Fast winning strategies in Avoider-Enforcer games

Dan Hefetz * Michael Krivelevich † Miloš Stojaković ‡ Tibor Szabó §

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Abstract

In numerous positional games the identity of the winner is easily determined. In this case one of the more interesting questions is not *who* wins but rather *how fast* can one win. These types of problems were studied earlier for Maker-Breaker games; here we initiate their study for unbiased Avoider-Enforcer games played on the edge set of the complete graph K_n on n vertices. For several games that are known to be an Enforcer's win, we estimate quite precisely the minimum number of moves Enforcer has to play in order to win. We consider the non-planarity game, the connectivity game and the non-bipartite game.

1 Introduction

Let $\mathcal{F}$ be a hypergraph. In an unbiased Avoider-Enforcer game $\mathcal{F}$ two players, called Avoider and Enforcer, take turns selecting previously unclaimed vertices of $\mathcal{F}$, with Avoider going first. Each player selects one vertex per turn, until all vertices are claimed. Enforcer wins if Avoider claims all the vertices of some hyperedge of $\mathcal{F}$; otherwise Avoider wins. We refer to the family of hyperedges of $\mathcal{F}$ as the family of losing sets. In this paper our

*Institute of Theoretical Computer Science, ETH Zurich, CH-8092 Switzerland; and School of Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: dan.hefetz@inf.ethz.ch. This paper is a part of the author's Ph.D. under the supervision of Prof. Michael Krivelevich.

†School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, 69978, Israel. Email: krivelev@post.tau.ac.il. Research supported in part by a USA-Israel BSF grant, a grant from the Israel Science Foundation and by a Pazy Memorial Award.

‡Department of Mathematics and Informatics, University of Novi Sad, Serbia. Email: smilos@inf.ethz.ch. Partially supported by the Ministry of Science, Republic of Serbia, and Provincial Secretariat for Science, Province of Vojvodina.

§Department of Mathematics and Statistics, McGill University, Montreal, QC H3A 2K6, Canada. Email: szabo@math.mcgill.ca.

attention is restricted to games which are played on the edges of the complete graph on n vertices, that is, the vertex set of $\mathcal{F}$ will always be $E(K_n)$.

Many positional games that were previously studied are known to be an easy win for Enforcer (for a comprehensive reference on positional games the reader is referred to [3]). For example, the non-planarity game, where the goal of Avoider is to avoid a non-planar graph, exhibits that kind of behavior – Avoider creates a non-planar graph and thus loses the game in the end, irregardless of his strategy, the simple reason being that every graph on n vertices with more than $3n - 6$ edges is non-planar. Thus, for games of this type, a more interesting question to ask is not *who wins* but rather *how long does it take* the winner to reach a winning position. This is the general problem we address in this paper. To the best of our knowledge, "fast winning" in Avoider-Enforcer games has not been studied before this paper. On the other hand, there are quite a few results concerning the analogous notion for Maker-Breaker games (see, e.g., [2, 4, 8, 9]).

For a hypergraph $\mathcal{F}$, let $\tau_E(\mathcal{F})$ denote the smallest integer t such that Enforcer has a strategy to win the game on $\mathcal{F}$ within t moves. For the sake of completeness, we define $\tau_E(\mathcal{F}) = \infty$ if the game is an Avoider's win.

One general way to approach the problem of determining the threshold $\tau_E(\mathcal{F})$ is by investigating the extremal properties of the hypergraph $\mathcal{F}$. For convenience, let us assume that the set of hyperedges of $\mathcal{F}$ is a monotone increasing family of sets. If this is not the case, we can extend it to an increasing family by adding all the supersets of its elements – this operation clearly does not change the outcome of the game. The *extremal number* (or Turán number) of the hypergraph $\mathcal{F}$ is defined by

$$\mathrm{ex}(\mathcal{F}) = \max \{|A| : A \subseteq V(\mathcal{F}),\ A \notin E(\mathcal{F})\}.$$

Knowing $\mathrm{ex}(\mathcal{F})$, the minimum move number $\tau_E(\mathcal{F})$ can be determined up to a factor of two.

Observation 1 *Given a monotone increasing family $\mathcal{F}$ of hyperedges, we have*

$$\frac{1}{2}\mathrm{ex}(\mathcal{F}) + 1 \leq \tau_E(\mathcal{F}) \leq \mathrm{ex}(\mathcal{F}) + 1.$$

Proof. To prove the lower bound, let Avoider fix an arbitrary subset A of $V(\mathcal{F})$ before the game starts, such that A is not an edge of $\mathcal{F}$ and $|A| = \mathrm{ex}(\mathcal{F})$. Then, during the game, Avoider just claims elements of A for as long as possible. This way he will be able to claim at least half of the elements of A without losing.

