Consistent digital line segments

Tobias Christ* Dömötör Pálvölgyi† Miloš Stojaković ‡

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Abstract

We introduce a novel and general approach for digitalization of line segments in the plane that satisfies a set of axioms naturally arising from Euclidean axioms. In particular, we show how to derive such a system of digital segments from any total order on the integers. As a consequence, using a well-chosen total order, we manage to define a system of digital segments such that all digital segments are, in Hausdorff metric, optimally close to their corresponding Euclidean segments, thus giving an explicit construction that resolves the main question of [1].

1 Introduction

One of the most fundamental challenges in digital geometry is to define a "good" digital representation of a geometric object. Of course, the meaning of the word "good" here heavily depends on particular conditions we may impose. Looking at the problem of digitalization in the plane, the goal is to find a set of points on the integer grid $\mathbb{Z}^2$ that approximates well a given object. The topology of the grid $\mathbb{Z}^2$ is commonly defined by the graph whose vertices are all the points of the grid, and each point is connected by an edge to each of the four points that are either horizontally or vertically adjacent to it.

Knowing that a straight line segment is one of the most basic geometric objects and a building block for many other objects, defining its digitalization in a satisfying manner is vital. Hence, it is no wonder that this has been a hot scientific topic in the last few decades, see [2] for a survey. For any pair of points p and q in the grid $\mathbb{Z}^2$ we want to define the digital line segment $S(p,q)$ connecting them, that is, $\{p,q\} \subseteq S(p,q) \subseteq \mathbb{Z}^2$. Chun et al. in [1] put forward the following four axioms that arise naturally from properties of line segments in Euclidean geometry.

(S1) *Grid path property:* For all $p, q \in \mathbb{Z}^2$, $S(p,q)$ is the vertex set of a path from p to q in the grid graph.

(S2) *Symmetry property:* For all $p, q \in \mathbb{Z}^2$, we have $S(p,q) = S(q,p)$.

(S3) *Subsegment property:* For all $p, q \in \mathbb{Z}^2$ and every $r \in S(p,q)$, we have $S(p,r) \subseteq S(p,q)$.

(S4) *Prolongation property:* For all $p, q \in \mathbb{Z}^2$, there exists $r \in \mathbb{Z}^2$, such that $r \notin S(p,q)$ and $S(p,q) \subseteq S(p,r)$.

*Institute of Theoretical Computer Science, ETH Zurich, Switzerland, email: tobias.christ@inf.ethz.ch
†Department of Combinatorial Geometry, EPF Lausanne, Switzerland, email: dom@cs.elte.hu.
‡Department of Mathematics and Informatics, University of Novi Sad, Serbia, email: milos.stojakovic@dmi.uns.ac.rs.

First, note that (S3) is not satisfied by the usual way a computer visualizes a segment. A natural definition of the digital straight segment between $p = (p_x, p_y)$ and $q = (q_x, q_y)$, where $p_x \leq q_x$ and $0 \leq q_y - p_y < q_x - p_x$ is $\left\{\left(x, \left\lfloor (x - p_x)\frac{q_y - p_y}{q_x - p_x} + p_y + 0.5\right\rfloor\right) : p_x \leq x \leq q_x\right\}$. This does not satisfy (S1), but it could be easily fixed by a slight modification of the definition. Still, it also does not satisfy (S3), for example, for $p = (0, 0), r = (1, 0), q = (4, 1)$, the subsegment from r to q is not contained in the segment from p to q.

Even though the set of axioms (S1)-(S4) seems rather natural, there are still some fairly exotic examples of digital segment systems that satisfy all four of them. For example, let us fix a double spiral D centered at an arbitrary point of $\mathbb{Z}^2$, traversing all the points of $\mathbb{Z}^2$. As it is a spanning path of the grid graph, we can set $S(p, q)$ to be the path between p and q on D, for every $p, q \in \mathbb{Z}^2$. It is easy to verify that this system satisfies axioms (S1)-(S4).

Another condition was introduced in [1] to enforce the monotonicity of the segments, ruling out pathological examples like the one above. Here, we phrase this monotonicity axiom differently, but still, the system of axioms (S1)-(S5) remains equivalent to the one given in [1].

(S5) *Monotonicity property:* If both $p, q \in \mathbb{Z}^2$ lie on a line that is either horizontal or vertical, then the whole segment $S(p, q)$ belongs to this line.

We call a system of digital line segments that satisfies the system of axioms (S1)-(S5) a *consistent digital line segments system (CDS)*. It is straightforward to verify that every CDS also satisfies the following three conditions.

(C1) If the slope of the line going through p and q is non-negative, then the slope of the line going through any two points of $S(p, q)$ is non-negative. The same holds for non-positive slopes.

(C2) For all $p, q \in \mathbb{Z}^2$, the grid-parallel box spanned by points p and q contains $S(p, q)$.

(C3) If the intersection of two digital segments contains two points $p, q \in \mathbb{Z}^2$, then their intersection also contains the whole digital segment $S(p, q)$.

