Fast winning strategies in Maker-Breaker games

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Abstract

We consider unbiased Maker-Breaker games played on the edge set of the complete graph K_n on n vertices. Quite a few such games were researched in the literature and are known to be Maker's win. Here we are interested in estimating the minimum number of moves needed for Maker in order to win these games.

We prove the following results, for sufficiently large n: (1) Maker can construct a Hamilton cycle within at most $n + 2$ moves. This improves the classical bound of $2n$ due to Chvátal and Erdős [6] and is almost tight; (2) Maker can construct a perfect matching (for even n) within $n/2 + 1$ moves, and this is tight; (3) For a fixed $k \geq 3$, Maker can construct a spanning k-connected graph within $(1 + o(1))kn/2$ moves, and this is obviously asymptotically tight. Several other related results are derived as well.

1 Introduction

Let $\mathcal{F}$ be a hypergraph. In an unbiased Maker-Breaker game $\mathcal{F}$ two players, called Maker and Breaker, take turns in selecting previously unselected vertices of $\mathcal{F}$, with Maker going

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first. Each player selects one vertex per turn, until all vertices are selected. Maker wins if he claims all the vertices of some hyperedge of $\mathcal{F}$; otherwise Breaker wins. In this paper our attention is restricted to games which are played on the edges of the complete graph on n vertices, that is, the vertex set of $\mathcal{F}$ will always be $E(K_n)$.

For quite a few Maker-Breaker games, it is rather easy to determine the identity of the winner. For example, it is not hard to see that Maker easily wins the *connectivity game*, in which Maker's goal is to occupy a connected and spanning subgraph. The *non-planarity game*, where Maker's goal is to create a non-planar graph is an even more convincing example — for $n \geq 11$, Maker creates a non-planar graph by the end and thus wins the game irregardless of his strategy for the prosaic reason that every graph with more than $3n-6$ edges on n vertices is non-planar. Thus, for games of this type, a more interesting question to ask is not *who* wins but rather *how long does it take* the winner to reach a winning position. This is the type of question we address in this paper.

For a hypergraph $\mathcal{F}$, let $\tau(\mathcal{F})$ denote the smallest integer t such that Maker has a strategy to win the game on $\mathcal{F}$ within t moves (for the sake of completeness, we define $\tau(\mathcal{F}) = \infty$ if the game is a Breaker's win). For example, it is easy to see that for the hypergraph $\mathcal{T}_n$ of the connectivity game (whose hyperedges are the spanning trees of K_n), we have $\tau(\mathcal{T}_n) = n-1$. The lower bound is trivial since Maker needs to occupy the $n-1$ edges of a spanning tree. For the upper bound, any strategy of Maker that does not call for occupying a cycle will do. Indeed, if Maker keeps maintaining a forest, then Breaker "does not have time" to fully occupy a cut, as occupying a cut would require $k(n-k) \geq n-1$ moves of Breaker, but by then Maker would already win by extending his forest to a spanning tree.

Several other results about fast wins in Maker-Breaker games appear in the literature. It is known that, playing on the edges of K_n, Maker can build a q-clique in a constant (depending on q, but not on n) number of moves, that is, $\tau(\mathcal{K}_n^q) = f(q)$, where the hyperedges of $\mathcal{K}_n^q$ are the q-cliques of K_n. The best upper bound, $f(q) = O((q-3)2^{q-1})$ is due to Pekeč [12]. Beck proved that the exponential dependency on q cannot be avoided, namely $f(q) = \Omega(\sqrt{2}^q)$ (see [3]). Note that Maker's strategy for the clique game provides him with a fast win in the non-planarity game and the non-r-colorability game as well, via building a copy of K_5 and K_{r+1}, respectively (for background on these games, see [10]). In [1] Beck discusses games played on almost disjoint n-uniform hypergraphs and proves that Breaker can always avoid losing for at least $2^{n-o(n)}$ moves. Beck also notes that his result is essentially tight: playing on the 3-chromatic almost disjoint n-uniform hypergraph constructed by Erdős and Lovász [8] on $n^4 2^n$ vertices, Maker wins and consequently does so in at most $n^4 2^{n-1}$ moves.

A general sufficient condition for Breaker's win in Maker-Breaker games was proved in [2]; it is based on the "potential function" method of Erdős and Selfridge [9]. This criterion, however, does not seem to be very useful for proving results concerning winning fast, as it is assumed that the game is played until every element of the board is claimed by some player. Nonetheless, using the "fake moves" trick (see [3]), it can be applied to get certain, usually rather weak, results. In this paper, in order to obtain stronger results, we will not rely on this criterion, but will rather use ad-hoc methods.