For the upper bound, observe that Enforcer will surely win after $\mathrm{ex}(\mathcal{F}) + 1$ rounds irregardless of his strategy. Indeed, at that point, Avoider has claimed $\mathrm{ex}(\mathcal{F}) + 1$ vertices, and every set with that many vertices is an edge of $\mathcal{F}$. □

1.1 Our results

As we have already mentioned, in the Avoider-Enforcer non-planarity game Avoider loses the game as soon as his graph becomes non-planar. The biased version of this game was studied in [7]. Denote by $\mathcal{NP}_n$ the hypergraph whose hyperedges are the edge-sets of all non-planar graphs on n vertices. From Observation 1, we obtain

$$\frac{3}{2}n - 2 \leq \tau_E(\mathcal{NP}_n) \leq 3n - 5.$$

The following theorem asserts that this upper bound is essentially tight, that is, Avoider can refrain from building a non-planar graph for at least $(3 - o(1))n$ moves. More precisely,

Theorem 2

$$\tau_E(\mathcal{NP}_n) > 3n - 28\sqrt{n}.$$

In the Avoider-Enforcer non-bipartite game Avoider loses the game as soon as his graph first becomes non-bipartite. Clearly, this game is equivalent to the game in which Avoider's goal is to avoid creating an odd cycle. Denote by $\mathcal{NC}_n^2$ the hypergraph whose hyperedges are the edge-sets of all non-bipartite graphs on n vertices. Mantel's Theorem asserts that the bipartite graph on n vertices which maximizes the number of edges is the complete bipartite graph with a balanced partition. Hence, it follows from Observation 1 that

$$\frac{1}{2}\left\lfloor \frac{n^2}{4} \right\rfloor + 1 \leq \tau_E(\mathcal{NC}_n^2) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

In the next theorem we improve the upper bound substantially and establish that the lower bound is asymptotically correct. We also slightly improve the lower bound and thus determine the order of magnitude of the second order term of $\tau_E(\mathcal{NC}_n^2)$.

Theorem 3

$$\tau_E(\mathcal{NC}_n^2) = \frac{n^2}{8} + \Theta(n).$$

Note that the non-bipartite game is just a special case of the non-k-colorability game $\mathcal{NC}_n^k$, where Avoider loses the game as soon as his graph becomes non-k-colorable. Observation 1 can be readily applied, but it would be interesting to obtain tighter bounds, as in the case $k = 2$.

Finally, we consider two Avoider-Enforcer games that turn out to be of similar behavior. In the positive min-degree game, Enforcer wins as soon as the minimum degree in Avoider's graph becomes positive, and in the connectivity game, Enforcer wins as soon as Avoider's graph becomes connected and spanning. Denote by $\mathcal{D}_n$ and $\mathcal{T}_n$ the hypergraphs whose hyperedges are the edge-sets of all graphs on n vertices with a positive minimum degree, and the edge-sets of all graphs on n vertices that are connected and spanning, respectively.

Clearly, we have $\tau_E(\mathcal{D}_n) \leq \tau_E(\mathcal{T}_n)$, since $\mathcal{D}_n \supseteq \mathcal{T}_n$. As $\mathrm{ex}(\mathcal{D}_n) = \mathrm{ex}(\mathcal{T}_n) = \binom{n-1}{2}$, Observation 1 implies

$$\frac{1}{2}\binom{n-1}{2} + 1 \leq \tau_E(\mathcal{D}_n) \leq \tau_E(\mathcal{T}_n) \leq \binom{n-1}{2} + 1.$$

Moreover, as Enforcer wins both games (see [6]), we have

$$\tau_E(\mathcal{D}_n) \leq \frac{1}{2}\binom{n}{2} \quad \text{and} \quad \tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n}{2},$$

which determines both parameters asymptotically and shows that they are "quite close to each other". This is somewhat reminiscent of the well-known property of random graphs, that the hitting time of being connected and the hitting time of having minimum positive degree are almost surely the same, and it motivates us to raise the following question.

Question 4 *Is it true that $\tau_E(\mathcal{D}_n) = \tau_E(\mathcal{T}_n)$ holds for sufficiently large n?*

The following theorem can be considered as a first step towards an affirmative answer to Question 4. We improve the aforementioned lower and upper bounds, determining in the process the second order term and the order of magnitude of the third for both of these parameters.

Theorem 5

$$\begin{aligned} \tau_E(\mathcal{D}_n) &= \frac{1}{2}\binom{n-1}{2} + \Theta(\log n), \\ \tau_E(\mathcal{T}_n) &= \frac{1}{2}\binom{n-1}{2} + \Theta(\log n). \end{aligned}$$

For the sake of simplicity and clarity of presentation, we omit floor and ceiling signs whenever these are not crucial. Some of our results are asymptotic in nature and, whenever necessary, we assume that n is sufficiently large. Throughout the paper, log stands for the natural logarithm. Our graph-theoretic notation is standard and follows that of [5].

2 The strategies

2.1 Keeping the graph planar for long

Proof of Theorem 2 We begin by introducing some terminology. Let v be a vertex, and let S be a set of vertices. Let $N_{\mathcal{A}}(v, S)$ denote the set of neighbors of v in Avoider's graph, belonging to S. Similarly, let $N_{\mathcal{E}}(v, S)$ denote the set of neighbors of v in Enforcer's graph, belonging to S.