We give a simple example of a CDS, where the segments follow the boundary of the grid-parallel box spanned by the endpoints. Let $p, q \in \mathbb{Z}^2$ be two points with coordinates $p = (p_x, p_y)$ and $q = (q_x, q_y)$. If $p_y \leq q_y$, we define $S(p, q) = S(q, p) = \{(x, p_y) : \min\{p_x, q_x\} \leq x \leq \max\{p_x, q_x\}\} \cup \{(q_x, y) : p_y \leq y \leq q_y\}\}$. If $p_y > q_y$, we swap the points p and q, and define the segment as in the previous case.

It can be easily verified that this way we defined a CDS, but the digital segments in this system visually still do not resemble well the Euclidean segments.

One of the standard ways to measure how close a digital segment is to a Euclidean segment is to use the Hausdorff distance. We denote by $\overline{pq}$ the Euclidean segment between p and q, and by $|\overline{pq}|$ the Euclidean length of $\overline{pq}$. For two plane objects A and B, by $H(A, B)$ we denote their Hausdorff distance.

The main question raised in [1] was if it is possible to define a CDS such that a Euclidean segment and its digitalization have a reasonably small Hausdorff distance. More precisely, the goal is to find a CDS satisfying the following condition.

(H) *Small Hausdorff distance property:* For every $p, q \in \mathbb{Z}^2$, we have that $H(\overline{pq}, S(p, q)) = O(\log |\overline{pq}|)$.

Note that in the CDS example we gave, the Hausdorff distance between a Euclidean segment of length n and its digitalization can be as large as $n/\sqrt{2}$.

While this question was not resolved in [1], a clever construction of a system of digital rays *emanating from the origin of* $\mathbb{Z}^2$ that satisfy (S1)-(S5) and (H) was presented. Moreover, it was shown using Schmidt's theorem [3] that already for rays emanating from the origin, the log-bound imposed in condition (H) is the best bound we can hope for, directly implying the following theorem.

Theorem 1 [1] *There exists a constant* $c > 0$*, such that for any CDS and any* $d > 0$*, there exist* $p, q \in \mathbb{Z}^2$ *with* $|\overline{pq}| > d$*, such that* $H(\overline{pq}, S(p,q)) > c \log |\overline{pq}|$.

In this paper, we introduce a novel and general approach for the construction of a CDS. Namely, for any total order $\prec$ on $\mathbb{Z}$, we show how to derive a CDS from $\prec$. (By total order we always mean a strict total order.) This process is described in Section 2. As a consequence, in Section 3, we manage to define a CDS that satisfies (H), deriving it from a specially chosen order on $\mathbb{Z}$, and thus giving the explicit construction that resolves the main question of [1].

Theorem 2 *There is a CDS that satisfies condition (H).*

Note that Theorem 1 ensures that such a CDS is optimal up to a constant factor in terms of the Hausdorff distance from the Euclidean segments. Finally, in Section 4, we make a step towards a characterization of CDSes, demonstrating their connection to total orders on $\mathbb{Z}$.

2 Digital line segments derived from a total order on $\mathbb{Z}$

Let $\prec$ be a total order on $\mathbb{Z}$. We are going to define a CDS $S_\prec$, deriving it from $\prec$. Let $p, q \in \mathbb{Z}^2$, $p = (p_x, p_y)$ and $q = (q_x, q_y)$. If $p_x > q_x$, we swap p and q. Hence, from now on we may assume that $p_x \leq q_x$.

If $p_y \leq q_y$, then $S_\prec(p,q)$ is defined as follows. We start at the point $p = (p_x, p_y)$ and we repeatedly go either up or to the right, collecting the points from $\mathbb{Z}^2$, until we reach q. Note that the sum of the coordinates $x + y$ increases by 1 in each step. In total we have to make $q_x + q_y - p_x - p_y$ steps and in exactly $q_y - p_y$ of them we have to go up. The decision whether to go up or to the right is made as follows: if we are at the point (x, y) for which $x + y$ is among the $q_y - p_y$ greatest elements of the interval $[p_x + p_y, q_x + q_y - 1]$ according to $\prec$, we go up, otherwise we go to the right. We will refer to this interval as the *segment interval*.

If $p_y > q_y$, that is, if p is the top-left and q the bottom-right corner of the grid-parallel box spanned by p and q, then we define $S_\prec(p,q)$ as the mirror reflection of $S_\prec((-q_x, q_y), (-p_x, p_y))$ in the y-axis.

Example. Suppose $p = (0,0)$ and $q = (2,2)$. Their segment interval consists of four numbers, $0, 1, 2, 3$. If $\prec$ is the natural order on $\mathbb{Z}$, then the two greatest elements of the segment interval are 2 and 3. Since $0 + 0$ is not one of these, at $(0,0)$ we go right, to $(1,0)$. At $(1,0)$ we again go to right, to $(2,0)$, from there to $(2,1)$ (since $2 + 0$ is one of the greater elements) and finally to $(2,2)$. In fact, it can be easily seen that using the natural order on $\mathbb{Z}$ we get the CDS mentioned in Section 1, the one that always follows the boundary of the box spanned by the endpoints.