1.1 Our results

In [6], Chvátal and Erdős studied the *Hamilton cycle game*, where Maker's goal is to occupy the edges of a Hamilton cycle. They proved that Maker can win the Hamilton cycle game on K_n within $2n$ rounds. Here we show that, for sufficiently large n, Maker can win this game much sooner, namely, he is able to build a Hamilton cycle within $n+2$ rounds. This bound is now only 1 away from the obvious lower bound. Indeed, in order to build a Hamilton cycle in n moves, Maker must build a Hamilton path by his $(n-1)$st move. But then, Breaker can claim the unique edge that closes this path into a cycle. Formally, define $\mathcal{H}_n$ to be the hypergraph whose hyperedges are the Hamilton cycles of K_n.

Theorem 1.1 *For sufficiently large n, we have*

$$n+1 \leq \tau(\mathcal{H}_n) \leq n+2.$$

The first phase of the strategy of Maker in Theorem 1.1 constitutes of building a perfect matching fast. This result is of independent interest, so we state it separately. Let $\mathcal{M}_n$ be the hypergraph whose hyperedges are the perfect matchings of K_n (or matchings that cover every vertex but one, if n is odd). Let $\mathcal{D}_n$ be the hypergraph whose hyperedges are the spanning subgraphs of K_n of positive minimum degree. We find the *exact* number of moves that Maker needs in order to win the games $\mathcal{M}_n$ and $\mathcal{D}_n$. Obviously, Maker needs to make at least $\lfloor \frac{n}{2} \rfloor$ moves to win the $\mathcal{M}_n$ game, as this is the size of a minimal element of $\mathcal{M}_n$. We show that if n is odd, then he does not need more moves, whereas if n is even, then he needs just one more move. A similar result showing the tightness of the obvious lower bound for the minimum degree game $\mathcal{D}_n$, easily follows.

Theorem 1.2

$$\tau(\mathcal{M}_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \textit{if } n \textit{ is odd} \\ \frac{n}{2}+1 & \textit{if } n \textit{ is even} \end{cases}$$

Corollary 1.3

$$\tau(\mathcal{D}_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Another byproduct of the proof of Theorem 1.1, is that Maker can build a Hamilton path in $n-1$ moves, which is clearly best possible.

Theorem 1.4 *For sufficiently large n, we have*

$$\tau(\mathcal{HP}_n) = n-1,$$

where $\mathcal{HP}_n$ is the hypergraph whose hyperedges are the Hamilton paths of K_n.

Let $\mathcal{C}_n^k$ be the hypergraph whose hyperedges are all spanning k-vertex-connected subgraphs of K_n. As we discussed in the introduction, Maker can build a 1-connected spanning graph in $n-1$ moves. From Theorem 1.1 it follows that Maker can build a 2-vertex-connected spanning graph by using just 3 more moves (that is, a total of $n+2$ moves).

In the following, we obtain a generalization of the latter fact for every $k \geq 3$. As every k-connected graph has minimum degree at least k, Maker needs at least $kn/2$ moves just for claiming an element of $\mathcal{C}_n^k$, even if Breaker does not play at all. The next theorem shows that this trivial lower bound is asymptotically tight, that is, there is a strategy for Maker to build a k-vertex-connected graph in $kn/2 + o_k(n)$ moves.

Theorem 1.5 *For every fixed $k \geq 3$ and sufficiently large n, we have*

$$kn/2 \leq \tau(\mathcal{C}_n^k) \leq kn/2 + (k+4)(\sqrt{n} + 2n^{2/3}\log n).$$

An interesting and a somewhat unusual feature of our proof of Theorem 1.5 is that, similarly to an argument of Bednarska and Łuczak from [4], the existence of a winning strategy for Maker is obtained via probabilistic tools (though the strategy itself is deterministic, which is always the case with positional games).

An easy consequence of Theorems 1.2, 1.1 and 1.5, is that for every fixed $k \geq 1$, Maker can build a graph with minimum degree at least k within $(1+o(1))kn/2$ moves. This is also clearly asymptotically optimal.

