Fast winning strategies in positional games

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Abstract

For the unbiased Maker-Breaker game, played on the hypergraph $\mathcal{H}$, let $\tau_M(\mathcal{H})$ be the smallest integer t such that Maker can win the game within t moves (if the game is a Breaker's win then set $\tau_M(\mathcal{H}) = \infty$). Similarly, for the unbiased Avoider-Enforcer game played on $\mathcal{H}$, let $\tau_E(\mathcal{H})$ be the smallest integer t such that Enforcer can win the game within t moves (if the game is an Avoider's win then set $\tau_E(\mathcal{H}) = \infty$). We investigate τ_M and τ_E and determine their value for various positional games.

1 Introduction

Let p and q be positive integers and let $\mathcal{H}$ be a hypergraph. In a $(p, q, \mathcal{H})$ Maker-Breaker game, two players, called Maker and Breaker, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Maker selects p vertices per move and Breaker selects q vertices per move. Maker wins if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Breaker wins. (Sometimes, when there is no risk of confusion, we will omit $\mathcal{H}$ in the notation above, calling a $(p, q, \mathcal{H})$-game simply a (p, q)-game.) For a $(1, 1, \mathcal{H})$ Maker-Breaker game, let $\tau_M(\mathcal{H})$ be the smallest integer t such that Maker can win the game within t moves (if the game is a Breaker's win, then set $\tau_M(\mathcal{H}) = \infty$).

Similarly, in a $(p, q, \mathcal{H})$ Avoider-Enforcer game two players, called Avoider and Enforcer, take turns selecting previously unclaimed vertices of $\mathcal{H}$. Avoider selects p vertices per move and Enforcer selects q vertices per move. Avoider loses if he claims all the vertices of some hyperedge of $\mathcal{H}$; otherwise Enforcer loses. For a $(1, 1, \mathcal{H})$ Avoider-Enforcer game, let $\tau_E(\mathcal{H})$ be the smallest integer t such that Enforcer can win the game within t rounds (if the game is an Avoider's win, then set $\tau_E(\mathcal{H}) = \infty$).

Our attention is restricted to games which are played on the edges of the complete graph on n vertices, that is, the vertex set of $\mathcal{H}$ will always be $E(K_n)$. For quite a few Maker-Breaker and Avoider-Enforcer games it is rather easy to determine the winner. For example, in the

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connectivity game played on the edges of the complete graph K_n on n vertices, Maker can easily construct a spanning tree by the end of the game. The Avoider-Enforcer planarity game, played on the edges of K_n for n sufficiently large, is an even more convincing example – Avoider creates a non-planar graph and thus loses the game in the end, irregardless of his strategy, the prosaic reason being that every graph on n vertices with more than $3n-6$ edges is non-planar. Thus, for games of this type, a more interesting question to ask is not who wins but rather how long it should take the winner to reach a winning position. This is the type of question we address here.

We start with providing a brief overview of known and relevant results about fast wins in Maker-Breaker and Avoider-Enforcer games. As an immediate consequence of the result of Lehman [9], Maker has a fast winning strategy in the connectivity game. That is, $\tau_M(\mathcal{T}_n) = n-1$, where $\mathcal{T}_n,\ n \geq 4$ is the hypergraph whose hyperedges are the (edge sets of the) spanning trees of K_n. This approach can be easily generalized to a fast winning strategy for Maker in the k-edge-connectivity game. Indeed, if K_n contains $2k$ pairwise edge disjoint spanning trees, then by partitioning them into k pairs and applying Lehman's strategy to each pair we get $\frac{1}{2}kn \leq \tau_M(\mathcal{T}_n^k) \leq k(n-1)$, where $\mathcal{T}_n^k,\ n \geq 4k$ is the hypergraph whose hyperedges are the spanning k-edge-connected subgraphs of K_n. The lower bound follows immediately since the minimum degree of a k-connected graph is at least k. We substantially reduce the gap between these two bounds. As another immediate consequence of Lehman's result, we get that Enforcer cannot win the Avoider-Enforcer cycle game faster than the trivial bound suggests, that is, $\tau_E(\mathcal{C}_n) = n$, where the hyperedges of $\mathcal{C}_n$ are all the cycles of K_n. A result of Bednarska [4] entails $\tau_M(\mathcal{TB}_n^k) = k-1$, where the hyperedges of $\mathcal{TB}_n^k$ are all the copies of complete binary trees on k vertices in K_n, and $k = o(n)$. In [5], Chvátal and Erdős provide Maker with a fast winning strategy for the $(1, 1, \mathcal{H}_n)$ Hamilton cycle game, showing that $\tau_M(\mathcal{H}_n) \leq 2n$, where $\mathcal{H}_n$ is the hypergraph whose hyperedges are the Hamilton cycles of K_n. We almost completely close the gap between this upper bound and the trivial lower bound of $n+1$. Maker can win the $(1, 1, \mathcal{K}_n^q)$ clique game in a constant (depending on q but not on n) number of moves, that is, $\tau_M(\mathcal{K}_n^q) = f(q)$, where the hyperedges of $\mathcal{K}_n^q$ are the q-cliques of K_n. The best upper bound, $f(q) = O((q-3)2^{q-1})$ is due to Pekeč (see [10]); Beck proved that the exponential dependency on q cannot be avoided, namely $f(q) = \Omega(\sqrt{2}^q)$ (see [3]). Note that Maker's strategy for the clique game provides him with a fast win in the non-planarity game and the non-r-colorability game by building a copy of K_5 and K_{r+1}, respectively (for background on these games, see [8]).