We will provide Avoider with a strategy for keeping his graph planar for at least $3n - 28\sqrt{n}$ rounds. The strategy consists of three stages.

Before the game starts, we partition the vertex set

$$V(K_n) = \{v_1\}\dot{\cup}\{v_2\}\dot{\cup}A\dot{\cup}N_{1,1}\dot{\cup}N_{1,2}\dot{\cup}N_{2,1}\dot{\cup}N_{2,2},$$

such that $|N_{1,1}| = |N_{1,2}| = |N_{2,1}| = |N_{2,2}| = \sqrt{n} - 1$ and $|A| = n - 4\sqrt{n} + 2$.

In the first stage, Avoider claims edges according to a simple pairing strategy. For every vertex $a \in A$, we pair up the edges (a, v_1) and (a, v_2). Whenever Enforcer claims one of the paired edges, Avoider immediately claims the other edge of that pair. If Enforcer claims an edge which does not belong to any pair, then Avoider claims the edge (a, v_1), for some $a \in A$, for which neither (a, v_1) nor (a, v_2) were previously claimed. He then removes the pair $(a, v_1), (a, v_2)$ from the set of considered edge pairs.

The first stage ends as soon as Avoider connects every $a \in A$ to either v_1 or v_2. Note that, at that point, Avoider's graph consists of two vertex-disjoint stars centered at v_1 and v_2, and the isolated vertices in $N_{1,1} \cup N_{1,2} \cup N_{2,1} \cup N_{2,2}$. Hence, during the first stage, Avoider has claimed exactly $n - 4\sqrt{n} + 2$ edges. Define $A_1 := N_{\mathcal{A}}(v_1, A)$, and $A_2 := N_{\mathcal{A}}(v_2, A)$.

Before the second stage starts, we pick four vertices $n_{1,1} \in N_{1,1}$, $n_{1,2} \in N_{1,2}$, $n_{2,1} \in N_{2,1}$ and $n_{2,2} \in N_{2,2}$, such that $|N_{\mathcal{E}}(n_{i,j}, A)| \leq \sqrt{n}$, for every $i, j \in \{1, 2\}$. Clearly, such a choice of vertices is possible as the total number of edges Enforcer has claimed during the first stage is $n - 4\sqrt{n} + 2 < \sqrt{n} \cdot (\sqrt{n} - 1)$. Define $G_1 := N_{\mathcal{E}}(n_{1,1}, A_1) \cup N_{\mathcal{E}}(n_{1,2}, A_1)$, and $G_2 := N_{\mathcal{E}}(n_{2,1}, A_2) \cup N_{\mathcal{E}}(n_{2,2}, A_2)$. Note that $|G_1| \leq 2\sqrt{n}$, $|G_2| \leq 2\sqrt{n}$, and $|N_{\mathcal{E}}(n_{1,1}, A_1 \setminus G_1)| = |N_{\mathcal{E}}(n_{1,2}, A_1 \setminus G_1)| = |N_{\mathcal{E}}(n_{2,1}, A_2 \setminus G_2)| = |N_{\mathcal{E}}(n_{2,2}, A_2 \setminus G_2)| = 0$.

Using a pairing strategy similar to the one used in the first stage, Avoider connects each vertex of $A_1 \setminus G_1$ to either $n_{1,1}$ or $n_{1,2}$, and each vertex of $A_2 \setminus G_2$ to either $n_{2,1}$ or $n_{2,2}$. More precisely, for every $a \in A_1 \setminus G_1$ we pair up the edges $(a, n_{1,1})$ and $(a, n_{1,2})$, and for every $a \in A_2 \setminus G_2$ we pair up edges $(a, n_{2,1})$ and $(a, n_{2,2})$. Avoider then proceeds as in the first stage.

The second stage ends as soon as Avoider connects every $a \in A_1 \setminus G_1$ to either $n_{1,1}$ or $n_{1,2}$, and every $a \in A_2 \setminus G_2$ to either $n_{2,1}$ or $n_{2,2}$. We define $A_{1,1} := N_{\mathcal{A}}(n_{1,1}, A_1)$, $A_{1,2} := N_{\mathcal{A}}(n_{1,2}, A_1)$, $A_{2,1} := N_{\mathcal{A}}(n_{2,1}, A_2)$ and $A_{2,2} := N_{\mathcal{A}}(n_{2,2}, A_2)$. Since $|A_{1,1}| + |A_{1,2}| = |A_1| - |G_1|$, $|A_{2,1}| + |A_{2,2}| = |A_2| - |G_2|$ and $|A_1| + |A_2| = |A|$, we infer that the number of edges Avoider has claimed in the second stage is at least $n - 8\sqrt{n}$. Note that during the first two stages Avoider did not claim any edge with both endpoints in one of the sets $A_{1,1}$, $A_{1,2}$, $A_{2,1}$, $A_{2,2}$.

In the third stage, Avoider claims only edges with both endpoints contained in the sets $A_{i,j}$, for some $i, j \in \{1, 2\}$. His goal in this stage is to build a "large" linear forest in $A_{1,1}$.

