathcal{T}_n$ and for every a.

Coming back to the Random Connectivity game, its outcome depends on the number of edges RandomMaker has at the end of the game rather than on the actual values of a and b. Hence, if $a=a(n)$ and $b=b(n)$ are positive integers satisfying $b \leq a(1-\varepsilon)n/\ln n$, for some constant $\varepsilon>0$, and b is not too large (clearly if for example $b \geq \binom{n}{2}$, then RandomBreaker wins regardless of the value of a), then almost surely RandomMaker wins the game. Similarly, if a is not too large and $b \geq a(1+\varepsilon)n/\ln n$, then almost surely RandomBreaker wins the game. Clearly the outcome of the random game and of the regular game could vary greatly for large values of a and b. For example, while Breaker wins the $(\binom{n}{2}:n)$ game $\mathcal{T}_n$ in one move, the corresponding random game is almost surely RandomMaker's win. We prove that for all "reasonable" values of a and b, the probabilistic intuition is maintained. In the following we state our results.

Theorem 1 *Let $\varepsilon > 0$ be a real number and let $n = n(\varepsilon)$ be a sufficiently large positive integer. If $a \leq \ln n$ and $b \geq (1+\varepsilon)\frac{an}{\ln(an)}$, then the $(a:b)$ game $\mathcal{T}_n$ is Breaker's win.*

Note that as a approaches $\ln n$, the lower bound on b in Theorem 1 exceeds n and is therefore trivial. Our next theorem improves that bound for large values of a.

Theorem 2 *Let $\varepsilon > 0$ be a real number. If $(1+\varepsilon)\ln n \leq a < \frac{n}{2e}$ and*

$$b \geq \frac{2a\left(n - 2 + \ln\left\lceil\frac{n}{2a}\right\rceil\right) + \ln\left\lceil\frac{n}{2a}\right\rceil - 1 + \frac{2a}{n}}{2a + \ln\left\lceil\frac{n}{2a}\right\rceil - 1 + \frac{2a}{n}},$$

then the $(a:b)$ game $\mathcal{T}_n$ is Breaker's win.

Finally, for very large values of a we obtain a nontrivial bound on b which suffices to ensure Breaker's win.

Theorem 3 *If $a < \frac{n}{2}$ and $b \geq n-2$, then the $(a:b)$ game $\mathcal{T}_n$ is Breaker's win.*

For Maker's win we prove the following sufficient conditions, covering the whole range of possible values of a.

Theorem 4 *If $a = o\left(\sqrt{\frac{n}{\ln n}}\right)$ and*

$$b < \frac{a\left(n - \frac{n}{a\ln n} + \frac{a-1}{2}\ln\left(\frac{n}{a^2\ln n}\right)\right)}{\ln n + a + \ln\ln n + 4},$$

or if $a = \Omega\left(\sqrt{\frac{n}{\ln n}}\right)$, $a \leq \frac{n-1}{2}$ and $b < \frac{an}{a + 2\ln n - 2\ln a + 4}$, then the $(a:b)$ game $\mathcal{T}_n$ is Maker's win.

A straightforward analysis of the results obtained in Theorems 1, 2, 3 and 4, yields the following estimates for the generalized threshold bias $b_0(a)$.

Corollary 5 *(i) If $a = o(\ln n)$, then $\frac{an}{\ln n} - (1+o(1))\frac{an(\ln\ln n + a)}{\ln^2 n} < b_0(a) < \frac{an}{\ln n} - (1-o(1))\frac{an\ln a}{\ln^2 n}$.*

(ii) If $a = c\ln n$ for some $0 < c \leq 1$, then $(1-o(1))\frac{cn}{c+1} < b_0(a) < \min\left\{cn, (1+o(1))\frac{2n}{3}\right\}$.

(iii) If $a = c\ln n$ for some $c > 1$, then $(1-o(1))\frac{cn}{c+1} < b_0(a) < (1+o(1))\frac{2cn}{2c+1}$.

(iv) If $a = \omega(\ln n)$ and $a = o\left(\sqrt{\frac{n}{\ln n}}\right)$, then $n - \frac{n\ln n}{a} < b_0(a) < n - (1-o(1))\frac{n\ln(n/a)}{2a}$.

(v) If $a = \Omega\left(\sqrt{\frac{n}{\ln n}}\right)$ and $a = o(n)$, then $n - (1+o(1))\frac{2n\ln(n/a)}{a} < b_0(a) < n - (1-o(1))\frac{n\ln(n/a)}{2a}$.

(vi) If $a = cn$ for $0 < c < \frac{1}{2e}$, then $n - \frac{2\ln(1/c)+4}{c} < b_0(a) < n - 2 - \frac{1-2c}{2c}\left(\ln(\frac{1}{2c}) - 1\right) + o(1)$.

(*vii*) *If* $a = cn$ *for* $\frac{1}{2e} \le c < \frac{1}{2}$, *then* $n - \frac{2\ln(1/c)+4}{c} < b_0(a) < n - 2$.

All lower bounds on the threshold bias $b_0(a)$ in Corollary 5 are obtained via Theorem 4. The upper bounds are obtained as follows. Theorem 1 is used in (*i*), Theorem 2 is used in (*iii*), (*iv*), (*v*) and (*vi*) and Theorem 3 is used in (*vii*). In (*ii*), the upper bound of cn is obtained from Theorem 1, whereas the upper bound $(1 + o(1))\frac{2n}{3}$ is obtained from (*iii*) by the bias monotonicity of Maker-Breaker games.