Theorem 3 $\mathrm{S}_\prec$*, defined as above, is a CDS.*

Proof. The conditions (S1) and (S5) follow directly from the definition of $S_\prec$. It remains to verify the axioms (S2), (S3), and (S4).

(S2) Let p, q be two points from $\mathbb{Z}^2$. If the first coordinates of p and q are different, then condition (S2) follows directly. Otherwise, p and q belong to the same vertical line, and from the construction we see that both $S_\prec(p,q)$ and $S_\prec(q,p)$ consist of all the points on that line between p and q.

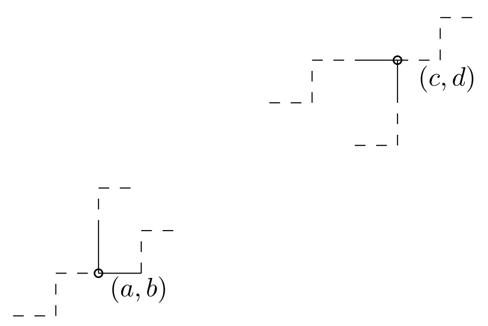

Figure 1: Two paths splitting up at (a, b) and meeting again at (c, d).

(S3) For a contradiction, assume that there are points $p = (p_x, p_y)$, $q = (q_x, q_y)$ and $r = (r_x, r_y)$, with $r \in S_\prec(p, q)$, such that $S_\prec(p, r) \not\subseteq S_\prec(p, q)$. W.l.o.g. we may assume that $\overline{pq}$ has a non-negative slope.

Case 1. $p_x \leq q_x$ and $p_y \leq q_y$. We also have $p_x \leq r_x$ and $p_y \leq r_y$, and going on each of the segments $S_\prec(p, r)$ and $S_\prec(p, q)$ point-by-point starting from p, we move either up or right. By assumption, these two segments separate at some point (a, b) and then meet again, for the first time after this separation, at some other point (c, d), see Figure 1. One of the segments goes up at (a, b) and enters (c, d) horizontally coming from the left, which implies that $a + b$ is among the greater numbers of the segment interval of this segment, while $c+d-1$ is not, thus $c+d-1 \prec a+b$. But the other segment goes horizontally at (a, b) and enters (c, d) vertically coming from below, which similarly implies $a + b \prec c + d - 1$, a contradiction.

Case 2. $q_x \leq p_x$ and $q_y \leq p_y$. We also have $q_x \leq r_x$ and $q_y \leq r_y$. By assumption, the two segments starting at q and r, $S_\prec(q, p)$ and $S_\prec(r, p)$, separate at some point (a, b) and then meet again, for the first time after this separation, at some other point (c, d). Using the same argument as before, we get a contradiction. Hence, (S3) holds.

(S4) To show that condition (S4) holds, consider the segment from $p = (p_x, p_y)$ to $q = (q_x, q_y)$. W.l.o.g. we can assume that $p_x \leq q_x$ and $p_y \leq q_y$. We distinguish two cases.

Case 1. If $q_x + q_y$ is among the $q_y - p_y + 1$ greatest numbers of $[p_x + p_y, q_x + q_y]$ according to $\prec$, then we can prolong the segment going one step vertically up, that is, the segment $S_\prec((p_x, p_y), (q_x, q_y + 1))$ contains the segment $S_\prec((p_x, p_y), (q_x, q_y))$ as a subsegment.

Case 2. If, on the other hand, $q_x + q_y$ is not among the $q_y - p_y$ greatest numbers of $[p_x + p_y, q_x + q_y]$, we can prolong the segment horizontally to the right, that is, $S_\prec((p_x, p_y), (q_x, q_y)) \subset S_\prec((p_x, p_y), (q_x + 1, q_y))$.

Note that if $q_x + q_y$ is exactly the $(q_y - p_y + 1)^{th}$ number in $[p_x + p_y, q_x + q_y]$, then the conclusions of both cases are true, and indeed the rays emanating from (p_x, p_y) split at (q_x, q_y). $\square$

Remark 4 *Even though our focus is on the digitalization of line segments, the present setup can be conveniently extended to a definition of digital lines. We say that a digital line is a path infinite in both directions in the $\mathbb{Z}^2$ base graph, such that the digital line segment between any two points on the digital line belongs to the digital line. It can be shown (using an argument similar to the proof of (S3) in Theorem 3) that there are no "touching" lines in* $\mathrm{S}_\prec$ *– if two different lines have a common point, then they either cross (having a common segment), or they have a common half-line.*