1.2 Preliminaries

For the sake of simplicity and clarity of presentation, we omit floor and ceiling signs whenever these are not crucial. Some of our results are asymptotic in nature and, whenever necessary, we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [7]. In particular, we use the following: for a graph G, denote its set of vertices by $V(G)$, and its set of edges by $E(G)$. Moreover, let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a graph $G = (V, E)$ and a set $A \subseteq V$ denote by $G[A]$ the subgraph of G induced by A. Let $N_G(A) = \{u \in V : \exists w \in A, (u, w) \in E\}$ be the neighborhood of A in G and let $\Gamma_G(A) = N_G(A) \setminus A$ be the external-neighborhood of A in G. Sometimes, when there is no risk of confusion, we abbreviate $N_G(A)$ to $N(A)$ and $\Gamma_G(A)$ to $\Gamma(A)$.

2 Fast strategies for Maker

In our definition of Maker-Breaker games, Maker starts the game. In the following, whenever proving a result of the form $\tau(\mathcal{F}) \leq a$, we will assume that Breaker starts the game (thus proving a statement which is stronger than the one asserted in the corresponding theorem).

2.1 Building a perfect matching fast

Proof of Theorem 1.2.

Assume first that n is even. Obviously Maker needs at least $n/2$ edges to build a perfect matching. In fact he will need at least one more, as Breaker, seeing the first $n/2-1$ moves of Maker, can occupy the unique edge (if no such edge exists, then our claim immediately follows) which would extend Maker's graph into a perfect matching. Hence $\tau(\mathcal{M}_n) \geq \frac{n}{2}+1$.

In the following we assume that Breaker starts the game and give a strategy for Maker to build his perfect matching in $\frac{n}{2}+1$ moves. A *round* of the game consists of a move by Breaker and a counter move by Maker. A vertex is considered *bad*, if it is isolated in Maker's graph but not in Breaker's graph.

We will provide Maker with a strategy to ensure that for every $3 \leq r \leq \frac{n}{2}$, the following three properties hold after his rth move:

(a) Maker's edges form a forest consisting of $r-1$ components: a path uvw of length two and $r-2$ paths of length one;

(b) every isolated vertex of Maker's graph is adjacent to neither u nor w in Breaker's graph;

(c) there are at most two bad vertices.

First, let us see that, if these properties hold after Maker's $\frac{n}{2}$th move, then Maker wins the perfect matching game on his next move. Observe that by property (a) after the $\frac{n}{2}$th move of Maker there is exactly one isolated vertex z in Maker's graph, which, by property (b), is connected to neither u nor w in Breaker's graph. Hence, no matter which edge Breaker claims in his $(\frac{n}{2}+1)$st move, Maker will be able to respond by claiming either (u,z) or (w,z). After that move Maker's graph is a spanning forest consisting of a path of length three and $\frac{n}{2}-2$ paths of length one; clearly such a graph contains a perfect matching.

Next, we prove that for every $n \geq 6$, Maker can maintain properties $(a)-(c)$. First, it is easy to see that Maker can execute his first three moves such that these three properties hold.

We will prove that on his rth move, where $\frac{n}{2} \geq r > 3$, Maker can select two vertices that are isolated in his graph and connect them by an edge, while ensuring that, right after his move, properties (b) and (c) hold. Note that this strategy automatically ensures that property (a) holds as well.

Let I_r be the set of vertices which are isolated in Maker's graph after the rth round. Property (a) ensures that $|I_r| = n - (2r - 1)$ and property (c) implies that there are at most two vertices in I_r which are not isolated in Breaker's graph; in particular there is at most one edge of Breaker with both endpoints in I_r. Assume that the rth round, where $r \leq n/2 - 1$, has just ended, then $|I_r| \geq 3$.

In case Breaker claims an edge of the form (x, u) or (x, w) where $x \in I_r$, then Maker responds by claiming an edge (x, y) where $y \in I_r$. Such a vertex y for which the edge (x, y) was not previously claimed by Breaker always exists as only one of Breaker's edges is spanned by I_r, and there are at least three vertices in I_r. Since the vertex x will not be bad at the end of the $(r + 1)$st round, the number of bad vertices does not increase and property (c) remains valid. Property (b) will also remain valid because the only new vertex which could dissatisfy it, x, is not isolated in Maker's graph anymore.

If Breaker does not claim an edge of the form (x, u) or (x, w), where $x \in I_r$, then Maker responds by claiming an edge with both endpoints in I_r such that property (c) remains valid. This can easily be done as there are at most two edges of Breaker with both endpoints in I_r, and $|I_r| \geq 3$. Property (b) was not affected by Breaker's move.