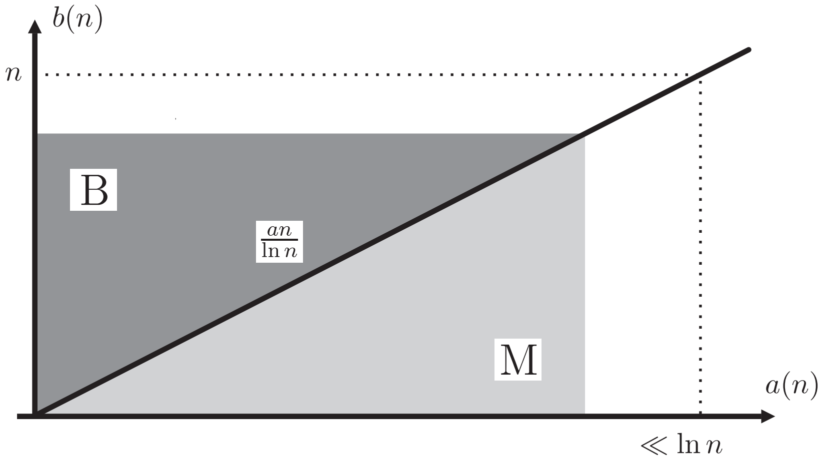

Figure 1: Leading term of the threshold bias for $a = o(\ln n)$.

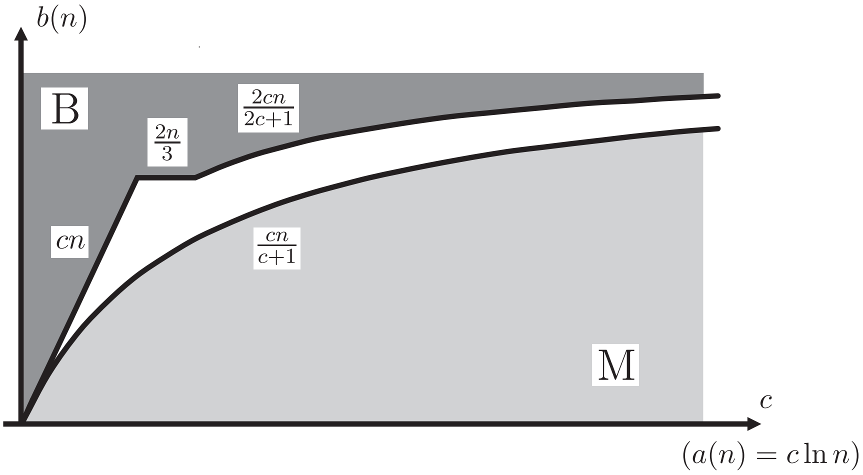

Figure 2: Bounds on the threshold bias for $a = c \ln n$, where c is a positive real number.

Corollary 5 gives fairly tight bounds for the threshold bias on the whole range of the bias a. In particular, for $a = o(\ln n)$, the leading term of the threshold bias is determined exactly; this is depicted in Figure 1. Then, if $a = c \ln n$, where c is a positive real number, (*ii*) and (*iii*) imply that the threshold bias is linear in n, and the upper bound we obtain is a constant factor away from the lower bound, as shown in Figure 2. If $a = \omega(\ln n)$ and $a = o(n)$, it follows by (*iv*) and (*v*) that the leading term of the threshold bias is n, and moreover, we obtain upper and lower bounds for the second order term which are a constant factor away from each other, see Figure 3. Finally, for $a = cn$, where $0 < c < 1/2$ is a real number, (vi) and (vii) imply that the threshold bias is just an additive constant away from n as shown in Figure 4. For larger values of a we have the trivial upper bound of $b_0(a) < n - 1$ and the same lower bound as in (vii) by monotonicity.

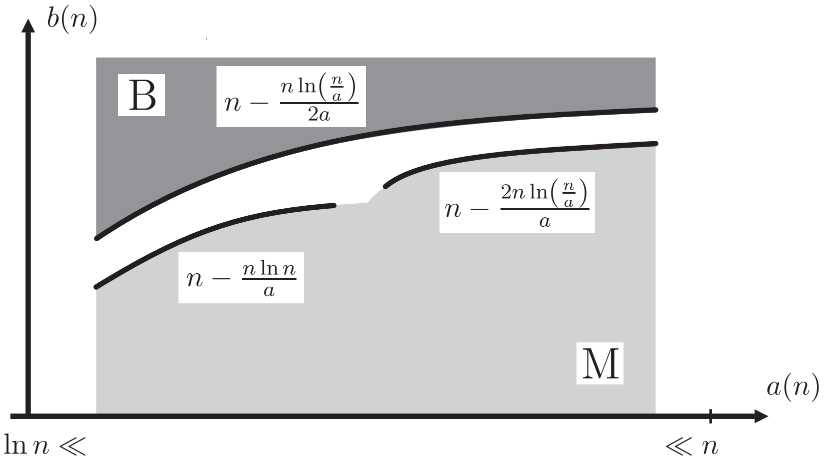

Figure 3: Bounds on the threshold bias for $a = \omega(\ln n)$ and $a = o(n)$.

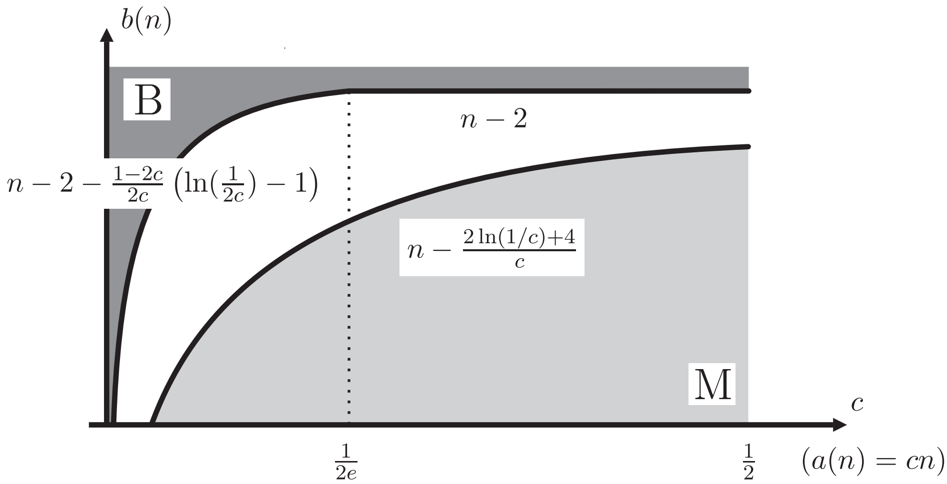

Figure 4: Bounds on the threshold bias for $a = cn$, where $0 < c < 1/2$ is a real number.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize some of the constants obtained in our proofs. We also omit floor and ceiling signs whenever these are not crucial. Most of our results are asymptotic in nature and whenever necessary we assume that n is sufficiently large. Throughout the paper, ln stands for the natural logarithm.