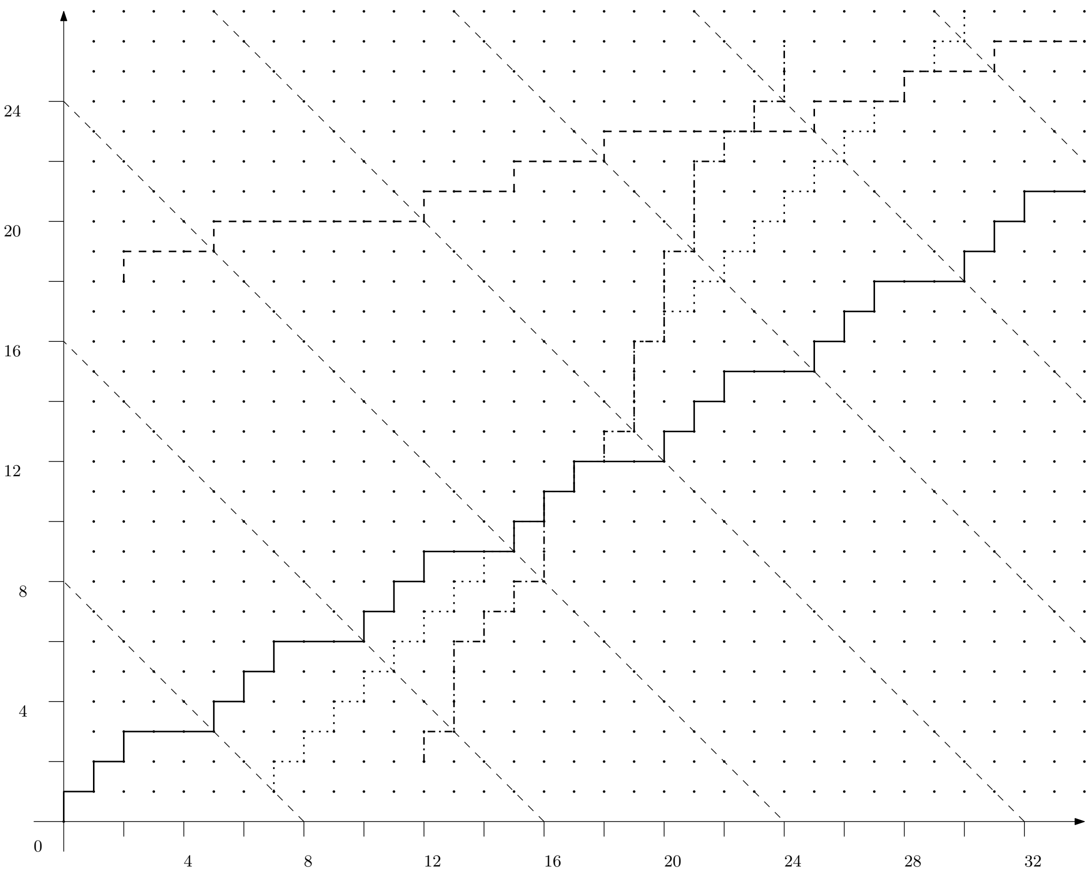

Figure 2: Some line segments.

3 Digital segments with small Hausdorff distance to Euclidean segments

For integers k and $l \geq 2$, let $|k|_l$ denote the number of times k is divisible by l, that is,

$$|k|_l = \sup\{m : l^m \mid k\}.$$

We define a total order on $\mathbb{Z}$ as follows. Let $a \prec b$ if and only if there exists a non-negative integer i such that $|a-i|_2 < |b-i|_2$, and for all $j \in \{0,\ldots,i-1\}$ we have $|a-j|_2 = |b-j|_2$. In plain words, for two integers a and b, we say that the one that contains a higher power of 2 is greater under $\prec$. In case of a tie, we repeatedly subtract 1 from both a and b, until at some point one of them contains a higher power of 2 than the other. Thus, for example, $-1 \prec -5 \prec 3 \prec -3 \prec 5 \prec 1 \prec -2 \prec 6 \prec -6 \prec 2 \prec 4 \prec -4 \prec 0$. Note that if we take the elements of an interval of the form $(-2^n, 2^n)$ in $\prec$-decreasing order and we apply the function $0.5 - x2^{-n-1}$ to them, then we get the first few elements of the Van der Corput sequence [4].

We will prove that using this total order to define the system of digital line segments $S_\prec$, as described in the previous section, we obtain a CDS which satisfies condition (H). In Figure 2 we give some examples of digital segments in this CDS, and Figure 3 shows the segments emanating from $(0,0)$ to some neighboring points, as well as the segments from $(2,3)$ to the neighboring points.

At first sight it may be surprising to observe that all the segments emanating from the origin in our construction coincide with the ones given in the construction of digital rays in [1]. However, it is not a coincidence, as the construction from [1] also relies on the same total order