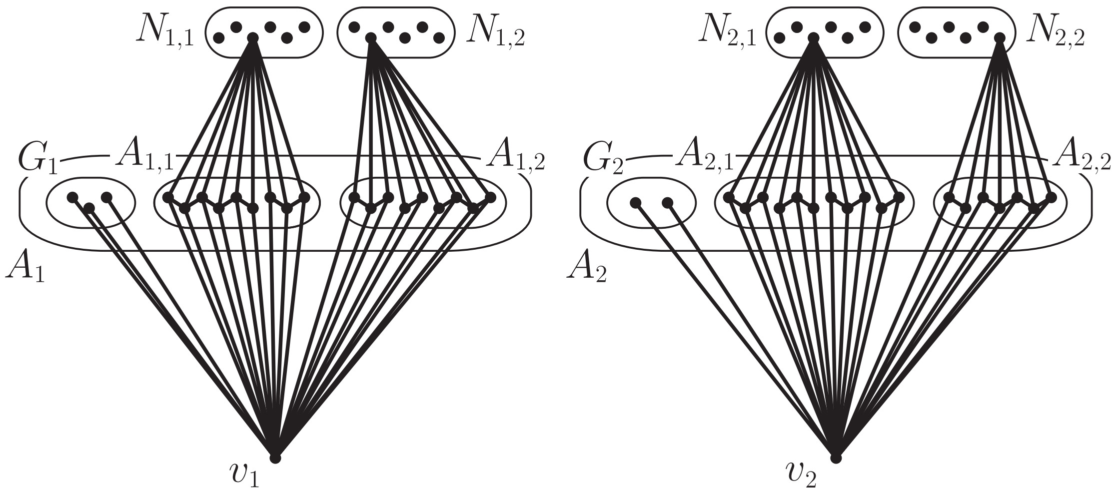

Figure 1: Avoider's graph.

(A *linear forest* is a vertex-disjoint union of paths.) In the beginning of the third stage, Avoider's graph induced on the vertices of $A_{1,1}$ is empty, that is, it consists of $|A_{1,1}|$ paths of length 0 each. For as long as possible, Avoider claims edges that connect endpoints of two of his paths in $A_{1,1}$, creating a longer path. When this is no longer possible, every edge that connects endpoints of two different paths must have been previously claimed by Enforcer. Since the total number of edges that Enforcer has claimed so far is at most $3n$, the number of paths of Avoider in $A_{1,1}$ is at most $2\sqrt{n}$. Hence, Avoider has claimed at least $|A_{1,1}| - 2\sqrt{n}$ edges to this point of the third stage.

Similarly, Avoider builds a "large" linear forest in $A_{1,2}$, $A_{2,1}$, and finally $A_{2,2}$, all in the same way. Thus, the total number of edges he claims during the third stage is at least

$$\begin{aligned}\sum_{i,j\in\{1,2\}} (|A_{i,j}| - 2\sqrt{n}) &\geq |A_1| - |G_1| + |A_2| - |G_2| - 8\sqrt{n} \\ &\geq |A| - 12\sqrt{n} \\ &\geq n - 16\sqrt{n}.\end{aligned}$$

The total number of edges claimed by Avoider during the entire game is therefore at least $(n-4\sqrt{n})+(n-8\sqrt{n})+(n-16\sqrt{n}) = 3n-28\sqrt{n}$. Moreover, at the end of the third stage (which is also the end of the game), Avoider's graph is the pairwise edge disjoint union of two stars, four other graphs - each being a subgraph of a union of K_{2,n_i} and a linear forest which is restricted to one side of the bipartition (see Figure 1). Clearly, such a graph is planar. $\square$

2.2 Forcing and avoiding odd cycles

Proof of Theorem 3

Forcing an odd cycle fast. First, we provide Enforcer with a strategy that will force Avoider to claim the edges of an odd cycle during the first $\frac{n^2}{8} + \frac{n}{2} + 1$ moves. In every stage of the game, each connected component of Avoider's graph is a bipartite graph with a unique bipartition of the vertices (we stop the game as soon as Avoider is forced to close an odd cycle). In every move, Enforcer's primary goal is to claim an edge which connects two opposite sides of the bipartition of one of the connected components of Avoider's graph. If no such edge is available, then Enforcer claims an arbitrary edge, and that edge is marked as "possibly bad". Clearly, in his following move Avoider cannot play inside any of the connected components of his graph either, and so he is forced to merge two of his connected components (that is, he has to claim an edge (x, y) such that x and y are in different connected components of his graph). As the game starts with n connected components, this situation can occur at most $n - 1$ times.

Therefore, when Avoider is not able to claim any edge without creating an odd cycle, his graph is bipartite, and all of Enforcer's edges, except some of the "possibly bad" ones, are compatible with the bipartition of Avoider's graph. The total number of edges that were claimed by both players to this point is at most $\frac{n^2}{4} + n - 1$, and so the total number of moves Avoider has played in the entire game is at most $\frac{n^2}{8} + \frac{n}{2} + 1$.