Our graph-theoretic notation is standard and follows that of [10]. In particular, we use the following. For a graph G, $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively,

$v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a vertex $u \in V(G)$, $d_G(u)$ denotes the degree of u in G. At any point during the game $\mathcal{T}_n$, we denote by M (respectively B) the graph which is spanned by the edges Maker (respectively Breaker) has claimed thus far.

For every positive integer j we denote the *jth harmonic number* by H_j, that is, $H_j = \sum_{i=1}^{j} 1/i$, for every $j \geq 1$. We will make use of the following known fact,

$$\ln j + 1/2 < H_j < \ln j + 2/3 \text{ for sufficiently large } j. \tag{1}$$

The rest of the paper is organized as follows. In Section 2 we analyze the $(a : b)$ Box Game – a classical game, introduced in [4] in order to analyze Breaker's strategy in Connectivity game. In Section 3 we prove Theorems 1, 2 and 3. In Section 4 we prove Theorem 4. Finally, in Section 5 we present some open problems.

2 $(a : b)$ Box Game

In order to present Breaker's winning strategy for the $(a : b)$ Maker-Breaker Connectivity game, we first look at the so-called Box Game. The Box Game was first introduced by Chvatál and Erdős in [4]. A hypergraph $\mathcal{H}$ is said to be of type (k, t) if $|\mathcal{H}| = k$, its hyperedges $e_1, e_2, \ldots, e_k$ are pairwise disjoint, and the sum of their sizes is $\sum_{i=1}^{k} |e_i| = t$. Moreover, the hypergraph $\mathcal{H}$ is said to be *canonical* if $||e_i| - |e_j|| \leq 1$ holds for every $1 \leq i, j \leq k$. The board of the Box Game $B(k, t, a, b)$ is a canonical hypergraph of type (k, t). This game is played by two players, called *BoxMaker* and *BoxBreaker*, with BoxMaker having the first move (in [4] the first player is actually BoxBreaker, but the version where BoxMaker is the first player is more suitable for our needs). BoxMaker claims a vertices of $\mathcal{H}$ per move, whereas BoxBreaker claims b vertices of $\mathcal{H}$ per move. BoxMaker wins the Box Game on $\mathcal{H}$ if he can claim all vertices of some hyperedge of $\mathcal{H}$, otherwise BoxBreaker wins this game.

In [7], Hamidoune and Las Vergnas have provided a sufficient and necessary condition for BoxMaker's win in the Box Game $B(k, t, a, b)$ for positive integers k, a, b and $t = kb + 1$. This result can be extended to all positive integers k, t, a and b. Unfortunately, this condition can rarely be used in practise. A more applicable criterion for BoxMaker's win was also provided in [7], but it turns out to be not so tight for certain values of a and b. Hence, in this section, we derive a sufficient condition for BoxMaker's win in the Box Game $B(k, t, a, b)$ which is better suited to our needs. In particular, it enables us to improve the additive constant in part (vi) of Corollary 5. Furthermore, it improves the low order terms of the bound obtained in Theorem 2. We will apply this new criterion whenever we use the Box Game in our solutions.

Given positive integers a and b, we define the following function,

$$f(k; a, b) := \begin{cases} (k-1)(a+1) & , \text{ if } 1 \leq k \leq b \\ ka & , \text{ if } b < k \leq 2b \\ \left\lfloor \frac{k(f(k-b;a,b)+a-b)}{k-b} \right\rfloor & , \text{ otherwise} \end{cases} .$$

First we prove the following technical result.

Lemma 6 *Let a, b and k be positive integers satisfying $k > b$ and $a - b - 1 \geq 0$, then*

$$f(k; a, b) \geq ka - 1 + \frac{k(a-b-1)}{b} \sum_{j=2}^{\lceil k/b \rceil - 1} \frac{1}{j} \ . \qquad (2)$$

Proof If $b < k \leq 2b$, then the assertion of the lemma holds since $ka \geq ka - 1$.

Otherwise, let $x = \lceil k/b \rceil - 2$. Note that x is the unique positive integer for which $b < k - xb \leq 2b$. For every $0 \leq y < x$ we have

$$\begin{aligned} \frac{k}{k-yb} \cdot f(k - yb; a, b) &= \frac{k}{k-yb} \left\lfloor \frac{(k-yb)(f(k-(y+1)b; a, b) + a - b)}{k-(y+1)b} \right\rfloor \\ &\geq \frac{k}{k-yb} \left(\frac{(k-yb)(f(k-(y+1)b; a, b) + a - b)}{k-(y+1)b} - 1 \right) \\ &= \frac{k(a-b)}{k-(y+1)b} - \frac{k}{k-yb} + \frac{k}{k-(y+1)b} \cdot f(k-(y+1)b; a, b). \qquad (3) \end{aligned}$$

Applying the substitution rule (3) repeatedly for every $0 \leq y < x$ and using the fact that $\frac{k}{k-xb} \cdot f(k - xb; a, b) = ka$, we obtain

$$\begin{aligned} f(k; a, b) &\geq ka - 1 + \sum_{i=1}^{\lceil k/b \rceil - 2} \frac{k(a-b)}{k-ib} - \sum_{j=1}^{\lceil k/b \rceil - 3} \frac{k}{k-jb} \\ &\geq ka - 1 + k(a-b-1) \sum_{i=1}^{\lceil k/b \rceil - 2} \frac{1}{k-ib} \ . \qquad (4) \end{aligned}$$

Since $\frac{1}{k-ib} \geq \frac{1}{(\lceil k/b \rceil - i)b}$ holds for every $1 \leq i \leq \lceil k/b \rceil - 2$, it follows by (4) that

$$\begin{aligned} f(k; a, b) &\geq ka - 1 + k(a-b-1) \sum_{i=1}^{\lceil k/b \rceil - 2} \frac{1}{(\lceil k/b \rceil - i)\, b} \\ &= ka - 1 + k(a-b-1) \sum_{j=2}^{\lceil k/b \rceil - 1} \frac{1}{jb} \\ &= ka - 1 + \frac{k(a-b-1)}{b} \sum_{j=2}^{\lceil k/b \rceil - 1} \frac{1}{j} \ . \end{aligned}$$

□

Lemma 7 *If $t \leq f(k; a, b) + a$, then BoxMaker has a winning strategy for $B(k, t, a, b)$.*

Proof We prove this lemma by induction on k.