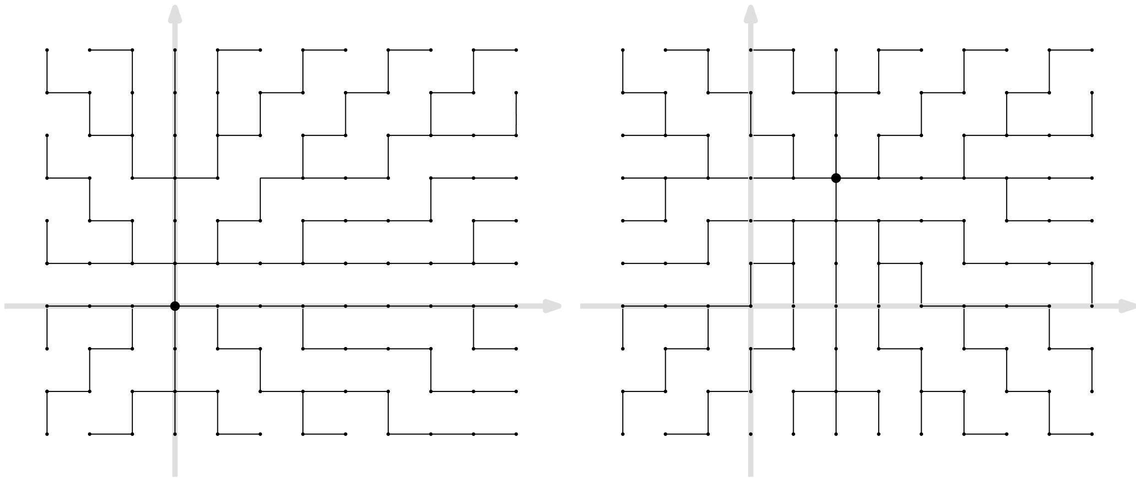

Figure 3: Digital line segments emanating from $(0,0)$ and from $(2,3)$.

on integers.

For points $v, w \in \mathbb{R}^2$ and $A \subseteq \mathbb{R}^2$, let $d(v,w) = |v-w|$ and $d(v,A) = \inf_{a\in A} d(v,a)$ denote the usual Euclidean distances between two points, and between a point and a set. For $p,q,r,s \in \mathbb{Z}^2$, by $\overline{pqrs}$ we denote the union of Euclidean linear line segments from p to q, from q to r, and from r to s.

Observation 5 *For any $p,q \in \mathbb{Z}^2$, $H(S_\prec(p,q), \overline{pq}) = max\{d(r,\overline{pq}) : r \in S_\prec(p,q)\}$.*

We proceed by proving two statements that we will use to ultimately prove Theorem 2.

Lemma 6 *If $p,q \in \mathbb{Z}^2$ and $r,s \in S_\prec(p,q)$. Then $H(\overline{rs}, S_\prec(r,s)) \leq 2H(\overline{pq}, S_\prec(p,q))$.*

Proof. We know that $d(r,\overline{pq}) \leq H(\overline{pq}, S_\prec(p,q)) =: h$ and $d(s,\overline{pq}) \leq h$, therefore $H(\overline{prsq},\overline{pq}) \leq h$. Hence, for all $v \in \overline{pq}$, $d(v,\overline{prsq}) \leq h$. Let $t \in S_\prec(r,s) \subseteq S_\prec(p,q)$ and $v \in \overline{pq}$ be such that $d(t,v) = d(t,\overline{pq}) \leq h$. Using the triangle inequality we conclude $d(t,\overline{prsq}) \leq d(t,v) + d(v,\overline{prsq}) \leq 2h$. Because of (C2), we have $d(t,\overline{rs}) = d(t,\overline{prsq}) \leq 2h$, and therefore $H(S_\prec(r,s),\overline{rs}) \leq 2h$. $\square$

Lemma 7 *Let $p,q,r,r' \in \mathbb{Z}^2$, such that $r_x - p_x = q_x - r'_x + \varepsilon$, $r_y - p_y = q_y - r'_y - \varepsilon$, with $\varepsilon \in \{0,1,-1\}$, $r'_x = r_x$ and $r'_y = r_y + 1$. Then $H(\overline{pq}, \overline{prr'q}) \leq c = \sqrt{5}/2$.*

Proof. Without loss of generality $p = (0,0)$. We have

$$H(\overline{pq}, \overline{prr'q}) = \max\{d(r,\overline{pq}), d(r',\overline{pq})\}.$$

By assumption $2r_x - \varepsilon = q_x$ and $2r_y + \varepsilon + 1 = q_y$. So we get

$$d(r,\overline{pq}) = \frac{q_y r_x - q_x r_y}{\sqrt{q_x^2+q_y^2}} = \frac{q_y \frac{q_x+\varepsilon}{2} - q_x \frac{q_y-\varepsilon-1}{2}}{\sqrt{q_x^2+q_y^2}} = \frac{1}{2\sqrt{q_x^2+q_y^2}}(q_y\varepsilon + q_x\varepsilon + q_x).$$

Similarly,

$$d(r',\overline{pq}) = \frac{1}{2\sqrt{q_x^2+q_y^2}}(q_y\varepsilon + q_x\varepsilon - q_x).$$

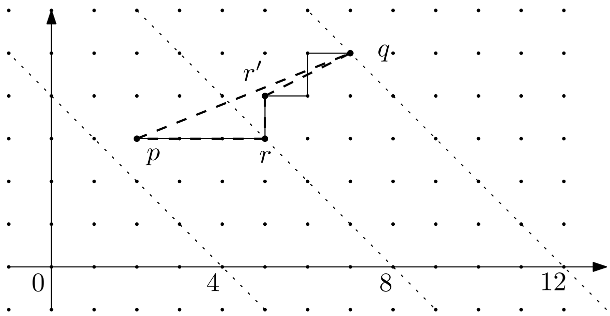

Figure 4: The digital line segment from $p = (2,3)$ to $q = (7,5)$ with the ordered segment interval: $8 \succ 10 \succ 6 \succ 9 \succ 5 \succ 11 \succ 7$.