This concludes our description of Maker's strategy and the proof if n is even.

If n is odd, then Maker's strategy is essentially the same as his strategy for even n (in fact it is a little simpler). The main difference is that property (b) is redundant, property (a) is replaced with:

(a') After Maker's rth round, his graph is a matching with r edges,

and we do not need to consider separately, Maker's first three moves. We omit the straightforward details. $\square$

Proof of Corollary 1.3. It is clear that $\tau(\mathcal{D}_n) \geq \lfloor n/2 \rfloor + 1$. Furthermore, if n is even, then by Theorem 1.2 we get $\tau(\mathcal{D}_n) \leq \tau(\mathcal{M}_n) = n/2 + 1$. If n is odd, then Maker can build a matching that covers all vertices but one in $\lfloor n/2 \rfloor$ rounds, and then claim an arbitrary edge incident with the last remaining isolated vertex. Hence, we get $\tau(\mathcal{D}_n) = \lfloor n/2 \rfloor + 1$ as claimed. $\square$

2.2 Building a Hamilton cycle fast

Proofs of Theorem 1.1 and Theorem 1.4.

In the proof, we use the method of Pósa rotations (see [13]). Let $P_0 = (v_1, v_2, \ldots, v_l)$ be a path of maximum length in a graph G. If $1 \leq i \leq l - 2$ and (v_l, v_i) is an edge of G then

$P' = (v_1, v_2, \ldots, v_i, v_l, v_{l-1}, \ldots, v_{i+1})$ is also of maximum length. We can then, in general, rotate P' to get more maximum length paths.

We will assume that Breaker starts the game. A *round* consists of a move by Breaker and a counter move by Maker. Assume first that n is even. Maker's strategy is divided into three stages.

In the first stage, Maker builds a perfect matching with one additional edge, that is, he builds a path of length 3 and $(n-4)/2$ paths of length 1. From Theorem 1.2 we know that Maker can do this in $n/2+1$ moves.

In the second stage, which lasts exactly $n/2-2$ rounds, Maker connects endpoints of the paths in his graph. In each move he connects two paths to form one longer path. Hence, in each round he decreases the number of paths by one, and thus, by the end of the second stage he will have a Hamilton path.

For every $0 \leq i \leq n/2-3$, let B'_i be the subgraph of Breaker's graph, induced by the endpoints of Maker's paths, just after the $(i+1)$st move of Breaker in the second stage (recall that Breaker starts the second stage). Let B_i be the graph obtained from B'_i by removing all edges (x,y) such that x and y are endpoints of the same path of Maker. The unclaimed edges $(x,y) \in \binom{V(B_i)}{2}$, for which x and y are endpoints of different paths of Maker are called *available*.

The first move of Maker in this stage is somewhat artificial, thinking ahead about stage three. Let $w \in V(B_0)$ be a vertex of maximum degree in Breaker's graph. On his first move of the second stage Maker claims an arbitrary available edge incident with w. Such an edge exists if n is large enough, since Breaker has $n/2+2$ edges, while there are $n-2$ endpoints in $V(B_0)$. Note that for any two vertices $z', z'' \in V(B_1)$, the sum of the degrees of z' and z'' in Breaker's graph is at most $n/3+4$ (we will use this observation only in stage three).

Maker's goal is now the following: he will make sure that $e(B_i) \leq v(B_i)-1$ for every $1 \leq i \leq n/2-3$. This easily holds for $i=1$ provided n is large enough. Assume that the statement holds for some $1 \leq i \leq n/2-4$ and let us prove that Maker can claim an available edge while ensuring that $e(B_{i+1}) \leq v(B_{i+1})-1$.

Case 1.j. (for every $0 \leq j \leq 3$). $e(B_i) \leq v(B_i)-1-j$ and there is an available edge incident with at least $3-j$ edges of B_i. Maker claims this edge entailing $e(B_{i+1}) \leq e(B_i)-(3-j)+1 \leq v(B_i)-3 = v(B_{i+1})-1$.

Case 2. There is a vertex v of degree at least 3 in B_i. Hence by Case 1.0 we can assume that there is no available edge incident with v, that is, the degree of v in B_i is exactly $v(B_i)-2$ (recall that there are no edges in B_i between the endpoints of the same path of Maker). Note that by the induction hypothesis there is at most one edge in B_i which is not incident with v. Since $i \leq n/2-4$, $v(B_i) \geq 6$, and so v has at least four neighbors in B_i.