If $1 \le k \le b$, then $t \le f(k; a, b) + a = (k-1)(a+1) + a = k(a+1) - 1$. Since the board is a canonical hypergraph, it follows that there exists a hyperedge of size at most a. In his first move, BoxMaker claims all vertices of such a hyperedge and thus wins.

If $b < k \le 2b$, then $t \le f(k; a, b) + a = ka + a$. In his first move, BoxMaker claims a vertices such that the resulting hypergraph is canonical of type (k, t'), where $t' = t - a \le ka$. It follows that every hyperedge is of size at most a. Subsequently, in his first move, BoxBreaker claims $b < k$ vertices. Hence, there must exist an hyperedge which BoxBreaker did not touch in his first move. In his second move, BoxMaker claims all free vertices of such an hyperedge and thus wins.

Assume then that $k > 2b$ and assume that the assertion of the lemma holds for every $k_1 < k$, that is, if $t_1 \le f(k_1; a, b) + a$, then BoxMaker has a winning strategy for $B(k_1, t_1, a, b)$. In his first move, BoxMaker claims a vertices such that the resulting hypergraph is canonical of type (k, t'), where $t' = t - a \le f(k; a, b)$. Subsequently, in his first move, BoxBreaker claims b board elements. Let $e_1, \ldots, e_{k-b}$ be arbitrary $k - b$ winning sets which BoxBreaker did not touch in his first move. Since BoxMaker's first move results in a canonical hypergraph, it follows that $\hat{t} := \sum_{i=1}^{k-b} |e_i| \le \frac{k-b}{k} \cdot (t' + b) \le \frac{k-b}{k} \cdot t' + b$. Moreover, it follows by the definition of f that $f(k; a, b) \le \frac{k}{k-b} \left(f(k-b; a, b) + a - b \right)$, implying that $f(k-b; a, b) \ge \frac{k-b}{k} \cdot f(k; a, b) + b - a \ge \hat{t} - a$. Hence, in order to prove that BoxMaker has a winning strategy for $B(k, t, a, b)$, it suffices to prove that BoxMaker has a winning strategy for $B(k - b, \hat{t}, a, b)$. This however follows by the induction hypothesis since $k - b < k$ and since, as noted above, $\hat{t} \le f(k - b; a, b) + a$. □

3 Breaker's strategy for the $(a : b)$ Connectivity Game

Proof of Theorem 1 Our proof relies on the approach of Chvatál and Erdős [4], who proved the special case $a = 1$.

For technical reasons we will assume that $\varepsilon < 1/3$. This is allowed by the bias monotonicity of Maker-Breaker games.

Before we describe Breaker's strategy in detail, we give its outline. Breaker's goal is to isolate some vertex $u \in V(K_n)$ in Maker's graph. His strategy consists of two phases. In the first phase, he claims all edges of a clique C on $k := \left\lceil \frac{an}{(a+1)\ln(an)} \right\rceil$ vertices, such that no vertex of C is touched by Maker, that is, $d_M(v) = 0$ for every $v \in C$. In the second phase, he claims all free edges which are incident with some vertex $v \in C$.

Breaker's strategy:

First Phase. For every $i \ge 1$, just before Breaker's ith move, let C_i denote the largest clique (breaking ties arbitrarily) in Breaker's graph such that $d_M(v) = 0$ for every $v \in C_i$. Let ℓ_i be the largest integer for which $b_i \le b$, where $b_i := \binom{a+\ell_i}{2} + (a + \ell_i)|C_i|$. If $|C_i| \ge k$, then the first phase is over and Breaker proceeds to the second phase of his strategy. Otherwise, in his ith move, Breaker picks $a + \ell_i$ vertices $u_1^i, \ldots, u_{a+\ell_i}^i$ of $V(K_n) \setminus V(C_i)$ such that $d_M(u_j^i) = 0$ for every $1 \le j \le a+\ell_i$, and then he claims all edges of $\{(u_{j_1}^i, u_{j_2}^i) : 1 \le j_1 < j_2 \le a+\ell_i\} \cup \{(u_j^i, w) : 1 \le j \le a+\ell_i,\ w \in V(C_i)\}$. He then claims additional $b - b_i$ arbitrary edges; we will disregard these additional edge in our analysis.

Since $b \ge \binom{a+1}{2} + (a+1)(k-1)$, it follows that $\ell_i \ge 1$ as long as $|C_i| < k$. Since Maker can touch at most a vertices of C_i in his ith move, it follows that $|C_{i+1}| \geq |C_i| + \ell_i$, assuming that, before Breaker's ith move, there are at least $a + \ell_i$ vertices in $V(K_n) \setminus V(C_i)$ which are isolated in Maker's graph.

It follows from the definition of ℓ_i that, if $|C_i| \leq \frac{b}{a+3} - \frac{a+2}{2}$, then $\ell_i \geq 3$. Similarly, if $|C_i| \leq \frac{b}{a+2} - \frac{a+1}{2}$, then $\ell_i \geq 2$ and if $|C_i| \leq k < \frac{b}{a+1} - \frac{a}{2}$, then $\ell_i \geq 1$. Hence, Breaker's clique reaches size k within at most

$$\frac{b}{3(a+3)} + \frac{b}{2(a+2)(a+3)} + \frac{b}{(a+1)(a+2)}$$

moves. Since Maker can touch at most $2a$ vertices in a single move, it follows that during the first phase, the number of vertices which are either in Breaker's clique or have positive degree in Maker's graph is at most

$$\frac{b}{3(a+3)}(2a+3) + \frac{b}{2(a+2)(a+3)}(2a+2) + \frac{b}{(a+1)(a+2)}(2a+1) \leq n,$$

where this inequality follows since $a \leq \ln n$ and $\varepsilon < 1/3$.