Avoiding odd cycles for long. Next, we provide Avoider with a strategy for keeping his graph bipartite for at least $\frac{n^2}{8} + \frac{n-2}{12}$ rounds. For technical reasons we assume that n is even; however, a similar statement holds for odd n as well. During the game Avoider will maintain a family of ordered pairs (V_1, V_2), where $V_1, V_2 \subseteq V(K_n)$, $V_1 \cap V_2 = \emptyset$ and $|V_1| = |V_2|$, which he calls *bi-bunch*es. We say that two bi-bunches (V_1, V_2) and (V_3, V_4) are disjoint if $(V_1 \cup V_2) \cap (V_3 \cup V_4) = \emptyset$. At any point of the game, Avoider calls a vertex *untouched* if it does not belong to any bi-bunch and all the edges incident with it are unclaimed. During the entire game, we will maintain a partition of the vertex set $V(K_n)$ into a number of pairwise disjoint bi-bunches, and a set of untouched vertices.

Avoider starts the game with n untouched vertices and no bi-bunches. In every move, his primary goal is to claim an edge *across* some existing bi-bunch, that is, an edge (x, y) where $x \in V_1$ and $y \in V_2$ for some bi-bunch (V_1, V_2). If no such edge is available, then he claims an edge joining two untouched vertices x and y, introducing a new bi-bunch $(\{x\}, \{y\})$. If he is unable to do that either, then he claims an edge connecting two bi-bunches, that is, an edge (x, y) such that there exist two bi-bunches (V_1, V_2) and (V_3, V_4) with $x \in V_1$ and $y \in V_3$. He then replaces these two bi-bunches with a single new one $(V_1 \cup V_4, V_2 \cup V_3)$.

Whenever Enforcer claims an edge (x, y) such that neither x nor y belong to any bi-bunch, we introduce a new bi-bunch $(\{x, y\}, \{u, v\})$, where u and v are arbitrary untouched vertices. If at that point of the game there are no untouched vertices (clearly this can happen at most once), then the new bi-bunch is just $(\{x\}, \{y\})$. If Enforcer claims an edge (x, y) such that there is a bi-bunch (V_1, V_2) with $x \in V_1$ and y is untouched, then the bi-bunch (V_1, V_2) is replaced with $(V_1 \cup \{y\}, V_2 \cup \{u\})$, where u is an arbitrary untouched vertex. Finally, if Enforcer claims an edge (x, y) such that there are bi-bunches (V_1, V_2) and (V_3, V_4) with $x \in V_1$ and $y \in V_3$, than these two bi-bunches are replaced with a single one $(V_1 \cup V_3, V_2 \cup V_4)$. Note that by following his strategy, and updating the bi-bunch partition as described, Avoider's graph will not contain an edge with both endpoints in the same side of a bi-bunch at any point of the game.

Observe that the afore-mentioned bi-bunch maintenance rules imply the following. If Enforcer claims an edge (x, y), such that before that move x was an untouched vertex, then the edge (x, y) will be contained in the same side of some bi-bunch, that is, after that move there will be a bi-bunch (V_1, V_2) with $x, y \in V_1$ (unless x and y were the last two isolated vertices).

Assume that in some move Avoider claims an edge (x, y), such that before that move x was an untouched vertex. It follows from Avoider's strategy that y was untouched as well, and there were no unclaimed edges across a bi-bunch at that point. Thus, in his next move, Enforcer will also be unable to claim an edge across a bi-bunch and so, by the bi-bunch maintenance rules for Enforcer's moves, the edge he will claim in that move will have both its endpoints in the same side of some bi-bunch.

By the previous paragraphs, we conclude that after every round in which at least one of the players claims an edge which is incident with an untouched vertex (which is not the next to last untouched vertex), the edge Enforcer claims in this round will be contained in the same side of some bi-bunch. By the bi-bunch maintenance rules, during every round the number of untouched vertices is decreased by at most 6. Hence, by the time all but two vertices are not untouched at least $(n-2)/6$ edges of Enforcer will be contained in the same side of a bi-bunch. Therefore, when Avoider can no longer claim an edge without creating an odd cycle, both players have claimed together all the edges of a balanced bipartite graph which is in compliance with the bi-bunch bipartition, and at least another $(n-2)/6$ edges. This gives a total of at least $\frac{n}{2} \cdot \frac{n}{2} + (n-2)/6$ edges claimed, which means that at least $\frac{n^2}{8} + \frac{n-2}{12}$ rounds were played to that point. □

2.3 Spanning trees and isolated vertices

Proof of Theorem 5. Clearly $\tau_E(\mathcal{D}_n) \le \tau_E(\mathcal{T}_n)$ and so it suffices to prove that $\tau_E(\mathcal{T}_n) \le \frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1$ and that, $\tau_E(\mathcal{D}_n) > \frac{1}{2}\binom{n-1}{2} + (1/4 - \varepsilon)\log n$, for every $\varepsilon > 0$ and sufficiently large n.