Setting $x := q_x/q_y$, we observe

$$H(\overline{pq}, \overline{prr'q}) \leq \frac{1}{2\sqrt{(xq_y)^2 + q_y^2}}(q_y + 2xq_y) = \frac{x + 1/2}{\sqrt{x^2+1}} \leq \sqrt{5}/2.$$

□

Proof. (of Theorem 2) Let $p, q \in \mathbb{Z}^2$. We may assume that $p_x < q_x$ and $p_y < q_y$. We are going to prove that $H(\overline{pq}, S_\prec(p,q)) \leq 2c\log(p_x + p_y - q_x - q_y)$ for $c = \sqrt{5}/2$. Let $r \in S_\prec(p,q)$ be the point with the property that $r_x + r_y$ is the greatest element of the segment interval, that is, $r_x + r_y \succ s$ for all $s \in [p_x + p_y, q_x + q_y)$, $s \neq r_x + r_y$, see Figure 4 for an example. Now let s' be the second greatest element of the segment interval according to $\prec$. Define $k := |s'|_2 + 1$.

We can extend the segment $S_\prec(p,q)$ over both endpoints, moving both p and q such that $|p_x + p_y - 1|_2 \geq k$ and $|q_x + q_y|_2 \geq k$, that is, we extend the segment as far as we can, so that k, defined as above, remains unchanged. From Lemma 6 we get that by this extension we decreased the Hausdorff distance by at most a factor of 2. Now the segment interval contains exactly $2^{k+1} - 1$ elements and $r_x + r_y$ is the element in the very middle. We call such a segment *normalized.*

We are going to proceed by induction on k to prove that for all normalized digital line segments $H(\overline{pq}, S_\prec(p,q)) \leq ck$ with $c = \sqrt{5}/2$. This will prove the theorem, as $k + 1 = \log(p_x + p_y - q_x - q_y + 1)$, the distance of the unnormalized original segment (we started from) is at most $2ck = 2c(\log(p_x + p_y - q_x - q_y + 1) - 1) \leq 2c\log(p_x + p_y - q_x - q_y)$.

In the base case $k = 1$, the segment interval consists of 3 numbers, so $S_\prec(p,q)$ is a path of length 3 and by checking all possibilities we see that $H(\overline{pq}, S_\prec(p,q)) < c$.

If $k > 1$, the idea is to split the segment at r into two subsegments which are similar in some sense and apply induction. Let $r' = (r_x, r_y + 1)$ be the point that comes after r in the segment $S_\prec(p,q)$. (We know that we go up at r, because we go up at least once and $r_x + r_y$ is the greatest element of the segment interval). Consider the subsegments $S_\prec(p,r)$ and $S_\prec(r',q)$ and partition the segment interval accordingly. The key observation is that picking the elements of the interval according to $\prec$ starting with the greatest, we first get r, and then alternately an element of the left and the right subsegment interval. Therefore, up to a difference of at most one, half of the $q_y - p_y - 1$ greatest elements (after $r_x + r_y$) belong to $[p_x + p_y, r_x + r_y)$ and half of them to $[r_x + r_y + 1, q_x + q_y)$. This implies that p, q, r, r' meet the conditions of Lemma 7, leading to $H(\overline{pq}, \overline{prr'q}) \leq c$. By the induction hypothesis we have $H(\overline{pr}, S_\prec(p,r)) \leq c(k-1)$ and $H(\overline{r'q}, S_\prec(r',q)) \leq c(k-1)$. Now $H(\overline{prr'q}, S_\prec(p,q)) = \max\{H(\overline{pr}, S_\prec(p,r)), H(\overline{r'q}, S_\prec(r',q))\} \leq c(k-1)$. Using the triangle inequality we conclude $H(\overline{pq}, S_\prec(p,q)) \leq ck$. □

4 A step towards a characterization of CDSes

Now we approach the same problem from a different angle, taking arbitrary CDSes and trying to find some common patterns in their structure. Knowing that condition (C1) holds for all CDSes, it is easy to verify that we can analyze the segments with non-positive and non-negative slopes separately, as they are completely independent. More precisely, the union of any CDS on segments with non-positive slopes and another CDS on segments with non-negative slopes is automatically a CDS. Having this in mind, in this section we will proceed with the analysis of only one half of the CDS, namely, of segments with non-negative slope.

We will show that, in a CDS, all the segments with non-negative slope emanating from one fixed point must be derived from a total order. However, as we will show later, these orders may differ for different points. Earlier, we used that these orders cannot be arbitrary when we proved Case 2 of (S3) in the proof of Theorem 3. Remember that when we defined $S_{\prec}(p,q)$ for $p_x \leq q_x$ in Section 2, we started from p and went towards q. Note that we could have done just the opposite, start from q and go towards p. This way we can define the rays emanating from p to both directions using only the total order that belongs to p.