Hence, for as long as $|C_i| < k$, there are vertices of degree 0 in $M[V(K_n) \setminus V(C_i)]$ and thus Breaker can follow the proposed strategy throughout the first phase.

Second Phase. Let C be the clique Breaker has built in the first phase, that is, $|C| \geq k$, $d_M(v) = 0$ holds for every $v \in V(C)$, and $(u, v) \in E(B)$ holds for every $u, v \in V(C)$. In this phase Breaker will isolate some vertex $v \in V(C)$ in Maker's graph; the game ends as soon as he achieves this goal (or as soon as $E(B \cup M) = E(K_n)$, whichever happens first). In order to do so, he restricts his attention to the part of the board spanned by the free edges of $E_2 := \{(u, v) : u \in V(C), v \in V(K_n) \setminus V(C)\}$. In order to choose which edges of E_2 to claim in each move, he consults an auxiliary Box Game $B(k, k(n-k), b, a)$ assuming the role of BoxMaker.

Since $|\{(u, v) \in E(K_n) \setminus (E(M) \cup E(B)) : v \in V(K_n)\}| \leq n - k$ holds for every $u \in V(C)$, it follows that, if BoxMaker has a winning strategy for $B(k, k(n-k), b, a)$, then Breaker, having built the clique C, has a winning strategy for the $(a : b)$ Connectivity game on K_n.

Finally, since $k > a$ and $b - a - 1 \geq 0$, it follows by Lemmas 6 and 7 that, in order to prove that BoxMaker has a winning strategy for $B(k, k(n-k), b, a)$, it suffices to prove that

$$k(n-k) \leq kb - 1 + \frac{k(b-a-1)}{a} \sum_{i=2}^{\lceil k/a \rceil - 1} \frac{1}{i} + b. \tag{5}$$

The latter inequality can be easily verified given our choice of k, the assumed bounds on a and b, and by applying (1) to $\sum_{i=2}^{\lceil k/a \rceil - 1} \frac{1}{i}$ while using the inequality $\ln n + 1/2 > \ln(n+1)$ which holds for every $n \geq 2$. $\square$

Proof of Theorem 2 Breaker aims to win Connectivity game on K_n by isolating a vertex in Maker's graph. While playing this game, Breaker plays (in his mind) an auxiliary Box Game $B(n, n(n-1), b, 2a)$ (assuming the role of BoxMaker). Let $V(K_n) = \{v_1, \ldots, v_n\}$ and let $e_1, e_2, \ldots, e_n$ be an arbitrary ordering of the winning sets of $B(n, n(n-1), b, 2a)$. In every move, Breaker claims b free edges of K_n according to his strategy for $B(n, n(n-1), b, 2a)$. That is, whenever he is supposed to claim an element of e_i, he claims an arbitrary free edge (v_i, v_j); if no such free edge exists, then he claims an arbitrary free edge. Whenever, Maker claims an edge (v_i, v_j) for some $1 \leq i < j \leq n$, Breaker (in his mind) gives BoxBreaker an arbitrary free element of e_i and an arbitrary free element of e_j. Note that every edge of K_n which Maker claims translates to two board elements of $B(n, n(n-1), b, 2a)$. This is why BoxBreaker's bias is set to be $2a$. It is thus evident that if BoxMaker has a winning strategy for $B(n, n(n-1), b, 2a)$, then Breaker has a winning strategy for the $(a : b)$ Connectivity game on K_n.

By Lemma 7, in order to prove that BoxMaker has a winning strategy for $B(n, n(n-1), b, 2a)$, it suffices to prove that $n(n-1) \leq f(n; b, 2a) + b$.

Since $b - 2a - 1 \geq 0$ and $n > 2a$, it follows by Lemma 6, (1) and the fact that for $n \geq 2$, $\ln n + 1/2 > \ln(n+1)$ that

$$f(n; b, 2a) \geq nb - 1 + \frac{n(b-2a-1)}{2a}\left(\ln\left\lceil\frac{n}{2a}\right\rceil - 1\right) \ .$$

Hence, it suffices to prove that

$$n(n-1) \leq \frac{n(b-2a-1)}{2a}\left(\ln\left\lceil\frac{n}{2a}\right\rceil - 1\right) + nb - 1 + b \ .$$

It is straightforward to verify that the above inequality holds for

$$b \geq \frac{2a(n - 2 + \ln\left\lceil\frac{n}{2a}\right\rceil) + \ln\left\lceil\frac{n}{2a}\right\rceil - 1 + \frac{2a}{n}}{2a + \ln\left\lceil\frac{n}{2a}\right\rceil - 1 + \frac{2a}{n}}.$$

□

Proof of Theorem 3 In his first move, Breaker claims the edges of some graph of positive minimum degree. This is easily done as follows. If n is even, then Breaker claims the edges of some perfect matching of K_n and then he claims additional $b - n/2$ arbitrary free edges. If n is odd, then Breaker claims the edges of a matching of K_n which covers all vertices of K_n but one, say u. He then claims a free edge (u, x) for some $x \in V(K_n)$ and additional $b - (n-1)/2 - 1$ arbitrary free edges.

In his first move, Maker cannot touch all vertices of K_n since $2a < n$. Let $w \in V(K_n)$ be an isolated vertex in Maker's graph after his first move. Since $d_B(w) \geq 1$ and $b \geq n - 2$, Breaker can claim all free edges which are incident with w in his second move and thus win. □

4 Maker's strategy for the $(a : b)$ Connectivity Game

Proof of Theorem 4 Our proof relies on the approach of Gebauer and Szabó [6], who proved the special case $a = 1$. First, let us introduce some terminology. For a vertex $v \in V(K_n)$ let $C(v)$ denote the connected component in Maker's graph which contains the vertex v. A connected component in Maker's graph is said to be *dangerous* if it contains at most $2b/a$ vertices. We define a *danger function* on $V(K_n)$ in the same way it was defined in [6],

$$\mathcal{D}(v) = \begin{cases} d_B(v), & \text{if } C(v) \text{ is dangerous} \\ -1, & \text{otherwise} \end{cases}.$$

We are now ready to describe Maker's strategy.