Forcing a spanning tree fast. Starting with the former inequality, we provide Enforcer with a strategy to force Avoider to build a connected spanning graph within $\frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1$ rounds. At any point of the game, we call an edge that was not claimed by Avoider *safe*, if both its endpoints belong to the same connected component of Avoider's graph. An edge which is not safe and was not claimed by Avoider is called *dangerous*. Denote by G_D the graph consisting of dangerous edges claimed by Enforcer. We will provide Enforcer with a strategy to make sure that, throughout the game, the maximum degree of the graph G_D does not exceed $4k$, where $k = \log_2 n$.

Assuming the existence of such a strategy, the assertion of the theorem readily follows. Indeed, assume for the sake of contradiction that after $\frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1$ rounds have been played (where Enforcer follows the afore-mentioned strategy), Avoider's graph is disconnected. Let $C_1, \ldots, C_r$, where $r \geq 2$ and $|C_1| \leq \ldots \leq |C_r|$, be the connected components in Avoider's graph at that point. By Enforcer's strategy, the maximum degree of the graph G_D does not exceed $4k$. Hence, the number of edges claimed by both players to this point does not exceed

$$\sum_{i=1}^{r} \binom{|C_i|}{2} + 4k \sum_{i=1}^{r-1} |C_i|.$$

Assuming that $r \geq 2$ and n is sufficiently large, this sum above attains its maximum for $r = 2$, $|C_1| = 1$ and $|C_2| = n-1$; that is, the sum is bounded from above by $\binom{n-1}{2} + 4\log_2 n$ - a contradiction.

Now we provide Enforcer with a strategy for making sure that, throughout the game, the maximum degree of the graph G_D does not exceed $4k$. In every move, if there exists an unclaimed safe edge, Enforcer claims it (if there are several such edges, Enforcer claims one arbitrarily). Hence, whenever Enforcer claims a dangerous edge, Avoider has to merge two connected components of his graph in the following move, and the number of Avoider's connected components is decreased by one. We will use this fact to estimate the number of dangerous edges at different points of the game.

When all edges within each of the connected components of Avoider's graph are claimed, Enforcer has to claim a dangerous edge. His strategy for choosing dangerous edges is divided into two phases. The first phase is divided into k stages. In the ith stage Enforcer will make sure that the maximum degree of the graph G_D is at most $2i$; other than that, he claims dangerous edges arbitrarily. He proceeds to the following stage only when it is not possible to play in compliance with this condition. Let c_i, $i = 1, \ldots, k$, denote the number of connected components in Avoider's graph after the ith stage. Let $c_0 = n$, be the number of components at the beginning of the first stage. During the ith stage, a vertex v is called *saturated*, if $d_{G_D}(v) = 2i$. Note that at the beginning of the first stage the maximum degree of G_D is $2 \cdot 0 = 0$.

We will prove by induction that $c_i \leq n2^{-i} + 2i$, for all $i = 0, 1, \ldots, k$. The statement trivially holds for $i = 0$.

Next, assume that $c_j \leq n2^{-j}+2j$, for some $0 \leq j < k$. At the beginning of the $(j+1)$st stage Avoider's graph has exactly c_j connected components, and at the end of this stage it has exactly c_{j+1} components. It follows that during this stage Avoider merged two components of his graph $c_j - c_{j+1}$ times. Hence, Enforcer has not claimed more than $c_j - c_{j+1}$ dangerous edges during the $(j+1)$st stage. As the maximum degree of the graph G_D before this stage was $2j$, the number of saturated vertices at the end of the $(j+1)$st stage is at most $c_j - c_{j+1}$. It follows that there are at least $n-(c_j-c_{j+1})$ non-saturated vertices at this point. The non-saturated vertices must be covered by at most $2(j+1)$ connected components of Avoider's graph. Indeed, assume for the sake of contradiction that there are non-saturated vertices $u_1, u_2, \dots, u_{2j+3}$ and connected components $U_1, U_2, \dots, U_{2j+3}$, such that $u_p \in U_p$ for every $1 \leq p \leq 2j+3$. Since $deg_{G_D}(u_p) \leq 2j+1$ for every $1 \leq p \leq 2j+3$, it follows that there must exist an unclaimed edge (u_r, u_s) for some $1 \leq r < s \leq 2j+3$, contradicting the fact that the $(j+1)$st stage is over. Therefore, there are at least $c_{j+1} - 2(j+1)$ connected components in Avoider's graph that do not contain any non-saturated vertex. Clearly every such component has size at least one, entailing $(c_{j+1} - 2j - 2) + (n - c_j + c_{j+1}) \leq n$. Applying the inductive hypothesis we get $c_{j+1} \leq c_j/2 + j + 1 \leq n2^{-(j+1)} + 2(j+1)$. This completes the induction step.

It follows, that at the end of the first phase, after the kth stage, the number of connected components in Avoider's graph, is at most $c_k \leq n2^{-k} + 2k \leq 2k+1$.