Proposition 8 *For any CDS and for any point $p = (p_x, p_y) \in \mathbb{Z}^2$, there is a total order $\prec_p$ that is uniquely defined on both $(-\infty, p_x + p_y - 1]$ and $[p_x + p_y, +\infty)$, such that the segments with non-negative slope emanating from p are derived from $\prec_p$ (in the way described in Section 2).*

Proof. We fix a CDS S and a point p. The segments with non-negative slope with $p = (p_x, p_y)$ as their upper-right point will induce an order on the integers smaller than $p_x + p_y$, the segments for which p is the lower-left endpoint will induce an order on the rest of the integers. In the following we will just look at the latter type of segments.

First we will show that it cannot happen that for two segments with non-negative slope having p as their lower-left endpoint, one of them goes up at $(a, C-a)$ and the other goes right at $(b, C-b)$, for some C and $a > b$. Let us, for a contradiction, assume the opposite, see Figure 5. We look at the $a-b+1$ segments between p and each of the points on the line $x+y=C$ between the points $(a, C-a)$ and $(b, C-b)$. It is possible to extend all of them through their upper-right endpoints, applying (S4). Note that each of the extended segments goes through a different point on the line $x+y=C$, and hence, because of condition (C3), no two of them can go through the same point on the line $x+y=C+1$. But, there are only $a-b$ available points on the line $x+y=C+1$ between the points $(a, C-a+1)$ and $(b+1, C-b)$, one less than the number of segments, a contradiction.

Now, we define the order $\prec_p$ in the following way. Whenever there is a segment in S starting at p going right at a point $(x, D-x)$ and going up at a point $(x', E-x')$ we set $D \prec_p E$. Assume that for some integers D and E we run into a conflict determining which one is greater in the order. That can happen only when there are two segments with non-negative slope having p as their lower-left endpoint, such that on the line $x+y=D$ one of them goes up, the other right, and then on $x+y=E$ they both go in different direction than at $x+y=D$. But then the situation described in the previous paragraph must occur on one of the two lines, a contradiction.

If $C \prec_p D$ and $D \prec_p E$, then we also have $C \prec_p E$ – we just take a segment starting from p that goes right at C and up at D, and (if necessary) extend it until it passes the line $x+y=E$. It must also go up at E because of $D \prec_p E$. This already proves that $\prec_p$ is a partial order on $[p_x + p_y, +\infty)$.

It remains to prove that it is in fact a total order. That is, for any pair of integers $p_x + p_y \leq D < E$, either $D \prec_p E$ or $E \prec_p D$ holds. Consider a segment from p to some point q on the line $x+y=E$, such that this segment splits at q, that is, there are two extensions of the segment,

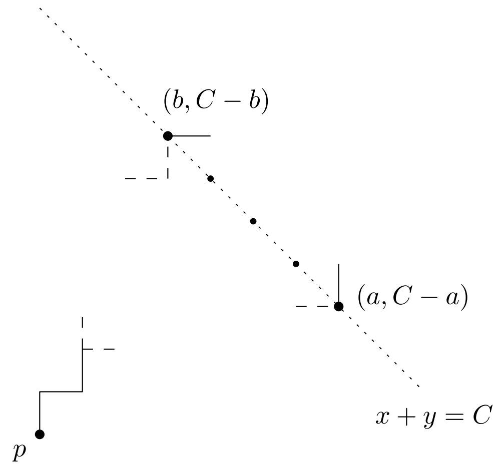

Figure 5:

one going up and another one going right. That segment exists since in the upper-right quadrant of p, the line $x+y=E+1$ contains one more point than the line $x+y=E$. If we look at all the segments between p and the points on $x+y=E+1$, the pigeonhole principle ensures that two of them will contain the same point q on the line $x+y=E$. Now the segment $S(p,q)$ crosses the line $x+y=D$ at some point q'. Depending on whether it goes up or right at the point q', either $E \prec_p D$ or $D \prec_p E$ holds. □

To see that these orders can differ for different points, consider the following example of a CDS, which we call the *waterline example* because of the special role of the x-axis. To connect two points with a segment, we do the following. Above the x-axis we go first right, then up, below the x-axis we go first up, then right, and when we have to traverse the x-axis, we go straight up to it, then travel on it to the right, and finally continue up, see Figure 6. It is easy to check that this construction satisfies all five axioms.

Now, if we consider a point $p=(a,b)$ below the waterline, $b<0$, the induced total order $\prec_p$ on $[a+b,+\infty)$ is $a \prec_p a+1 \prec_p \ldots \prec_p (+\infty) \prec_p a-1 \prec_p a-2 \prec_p \ldots \prec_p a+b$, and the order on $(-\infty, a+b-1]$ is $a+b-1 \prec_p a+b-2 \prec_p \ldots \prec_p -\infty$. If p is above the waterline, $b \geq 0$, the induced total orders are $a-1 \prec a-2 \prec \ldots \prec (-\infty) \prec a \prec a+1 \prec \ldots \prec a+b-1$ and $a+b \prec a+b+1 \prec \ldots \prec +\infty$. Obviously, there is no total order on $\mathbb{Z}$ compatible with these orders for all possible choices of p.