Maker's strategy: Throughout the game, Maker maintains a set $A \subseteq V(K_n)$ of *active* vertices. Initially, $A = V(K_n)$.

For as long as Maker's graph is not a spanning tree, Maker plays as follows. For every $i \geq 1$, Maker's ith move consists of a steps. For every $1 \leq j \leq a$, in the jth step of his ith move Maker chooses an active vertex $v_i^{(j)}$ whose danger is maximal among all active vertices (breaking ties arbitrarily). He then claims a free edge (x, y) for arbitrary vertices $x \in C(v_i^{(j)})$ and $y \in V(K_n) \setminus C(v_i^{(j)})$. Subsequently, Maker deactivates $v_i^{(j)}$, that is, he removes $v_i^{(j)}$ from A. If at any point during the game Maker is unable to follow the proposed strategy, then he forfeits the game.

Note that by Maker's strategy his graph is a forest at any point during the game. Hence, there are at most $n-1$ steps in the entire game. It follows that the game lasts at most $\lceil \frac{n-1}{a} \rceil$ rounds.

In order to prove Theorem 4, it clearly suffices to prove that Maker is able to follow the proposed strategy without ever having to forfeit the game. First, we prove the following lemma.

Lemma 8 *At any point during the game there is exactly one active vertex in every connected component of Maker's graph.*

Proof Our proof is by induction on the number of steps r which Maker makes throughout the game.

Before the game starts, every vertex of K_n is a connected component of Maker's graph, and every vertex is active by definition. Hence, the assertion of the lemma holds for $r = 0$.

Let $r \geq 1$ and assume that the assertion of the lemma holds for every $r' < r$. In the rth step, Maker chooses an active vertex v and then claims an edge (x, y) such that $x \in C(v)$ and $y \notin C(v)$ hold prior to this move. By the induction hypothesis there is exactly one active vertex $z \in C(y)$ and v is the sole active vertex in $C(v)$. Since Maker deactivates v after claiming (x, y), it follows that z is the unique active vertex in $C(x) \cup C(y)$ after Maker's rth step. Clearly, every other component still has exactly one active vertex. $\square$

We are now ready to prove that Maker can follow his strategy (without forfeiting the game) for $n-1$ steps. Assume for the sake of contradiction that at some point during the game Maker chooses an active vertex $v \in C$ and then tries to connect C with some component of $M \setminus C$, but fails. It follows that Breaker has already claimed all the edges of K_n with one endpoint in C and the other in $V \setminus C$. Assume that Breaker has claimed the last edge of this cut in his sth move. As noted above, $s \leq \lceil \frac{n-1}{a} \rceil$ must hold. It follows that $|C| \leq 2b/a$ as otherwise Breaker would have had to claim at least $\frac{2b}{a}(n - \frac{2b}{a}) > sb$ edges in s moves. It follows that at any point during the first s rounds of the game there is always at least one dangerous connected component.

In his sth move, Breaker claims at most b edges. Hence, just before Breaker's sth move, $e_B(V(C), V(K_n) \setminus V(C)) \geq |C|\,(n-|C|) - b$ must hold. In particular, $d_B(v) \geq n - \frac{2b}{a} - b$, where v is the unique active vertex of C. Since Maker did not connect C with $M \setminus C$ in his $(s-1)$st move, it follows that, just before this move, there must have been at least $a+1$ active vertices $v, v_1, \ldots, v_a$ such that the components $C, C(v_1), \ldots, C(v_a)$ were dangerous and $d_B(u) \geq n - \frac{2b}{a} - b$ for every $u \in \{v, v_1, \ldots, v_a\}$.

For every $1 \leq i \leq s$, let M_i and B_i denote the ith move of Maker and of Breaker, respectively. By Maker's strategy $v_1^{(1)}, \ldots, v_1^{(a)}, v_2^{(1)}, \ldots, v_2^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}$ are of maximum degree in Breaker's graph among all active vertices at the appropriate time, that is, just before Maker's jth step of his ith move, $d_B(v_i^{(j)})$ is maximal among all active vertices. Let v_s be an active vertex of maximum degree in Breaker's graph just before Maker's sth move. Note that, for every $u \in \{v_1^{(1)}, \ldots, v_1^{(a)}, v_2^{(1)}, \ldots, v_2^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$, if u is active, then $C(u)$ is a dangerous component. For every $1 \leq i \leq s-1$, let $A_{s-i} = \{v_{s-i}^{(1)}, \ldots, v_{s-i}^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$ denote the subset of vertices of $\{v_1^{(1)}, \ldots, v_1^{(a)}, v_2^{(1)}, \ldots, v_2^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$ that are still active just before Maker's $(s-i)$th move and let $A_s = \{v_s\}$. For every $A \subseteq V$, let $\overline{\mathcal{D}}_{B_i}(A) = \frac{\sum_{v \in A} \mathcal{D}(v)}{|A|}$ denote the average danger value of the vertices of A, immediately before Breaker's move B_i. The average danger $\overline{\mathcal{D}}_{M_i}(A)$ is defined analogously.

Since Maker always deactivates vertices of maximum danger, thus reducing the average danger value of active vertices, we have the following lemma.

Lemma 9 *$\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) \geq \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})$ holds for every $1 \leq i \leq s-1$.*

Proof Let $u \in A_{s-i+1} = \{v_{s-i+1}^{(1)}, \ldots, v_{s-i+1}^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$ be an arbitrary vertex. Since $C(u)$ is a dangerous component immediately before B_{s-i+1}, the danger $\mathcal{D}(u)$ does not change during M_{s-i}. It follows that $\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i+1}) = \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})$.

Note that the vertices contained in A_{s-i+1} were still active before M_{s-i}.