In the second phase, whenever Enforcer is forced to claim a dangerous edge, he claims one arbitrarily. Since at the beginning of the second phase, there are at most $2k+1$ connected components in Avoider's graph, Enforcer will claim at most $2k$ dangerous edges during this phase.

It follows that at the end of the game, the maximum degree in G_D will be at most $4k$, as claimed.

Keeping an isolated vertex for long. Fix $\varepsilon > 0$ and set $l := \frac{1-4\varepsilon}{2} \log n$. We provide Avoider with a strategy to keep an isolated vertex in his graph for at least $\frac{1}{2}\binom{n-1}{2} + \frac{l}{2}$ rounds.

Throughout the game, Avoider's graph will consist of one connected component, which we denote by C, and $n - |C|$ isolated vertices. A vertex $v \in V(K_n) \setminus C$ is called *bad*, if there is an even number of unclaimed edges between v and C; otherwise, v is called *good*.

For every vertex $v \in V(K_n)$ let $d_{\mathcal{E}}(v)$ denote the degree of v in Enforcer's graph. If at any point of the game there exists a vertex $v \in V(K_n) \setminus C$ such that $d_{\mathcal{E}}(v) \geq l$, then Avoider simply proceeds by arbitrarily claiming edges which are not incident with v, for as long as possible. The total number of rounds that will be played in that case is at least $\frac{1}{2}\binom{n-1}{2} + \frac{l}{2}$, which proves the theorem. We will show that Avoider can make sure that such a vertex $v \in V(K_n) \setminus C$, with $d_{\mathcal{E}}(v) \geq l$, will appear before the order of his component C reaches $n - l\varepsilon^{-1} - 1$. Hence, from now on, we assume that $|C| \leq n - l\varepsilon^{-1} - 2$.

Whenever possible, Avoider will claim an edge with both endpoints in C. If this is not possible, he will join a new vertex to the component, that is, he will connect it by an edge to an arbitrary vertex of C. Note that this is always possible. Indeed, assume that every edge between C and $V(K_n) \setminus C$ was already claimed by Enforcer. If $|C| \geq l$ then there exists a vertex $v \in V(K_n)$ such that $d_{\mathcal{E}}(v) \geq l$ and so we are done by the previous paragraph. Otherwise, $|C| < l$ and thus, until this point, Enforcer has claimed at most $l^2 < l(n-l)$ edges. As for the way he chooses this new vertex, we consider three cases. Let $\overline{\overline{d}}$ denote the average degree in Enforcer's graph, taken over all the vertices of $V(K_n) \setminus C$, that is,

$$\overline{\overline{d}} := \frac{\sum_{v \in V(K_n) \setminus C} d_{\mathcal{E}}(v)}{n - |C|}.$$

Throughout the case analysis, C and $\overline{\overline{d}}$ represent the values as they are just before Avoider makes his selection.

1. There exists a vertex $v \in V(K_n) \setminus C$, such that $d_{\mathcal{E}}(v) \leq \overline{\overline{d}} - 1$.

   Avoider joins v to his component C. Then $|C|$ increases by one, and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n - |C|)\overline{\overline{d}} - (\overline{\overline{d}} - 1)}{n - |C| - 1} \quad = \quad \overline{\overline{d}} + \frac{1}{n - |C| - 1}.$$

2. Every vertex $v \in V(K_n) \setminus C$ satisfies $d_{\mathcal{E}}(v) > \overline{\overline{d}} - 1$, and $\overline{\overline{d}} < \lfloor \overline{\overline{d}} \rfloor + 1 - \varepsilon$.

   Let D denote the set of vertices $u \in V(K_n) \setminus C$ such that $d_{\mathcal{E}}(u) = \lfloor \overline{\overline{d}} \rfloor$. Note that there must be at least $\varepsilon(n - |C|)$ vertices in D. We distinguish between the following two subcases.

   (a) There is a good vertex in D. Avoider joins it to his component C (if there are several good vertices, then he picks one arbitrarily). Since v was a good vertex, Enforcer must claim at least one edge (x, y) such that $x \notin C \cup \{v\}$, before Avoider is forced again to join another vertex to his component. After this move of Enforcer $|C|$ is (still) increased by (just) one, and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n - |C|)\overline{\overline{d}} - \lfloor \overline{\overline{d}} \rfloor + 1}{n - |C| - 1} \quad \geq \quad \overline{\overline{d}} + \frac{1}{n - |C| - 1}.$$

   (b) All vertices in D are bad. Knowing that $d_{\mathcal{E}}(v) \leq l - 1$ for all vertices $v \in V(K_n) \setminus C$, and $|C| \leq n - l\varepsilon^{-1} - 2$, we have

$$\max_{v \in D} d_{\mathcal{E}}(v) = \lfloor \overline{\overline{d}} \rfloor < l - 1 + 2\varepsilon \leq \varepsilon(n - |C|) - 1 \leq |D| - 1$$

and hence there have to be two vertices $u, w \in D$ such that (u, w) is unclaimed. Avoider joins u to his component C, and thus w becomes good. If Enforcer, in his next move, claims an edge (w, v) for some $v \in C$, then $|C|$ is increased by one and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n-|C|)\overline{\overline{d}} - \lfloor\overline{\overline{d}}\rfloor + 1}{n-|C|-1} \geq \overline{\overline{d}} + \frac{1}{n-|C|-1}.$$