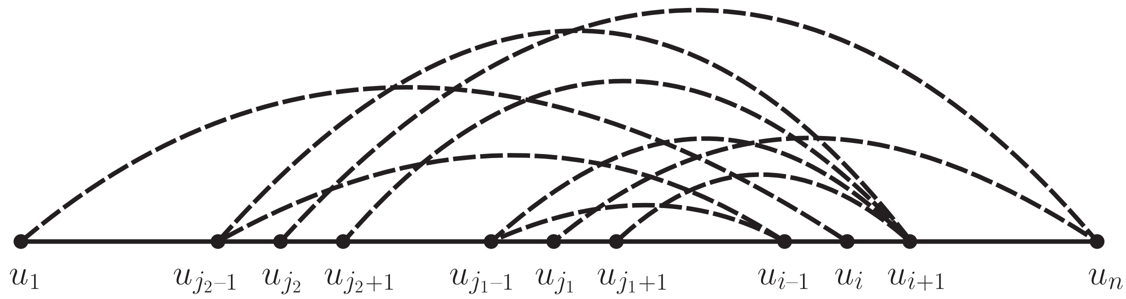

Figure 1: Dashed edges are unclaimed by Breaker.

Assume first that every edge of B_i is incident with v, entailing $e(B_i) = v(B_i) - 2$. Among the four neighbors of v there has to be at least one available edge. This edge is incident with two edges of Breaker and so Case 1.1 applies.

Suppose now that there is an edge of B_i which is not incident with v. One of its endpoints z is a neighbor of v. Hence, since $v(B_i) \geq 6$, there must exist an available edge between z and another neighbor of v; thus Case 1.0 applies.

Case 3. The maximum degree of B_i is at most 2. Hence every connected component of B_i is either a path or a cycle. By Case 1.3 we can assume that $e(B_i) > v(B_i) - 4$. If $e(B_i) = v(B_i) - 3$, then by Case 1.2 Maker can claim any available edge which is incident with some edge of Breaker. If $e(B_i) = v(B_i) - 2$, then there is a vertex x of degree 2, since $v(B_i) \geq 6$. By Case 1.1 Maker can claim any available edge which is incident with x. Finally, if $e(B_i) = v(B_i) - 1$, then again there is a vertex x of degree 2. Moreover, there is an available edge incident with x whose other endpoint y is a non-isolated vertex in B_i (such a non-isolated vertex exists, since $v(B_i) \geq 6$ and $e(B_i) = v(B_i) - 1$). Maker claims the edge (x, y) and Case 1.0 applies.

This means that after $n/2 - 3$ moves in the second stage Maker has successfully built a spanning forest consisting of two paths such that Breaker's graph $B_{n/2-3}$ on the four endpoints of these two paths satisfies $e(B_{n/2-3}) \leq v(B_{n/2-3}) - 1$. Hence, there exists at least one available edge in $B_{n/2-3}$. Maker claims this edge, thus creating his Hamilton path.

In the third stage, Maker uses Pósa rotations to close his Hamilton path $u_1, u_2, \ldots, u_n$ to a Hamilton cycle. Let u_i, u_{j_1}, u_{j_2} be three vertices on this path such that $i-1 > j_1+1 > j_2+1$ and, just before Maker's first move in this stage, none of the edges (u_1, u_i), (u_{j_1}, u_n), (u_{j_2}, u_n), (u_{i+1}, u_{j_1-1}), (u_{i-1}, u_{j_1-1}), (u_{i+1}, u_{j_1+1}), (u_{i+1}, u_{j_2-1}), (u_{i-1}, u_{j_2-1}), (u_{i+1}, u_{j_2+1}) were previously claimed by Breaker (see Figure 1). In his first move of the third stage, Maker claims the edge (u_1, u_i). In his next move, Breaker cannot claim both (u_{j_1}, u_n) and (u_{j_2}, u_n). Assume without loss of generality that he does not claim (u_{j_1}, u_n). In his next move Maker claims (u_{j_1}, u_n), and then he claims either (u_{i+1}, u_{j_1-1}) or (u_{i-1}, u_{j_1-1}) or (u_{i+1}, u_{j_1+1}) (Breaker cannot neutralize these three simultaneous threats with only two edges). This yields a Hamilton cycle. Note that stage three lasts exactly 3 rounds.