Following his strategy, Maker deactivated all the vertices of $\{v_{s-i}^{(1)}, \ldots, v_{s-i}^{(a)}\}$, because their danger values were the largest among all the active vertices of $\{v_{s-i}^{(1)}, \ldots, v_{s-i}^{(a)}, \ldots, v_{s-1}^{(1)}, \ldots, v_{s-1}^{(a)}, v_s\}$. It follows that

$$\min\{\mathcal{D}(v_{s-i}^{(1)}), \ldots, \mathcal{D}(v_{s-i}^{(a)})\} \geq \max\{\mathcal{D}(v_{s-i+1}^{(1)}), \ldots, \mathcal{D}(v_{s-i+1}^{(a)}), \ldots, \mathcal{D}(v_{s-1}^{(1)}), \ldots, \mathcal{D}(v_{s-1}^{(a)}), \mathcal{D}(v_s)\},$$

and thus $\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) \geq \overline{\mathcal{D}}_{M_{s-i}}(A_{s-i+1})$, as claimed. □

The following lemma gives two estimates on the change of the danger value caused by Breaker's moves.

Lemma 10 *The following two inequalities hold for every $1 \leq i \leq s-1$.*

(i) $\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i}) \leq \frac{2b}{ai+1} < \frac{2b}{ai}$.

(ii) Define a function $g : \{1, \ldots, s\} \to \mathbb{N}$ by setting $g(i)$ to be the number of edges with both endpoints in A_i which Breaker has claimed during the first $i-1$ moves of the game. Then,

$$\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i}) \leq \frac{b + a^2(i-1) + a + \binom{a}{2} + g(s-i+1) - g(s-i)}{ai+1}.$$

Proof

(i) The components $C(v_{s-i}^{(1)}), \dots, C(v_{s-i}^{(a)}), \dots, C(v_{s-1}^{(1)}), \dots, C(v_{s-1}^{(a)}), C(v_s)$ are dangerous before Maker's $(s-i)$th move. During Breaker's moves, the components of Maker's graph do not change. Hence, the change of the danger values of the vertices of A_{s-i}, caused by Breaker's $(s-i)$th move, depend solely on the change of their degrees in Breaker's graph. In his $(s-i)$th move, Breaker claims b edges and thus the increase of the sum of the degrees of the vertices of $\{v_{s-i}^{(1)}, \dots, v_{s-i}^{(a)}, \dots, v_{s-1}^{(1)}, \dots, v_{s-1}^{(a)}, v_s\}$ is at most $2b$. The size of A_{s-i} is $ai+1$. Thus $\overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})$ increases by at most $\frac{2b}{ai+1}$ during B_{s-i}.

(ii) Let p denote the number of edges (x,y) claimed by Breaker during B_{s-i} such that $\{x,y\} \subseteq A_{s-i}$ and let $q = b-p$. Hence, the increase of the sum $\sum_{u \in A_{s-i}} d_B(u)$ during B_{s-i} is at most $2p+q = p+b$. It follows that $\overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})$ increases by at most $\frac{b+p}{ai+1}$ during B_{s-i}. It remains to prove that $p \le a^2(i-1) + a + \binom{a}{2} + g(s-i+1) - g(s-i)$.

During his first $s-i-1$ moves, Breaker has claimed exactly $g(s-i)$ edges with both their endpoints in A_{s-i}. Hence, during his first $s-i$ moves, Breaker has claimed exactly $g(s-i)+p$ edges with both their endpoints in A_{s-i}. Exactly $g(s-i+1)$ of these edges have both their endpoints in $A_{s-i+1} = A_{s-i} \setminus \{v_{s-i}^{(1)}, \dots, v_{s-i}^{(a)}\}$. There can be at most $\binom{a}{2}$ edges connecting two vertices of $\{v_{s-i}^{(1)}, \dots, v_{s-i}^{(a)}\}$. Moreover, each vertex of $\{v_{s-i}^{(1)}, \dots, v_{s-i}^{(a)}\}$ is adjacent to at most $a(i-1)+1$ vertices of A_{s-i+1}. Combining all of these observations, we conclude that $g(s-i)+p \le g(s-i+1) + a^2(i-1) + a + \binom{a}{2}$, entailing $p \le \binom{a}{2} + a^2(i-1) + a + g(s-i+1) - g(s-i)$ as claimed.

□

Clearly, before the game starts $\mathcal{D}(u) = d_B(u) = 0$ holds for every vertex u. In particular $\overline{\mathcal{D}}_{B_1}(A_1) = 0$. Using our assumption that Breaker wins the game, we will obtain a contradiction by showing that $\overline{\mathcal{D}}_{B_1}(A_1) > 0$.

Note that, as previously observed, $\overline{\mathcal{D}}_{B_s}(A_s) \ge n - \frac{2b}{a} - b$. We will use this fact, Lemma 9, Lemma 10, the inequalities $\frac{1}{ai+1} < \frac{1}{ai}$ and $b + a^2(i-1) + a + \binom{a}{2} + g(s-i+1) - g(s-i) \ge 0$ (which hold for every $1 \le i \le s-1$), and (1), in order to reach the aforementioned contradiction.

Let $k := \left\lfloor \frac{n}{a^2 \ln n} \right\rfloor$.

First, assume that $a = o(\sqrt{n/\ln n})$. We split the game into two parts – the main game and the last k moves. In these last moves, we will use a more delicate estimate on the effect of Breaker's move on the average danger. We distinguish between two cases.