Otherwise, in his next move Avoider joins w to C. Since w was good, then, as in the previous subcase, Enforcer will be forced to claim an edge (x, y) such that $x \notin C \cup \{w\}$. After that move of Enforcer, we will have that $|C|$ is still increased just by two and the new value of $\overline{\overline{d}}$ is at least

$$\frac{(n-|C|)\overline{\overline{d}} - \lfloor\overline{\overline{d}}\rfloor - \lfloor\overline{\overline{d}}\rfloor + 1}{n-|C|-2} \geq \overline{\overline{d}} + \frac{1}{n-|C|-2}.$$

3. Every vertex $v \in V(K_n) \setminus C$ satisfies $d_{\mathcal{E}}(v) > \overline{\overline{d}} - 1$, and $\overline{\overline{d}} \geq \lfloor\overline{\overline{d}}\rfloor + 1 - \varepsilon$.

   Let D denote the set of vertices in $V(K_n) \setminus C$ with degree either $\lfloor\overline{\overline{d}}\rfloor$ or $\lfloor\overline{\overline{d}}\rfloor + 1$. Clearly, $|D| \geq \frac{1}{2}(n - |C|)$. We distinguish between the following two subcases.

   (a) There is a good vertex in D. Similarly to subcase 2(a), Avoider joins that vertex to his component C, and after Enforcer claims some edge with at least one endpoint outside C, we have that $|C|$ is increased by one and the new value of $\overline{\overline{d}}$ is at least

   $$\frac{(n-|C|)\overline{\overline{d}} - (\overline{\overline{d}} + \varepsilon) + 1}{n-|C|-1} = \overline{\overline{d}} + \frac{1-\varepsilon}{n-|C|-1}.$$

   (b) All vertices in D are bad. Similarly to subcase 2(b), Avoider can find two vertices in D such that the edge between them is unclaimed. He joins them to his component C, one after the other. After Enforcer claims some edge with at least one endpoint outside C, we have that $|C|$ increased by two and the new value of $\overline{\overline{d}}$ is at least

   $$\frac{(n-|C|)\overline{\overline{d}} - (\overline{\overline{d}} + \varepsilon) - (\overline{\overline{d}} + \varepsilon) + 1}{n-|C|-2} = \overline{\overline{d}} + \frac{1-2\varepsilon}{n-|C|-2}.$$

It follows that in all cases the value of $\overline{\overline{d}}$ grows by at least $\frac{1-2\varepsilon}{n-|C|-1}$, whenever $|C|$ grows by at most 2. Hence, when the size of C reaches $n - l\varepsilon^{-1} - 2$, we have

$$\begin{aligned}
\overline{\overline{d}} &\geq \sum_{i=2}^{n/2-\frac{1}{2\varepsilon}l-1} \frac{1-2\varepsilon}{n-2i-1} \\
&\geq \frac{1-2\varepsilon}{2} \sum_{i=4}^{n-l\varepsilon^{-1}-2} \frac{1}{n-i-1} \\
&\geq \frac{1-2\varepsilon}{2} \left(\sum_{i=1}^{n-5} \frac{1}{i} - \sum_{i=1}^{l\varepsilon^{-1}} \frac{1}{i} \right) \\
&\geq \frac{1-3\varepsilon}{2} \left(\log n - \log(l\varepsilon^{-1}) \right) \\
&\geq l,
\end{aligned}$$

which concludes the proof of the theorem. □

3 Concluding remarks and open problems

Recently, the approach we used to prove Theorem 2 was enhanced [1], and the error term was improved to a constant.

It was proved in Theorem 3 that $\tau_E(\mathcal{NC}_n^2) = \frac{n^2}{8} + \Theta(n)$. For $k \geq 3$, we know only the simple bounds $\frac{(k-1)n^2}{4k} \leq \tau_E(\mathcal{NC}_n^k) \leq \frac{1}{2}\binom{n}{2}$. Here the lower bound follows from Turán's Theorem and Observation 1 and the upper bound is the consequence of Enforcer being able to win. It would be interesting to close, or at least reduce, the gap between these bounds. It seems reasonable that, as in the case $k = 2$, the truth is closer to the lower bound, and maybe $\tau_E(\mathcal{NC}_n^k) = (1 + o(1))\frac{(k-1)n^2}{4k}$, for every $k \geq 3$.

In Question 4 we ask whether $\tau_E(\mathcal{D}_n) = \tau_E(\mathcal{T}_n)$ holds for sufficiently large n. It would be interesting to consider related families, with a similar random graph hitting time, like the hypergraph $\mathcal{M}_n$ of perfect matchings or that of Hamilton cycles $\mathcal{H}_n$, and obtain estimates on their minimum Avoider-Enforcer move number $\tau_E(\mathcal{M}_n)$ and $\tau_E(\mathcal{H}_n)$.

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