The special role played by the x-axis in the waterline example can be fulfilled by any other monotone digital line with a positive slope; above the line go right, then up, below the line go up, then right, and whenever the line is hit, follow it until either the x- or the y-coordinate matches that of the final destination, see Figure 7.

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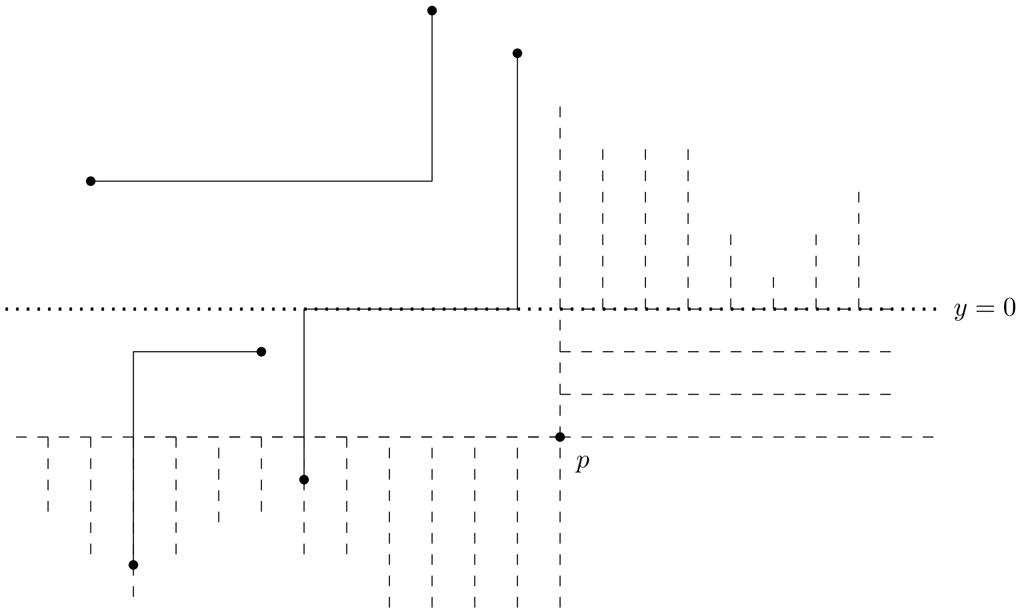

Figure 6: The waterline example: Examples of three characteristic segments, and the rays emanating from a point p, which is below the waterline

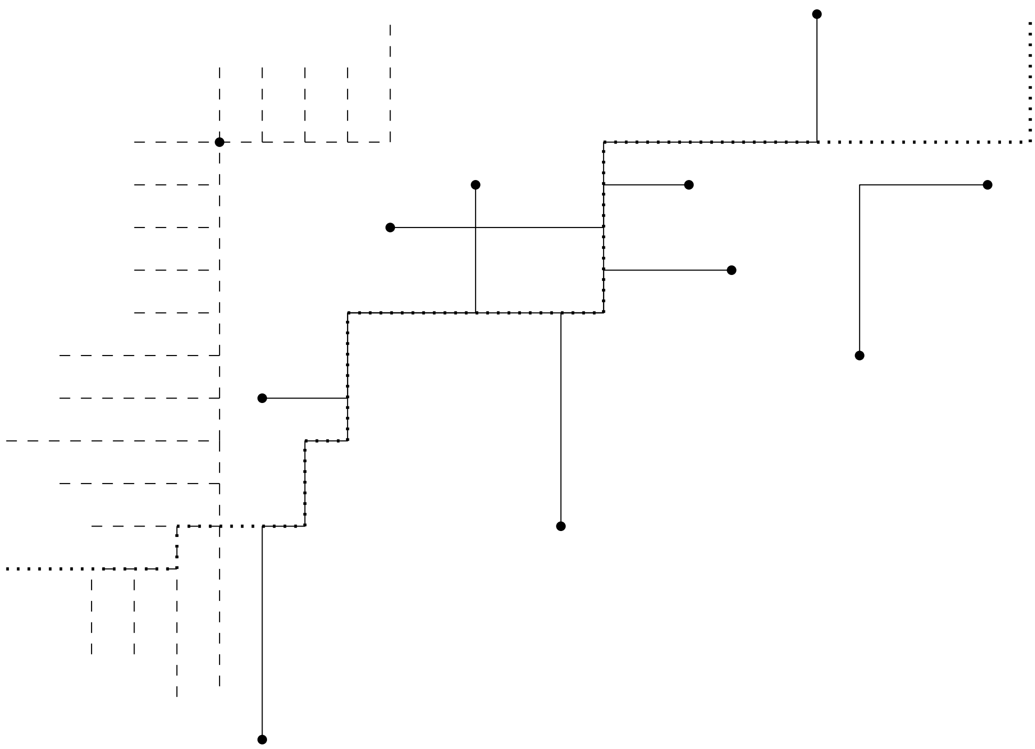

Figure 7: A more exotic example of a CDS with an arbitrary "special" line (bold and dotted).

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