It remains to prove that the three vertices u_i, u_{j_1}, u_{j_2} with the desired properties exist. Recall that, by Maker's first move in the second stage, we have $deg_{B_1}(u_1)+deg_{B_1}(u_n) \leq n/3+4$. In the second and third stages Breaker adds $n/2$ more edges, entailing $deg_{B_{n/2-3}}(u_1) + deg_{B_{n/2-3}}(u_n) \leq 5n/6 + 4$. Hence, for sufficiently large n, there are at least $n/7$ vertices u_k such that neither (u_1, u_k) nor (u_k, u_n) was claimed by Breaker. Thus there are at least $n^2/200$ pairs of vertices u_i, u_j such that $i - 1 > j + 1$ and both (u_1, u_i) and (u_j, u_n) were not claimed by Breaker. Moreover, Breaker has only $O(n)$ edges and every edge (u_p, u_q) he claims affects at most four of the pairs (u_i, u_j), namely $(u_{p-1}, u_{q-1}), (u_{p-1}, u_{q+1}), (u_{p+1}, u_{q-1})$ and (u_{p+1}, u_{q+1}). Hence, there exist two such pairs u_i, u_{j_1} and u_i, u_{j_2}.

If n is odd, then the proof is essentially the same, with just a few small technical changes:

1. The first stage lasts $\lfloor n/2 \rfloor + 1$ rounds and, when it ends, Maker has one path of length 2 and $(n-3)/2$ paths of length 1.

2. The second stage lasts exactly $\lceil n/2 \rceil - 2$ rounds.

3. In B_0 there are $n - 1$ vertices and at most $\lfloor n/2 \rfloor + 2$ edges.

□

2.3 Building a k-connected graph fast

Proof of Theorem 1.5. Let $K_n = (V, E)$ where $V = \{1, 2, \ldots, n\}$. Assume first that n is even and let $m = kn/2$. We will present a random strategy for Maker, which enables him to build a k-vertex-connected graph within $kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n)$ rounds, with positive probability. This, however, will imply the existence of a deterministic strategy for Maker with the same outcome.

Before we start with a detailed description of Maker's strategy, we give a short overview of his actions. The game consists of two stages (it is possible that the second stage will not take place). In the first stage most of Maker's moves are used for building a graph which is "not far" from being a random k-regular graph. The motivation for this approach is that random k-regular graphs are known to be k-vertex-connected a.s. (for more on random regular graphs, the reader is referred to [5], [11] and [14]). In this stage Maker also has to watch out for Breaker's maximum degree growing too large; he will handle this by momentarily abandoning the creation of the pseudo-random graph in order to occupy some edges incident with the "dangerous vertex" (that is, a vertex of high degree in Breaker's graph). In the second stage, Maker occupies some more edges to neutralize possible damage to his pseudo-random graph, caused by Breaker during the first stage.

Before the beginning of the game, Maker does the following. With every $1 \leq i \leq n$, he associates a set $W_i = \{i_1, i_2, \ldots, i_k\}$ of "copies" of i, the sets being pairwise disjoint. Maker then draws uniformly at random a perfect matching P of the $2m$ elements of $W = \bigcup_{i=1}^{n} W_i$. Let $S = ((a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m))$ be an arbitrary ordering of the matched pairs. Note that the selection of the perfect matching P, can be done equivalently by choosing the pairs one at a time. That is, Maker repeatedly draws a pair randomly, uniformly on all unmatched elements of W. Sometimes this point of view is more convenient for our analysis. If $a_r \in W_i$ and $b_r \in W_j$, then we say that the pair (a_r, b_r) corresponds to the edge (i, j). Clearly, different pairs can correspond to the same edge, and so it is possible to get parallel edges. Furthermore, it is possible that $\{a_r, b_r\} \subseteq W_i$ and so the pair (a_r, b_r) corresponds to the loop (i, i). Thus the pairing P corresponds to a k-regular multi-graph. We will discard loops and parallel edges and thus obtain a simple graph of maximum degree at most k.

A vertex $i \in V$ will be called *dangerous* if its degree in Breaker's graph is at least $k\sqrt{n}$. As soon as such a vertex appears, Maker "treats" it immediately (this process will be described in the following paragraph). Throughout the game, let D denote the set of all dangerous vertices which were already "treated". Before the game starts we set $D = \emptyset$.