Case 1: $s < k$.

$$\overline{\mathcal{D}}_{B_1}(A_1) = \overline{\mathcal{D}}_{B_s}(A_s) + \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1}) \right)$$
$$- \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i}) \right)$$

$$
\begin{aligned}
&\geq n - \frac{2b}{a} - b + \sum_{i=1}^{s-1} 0 - \sum_{i=1}^{s-1} \frac{b + a^2(i-1) + a + \binom{a}{2} + g(s-i+1) - g(s-i)}{ai} \\
&\geq n - \frac{b}{a}(H_{s-1} + 2 + a) - a(s-1) + aH_{s-1} - H_{s-1} - \frac{a-1}{2}H_{s-1} \\
&- \frac{g(s)}{a} + \sum_{i=1}^{s-2} \frac{g(s-i)}{ai(i+1)} + \frac{g(1)}{a(s-1)} \\
&\geq n - \frac{b}{a}(H_{s-1} + 2 + a) + \frac{a-1}{2}H_{s-1} - a(s-1) \\
&\qquad\qquad\qquad\qquad (\text{since } g(s) = 0 \text{ and } g(s-i) \geq 0) \\
&> n - \frac{b}{a}(H_s + 2 + a) + \frac{a-1}{2}H_s - as \\
&> n - \frac{b}{a}(\ln k + 3 + a) + \frac{a-1}{2}\ln k - ak \\
&\qquad\qquad\qquad\qquad (\text{since } s < k) \\
&> n - \frac{n - \frac{n}{a\ln n} + \frac{a-1}{2} \cdot \ln\left(\frac{n}{a^2 \ln n}\right)}{\ln n + a + \ln\ln n + 4} (\ln n - \ln a - \ln\ln n + a + 3) \\
&- \frac{n}{a \ln n} + \frac{a-1}{2} \cdot \ln\left(\frac{n}{a^2 \ln n}\right) \\
&> 0.
\end{aligned}
$$

Case 2: $s \geq k$.

$$
\begin{aligned}
\overline{\mathcal{D}}_{B_1}(A_1) &= \overline{\mathcal{D}}_{B_s}(A_s) + \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})\right) \\
&- \sum_{i=1}^{k-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})\right) - \sum_{i=k}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})\right) \\
&\geq n - \frac{2b}{a} - b + \sum_{i=1}^{s-1} 0 \\
&- \sum_{i=1}^{k-1} \frac{b + a^2(i-1) + a + \binom{a}{2} + g(s-i+1) - g(s-i)}{ai} - \sum_{i=k}^{s-1} \frac{2b}{ai} \\
&\geq n - \frac{2b}{a} - b - \frac{b}{a}H_{k-1} + \frac{a-1}{2}H_{k-1} - a(k-1) \\
&- \frac{g(s)}{a} + \sum_{i=1}^{k-2} \frac{g(s-i)}{ai(i+1)} + \frac{g(s-k+1)}{a(k-1)} - \frac{2b}{a}(H_{s-1} - H_{k-1}) \\
&\geq n - \frac{b}{a}(2H_s - H_{k-1} + 2 + a) + \frac{a-1}{2}H_{k-1} - a(k-1) \\
&\qquad\qquad\qquad\qquad (\text{since } g(s) = 0 \text{ and } g(s-i) \geq 0) \\
&> n - \frac{b}{a}\left(2\ln\left(\frac{n-1}{a}\right) - \ln k + a + 4\right) + \frac{a-1}{2} \cdot \ln k - ak
\end{aligned}
$$

$$
\begin{aligned}
&\geq n - \frac{b}{a}(\ln n + \ln\ln n + a + 4) + \frac{a-1}{2} \cdot \ln\left(\frac{n}{a^2 \ln n}\right) - \frac{n}{a \ln n} \\
&> n - \frac{n - \frac{n}{a \ln n} + \frac{a-1}{2} \cdot \ln\left(\frac{n}{a^2 \ln n}\right)}{\ln n + a + \ln\ln n + 4}(\ln n + \ln\ln n + 4 + a) \\
&+ \frac{a-1}{2} \cdot \ln\left(\frac{n}{a^2 \ln n}\right) - \frac{n}{a \ln n} \\
&= 0\,.
\end{aligned}
$$

Next, assume that $a = \Omega(\sqrt{n/\ln n})$ and $a \leq \frac{n-1}{2}$. In this case, the game does not last long.

$$
\begin{aligned}
\overline{\mathcal{D}}_{B_1}(A_1) &= \overline{\mathcal{D}}_{B_s}(A_s) + \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i+1}}(A_{s-i+1})\right) \\
&\quad - \sum_{i=1}^{s-1} \left(\overline{\mathcal{D}}_{M_{s-i}}(A_{s-i}) - \overline{\mathcal{D}}_{B_{s-i}}(A_{s-i})\right) \\
&> n - \frac{2b}{a} - b + \sum_{i=1}^{s-1} 0 - \sum_{i=1}^{s-1} \frac{2b}{ai} \\
&> n - \frac{2b}{a} - b - \frac{2b}{a}(\ln s + 1) \\
&= n - \frac{2b}{a}(2 + \frac{a}{2} + \ln s) \\
&\geq n - \frac{2b}{a}\left(2 + \frac{a}{2} + \ln\left(\frac{n-1}{a}\right)\right) \\
&> n - \frac{n}{\ln n - \ln a + 2 + \frac{a}{2}}\left(2 + \frac{a}{2} + \ln\left(\frac{n-1}{a}\right)\right) \\
&\geq 0\,.
\end{aligned}
$$

□

5 Concluding remarks and open problems

Determining the threshold bias. In this paper we have tried to determine the winner of the $(a : b)$ Connectivity game on $E(K_n)$ for all values of a and b. We have established lower and upper bounds on the threshold bias $b_0(a)$ for every value of a. For most values, these bounds are quite sharp. However, for $a = c \ln n$, where $c > 0$ is fixed, the first order terms in the upper bound and the lower bound differ. For that reason, we feel that an improvement of the bounds in this case would be particularly interesting.

Analyzing other games. There are many well-studied Maker-Breaker games played on the edge set of the complete graph for which, in the biased $(a : b)$ version, the identity of the winner is known for $a = 1$ and almost all values of b. Some examples are the Hamilton cycle game (see [4] and [8]) and the H-game, where H is some fixed graph (see [3]). It would be interesting to analyze these games for other values of a (and corresponding values of b) as well.

We note that all the results of the present paper also hold for the Positive Minimum Degree game, where Maker's goal is to touch all n vertices of the board K_n, and Breaker's goal is to prevent Maker from doing so. Indeed, if Maker wins the Connectivity game, then he clearly wins the Positive Minimum Degree game with the same parameters as well. On the other hand, in all our results that guarantee Breaker's win in the Connectivity game, we in fact prove that Breaker can isolate a vertex in Maker's graph, which clearly also ensures Breaker's win in the Positive Minimum Degree game.

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