Stage 1: During this stage, if there are no dangerous vertices outside D, then Maker claims edges of K_n according to the ordering S (note that the matching P and its ordering S are not known to Breaker). That is, let r be the smallest positive integer such that the pair (a_r, b_r) was not considered by Maker before. Maker then claims the edge (i, j), where $(a_r, b_r) = (i_p, j_q)$ for some $1 \leq i, j \leq n$ and $1 \leq p, q \leq k$. If $i = j$ or the edge (i, j) was previously claimed, either by him or by Breaker, then Maker skips his turn (that is, he claims an arbitrary edge which will not be considered in the analysis) and the pair (a_r, b_r) is marked a *failure*. As soon as some $u \in V$ becomes dangerous (if there are several dangerous vertices, then Maker picks one arbitrarily), Maker suspends the above mentioned strategy and plays as follows. He arbitrarily picks $2k + 8$ vertices $w_1, w_2, \ldots, w_{2k+8} \notin D$ such that the edges (u, w_j) are unclaimed for every $1 \leq j \leq 2k + 8$ and, at that point, no w_j is adjacent in Maker's graph to any vertex in D. This is always possible since the first stage lasts less than $kn/2$ moves, so there can be at most $\sqrt{n}$ dangerous vertices. Handling each such vertex takes $k + 4$ moves, so any dangerous vertex, when handled, has degree at most $k\sqrt{n} + (k + 4)\sqrt{n}$ in Breaker's graph, and every vertex which is not in D has degree at most $k + 1$ in Maker's graph. During his next $k + 4$ moves, Maker claims some $k + 4$ edges from the set $\{(u, w_1), (u, w_2), \ldots, (u, w_{2k+8})\}$. He then labels u treated, adds it to D and returns to his usual strategy. The first stage ends as soon as every dangerous vertex is treated and all but $kn^{2/3}$ pairs of S are considered by Maker. The last $kn^{2/3}$ pairs of S are also considered to be failures.

Lemma 2.1 *During the first stage there are at most $n^{2/3} \log n$ failures almost surely.*

Proof of Lemma 2.1: It is well-known that for every fixed k, an n-vertex k-regular multi-graph that corresponds to a random pairing, almost surely contains at most $n^{2/3}$ loops and parallel edges (see, e.g., [11]). Hence, it suffices to bound from above the number of failures that correspond to edges that were previously claimed by Breaker. Throughout the first stage, there are at most $\sqrt{n}$ vertices in D. Hence, after considering at most $kn/2 - kn^{2/3}$ pairs of S, there are at least $n^{2/3} < 2n^{2/3} - \sqrt{n} - (k+4)\sqrt{n}$ vertices of degree strictly smaller than k in Maker's graph. It follows that at any point during the first stage there are at least $\binom{n^{2/3}}{2} - kn/2$ edges available for Maker to continue his configuration (following S). Since Breaker has claimed at most $kn/2$ edges to this point, the probability that any specific pair (a_i, b_i) corresponds to an edge that was previously claimed by Breaker (here we view S as if it was built sequentially) is at most

$$\frac{kn/2}{\binom{n^{2/3}}{2} - kn/2} \leq \frac{2k}{n^{1/3}}.$$

Let F be the random variable that counts the number of the first $kn/2 - kn^{2/3}$ pairs of S, that correspond to edges that were previously claimed by Breaker. Then

$$\mathbb{E}(F) \leq \frac{kn}{2} \cdot \frac{2k}{n^{1/3}} \leq k^2 n^{2/3}.$$

Using Markov's inequality we obtain

$$Pr(F \geq n^{2/3}(\log n - k - 1)) = o(1).$$

It follows that almost surely throughout Stage 1 there are at most $n^{2/3} \log n$ failures ($n^{2/3}(\log n - k - 1)$ for hitting Breaker's edges, $n^{2/3}$ for loops and parallel edges and $kn^{2/3}$ for the last $kn^{2/3}$ pairs of S), which proves the statement of the lemma. □

Let $G_1 = (V, E)$ denote the graph that Maker has built in the first stage, following his random strategy. Let X be the set of all vertices of $V \setminus D$ that are incident with at least one edge, that corresponds to a failure pair, and let $V = V_1 \cup V_2$ be a partition of V, where $V_1 = D \cup X$. Observe that each vertex of V_2 is incident with k random edges of the random graph defined by P. We can thus derive expansion properties of subsets of V_2 from those of the random k-regular graph. This is done in the following claim.

Claim 2.2 *The following holds almost surely. There exists a constant $c > 0$ such that if $A \subseteq V_2$ and $|A| < c \log n$, then $|\Gamma(A)| \geq (k-2)|A|$, and if $A \subseteq V_2$, $B \subseteq V \setminus A$, where $c \log n \leq |A| \leq |B|$ and $|B| \geq n - k - |A|$, then there is an edge between a vertex of A and a vertex of B. Moreover, if $|A| = 1$, then $|\Gamma(A)| \geq k$, and if $|A| = 2$, then $|\Gamma(A)| \geq 2k - 3$.*

The proof of Claim 2.2 is essentially the same as the proof of Theorem 7.32 from [5]. We omit the straightforward details.

As we already mentioned, since we are looking at a finite, perfect information game with no chance moves, it follows that Maker has a deterministic strategy to build $G_1 = (D \cup X \cup V_2, E)$ within $kn/2 + (k+4)\sqrt{n}$ moves, such that $|D| \leq \sqrt{n}$, $|X| \leq 2n^{2/3} \log n$, and V_2 satisfies the properties described in Claim 2.2.

Stage 2: For every $u \in X$, Maker arbitrarily picks $2k+8$ vertices $w_1^u, w_2^u, \ldots, w_{2k+8}^u \in V \setminus N(D)$, such that the edges (u, w_j^u) are unclaimed for every $1 \leq j \leq 2k+8$ and $\{w_1^u, w_2^u, \ldots, w_{2k+8}^u\} \cap \{w_1^v, w_2^v, \ldots, w_{2k+8}^v\} = \emptyset$ for every $u \neq v \in X$. This is possible as $|X| \leq 2n^{2/3} \log n$, $|D| \leq \sqrt{n}$, and each vertex in X has $n - o(n)$ unclaimed edges incident with it, as $X \cap D = \emptyset$. Using an obvious pairing strategy, Maker claims $k+4$ of the edges (u, w_j^u) for every $u \in X$.

Let G_M denote the graph built by Maker during the entire game. We claim that it is k-vertex-connected. Assume for the sake of contradiction, that a small set separates G_M, that is, $V = A \cup S \cup B$, where $1 \leq a = |A| \leq |B|$, $|S| = s < k$ and there are no edges between A and B in G_M. If $a \leq 5$ and $x \in A \cap V_1$, then by Maker's strategy $|(\Gamma(A) \cup A) \setminus \{x\}| \geq |\Gamma(x)| \geq k+4 > |(A \cup S) \setminus \{x\}|$ which is a contradiction as $(\Gamma(A) \cup A) \setminus \{x\} \subseteq (A \cup S) \setminus \{x\}$. On the other hand, if $A \cap V_1 = \emptyset$, then $|\Gamma(A)| \geq k$ by Claim 2.2 (recall that $k \geq 3$). Hence, from now on we assume that $6 \leq a < c \log n$. If $|A \cap V_1| \geq a/4$, then by Maker's strategy $|N(A \cap V_1)| \geq (k+4)a/4 > a + k \geq |A \cup S|$ which is a contradiction as $N(A \cap V_1) \subseteq A \cup S$. Otherwise, $|A \cap V_1| < a/4$ and so by Claim 2.2 we have $|\Gamma(A \cap V_2)| \geq (k-2)3a/4 \geq a/4 + k > |(A \cap V_1) \cup S|$, where the second inequality follows since $a \geq 6$ and $k \geq 3$. Again, this is a contradiction.

If n is odd, then Maker plays as follows. He arbitrarily picks some vertex u and then plays two disjoint games in parallel. One is on the board $\{(u, v) : v \in V \setminus \{u\}\}$, which is played until he claims exactly k of its elements, and the other is on $K_n[V \setminus \{u\}] \cong K_{n-1}$, where Maker plays according to the above strategy. It is easy to see that the resulting graph is k-vertex-connected (adding a vertex to a k-connected graph and then connecting it to k arbitrary vertices of the graph produces a k-connected graph).

Finally, note that by Maker's strategy and by Lemma 2.1, in both stages Maker plays at most $kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n)$ moves. $\square$

3 Concluding remarks and open problems

It was stated in Theorem 1.1 that $n + 1 \leq \tau(\mathcal{H}_n) \leq n + 2$ holds for sufficiently large n. It would be interesting to decide which of the two values is the correct answer.

We know from Theorem 1.5 that Maker can win the k-vertex-connectivity game on K_n within $kn/2 + o(n)$ moves. We are curious whether the $o(n)$ term can be replaced with some function of k, if not for this game, then for the k-edge-connectivity game or the minimum-degree-k game.

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