Some general sufficient conditions for winning Maker-Breaker games and Avoider-Enforcer games were proved in [2] and [7], respectively. Both are based on the "potential" method of Erdős and Selfridge [6]. These criteria, however, seem not to be very useful for winning quickly, as it is assumed that the game is played until every element of the board is claimed by some player. Nonetheless, using some "fake move" trick (see [3]), they can be used to get certain, usually rather weak, results.

If Maker wins a $(1, q, \mathcal{H})$ Maker-Breaker game for some positive integer q, then $\tau_M(\mathcal{H}) \leq v(\mathcal{H})/(q+1)$, where $v(\mathcal{H})$ is the number of vertices in $\mathcal{H}$. Indeed, when playing the $(1, 1, \mathcal{H})$ game, Maker can use his winning strategy in the $(1, q, \mathcal{H})$ game. In every round, he imagines that additional $q-1$ arbitrary unclaimed vertices were claimed by Breaker. Whenever Breaker claims a vertex which is already his in Maker's imagination, Maker imagines that another (arbitrary still unclaimed) vertex was claimed by Breaker. Clearly, after all vertices have been claimed (including the ones in Maker's imagination), Maker has already won, and the number of rounds played is $v(\mathcal{H})/(q+1)$. Equivalently, this shows that if Breaker can keep from losing the $(1, 1, \mathcal{H})$ game within t rounds, then he can win the $(1, \frac{v(\mathcal{H})}{t} - 1, \mathcal{H})$ game. It was proved by Beck in [1] that Breaker, playing the $(1, 1, \mathcal{H})$ game on an almost disjoint n-uniform hypergraph $\mathcal{H}$, can keep from losing for at least $(2 - \varepsilon)^n$ moves, for any $\varepsilon > 0$. Hence, we can immediately deduce that Breaker can win the $(1, \frac{v(\mathcal{H})}{(2-\varepsilon)^n} - 1)$ game, on any almost disjoint n-uniform hypergraph $\mathcal{H}$ and for every $\varepsilon > 0$. Similarly, if Avoider wins the $(1, q, \mathcal{H})$ game for some positive integer q, then $\tau_E(\mathcal{H}) > v(\mathcal{H})/(q + 1)$. Indeed, when playing the $(1, 1, \mathcal{H})$ game, Avoider can use his winning strategy from the $(1, q, \mathcal{H})$ game. Equivalently, this shows that if Enforcer can win the game on $\mathcal{H}$ within t rounds, then he can also win the $(1, \frac{v(\mathcal{H})}{t} - 1, \mathcal{H})$ game.

To conclude, in order to say something non-trivial about the games we analyze, we will have to find winning strategies for Maker and Enforcer that are faster than the known strategies mentioned above (in case they exist).

1.1 Fast strategies for Maker and slow strategies for Breaker

We now turn back to the analysis of some concrete games. All games we consider here are played on the edges of the complete graph K_n.

Let $\mathcal{M}_n$ be the hypergraph whose hyperedges are all perfect matchings of K_n (or matchings that cover every vertex but one, if n is odd). Let $\mathcal{D}_n$ be the hypergraph whose hyperedges are all spanning subgraphs of K_n of positive minimum degree. We find the *exact* number of moves that Maker needs, in order to win the $(1, 1, \mathcal{M}_n)$ game and the $(1, 1, \mathcal{D}_n)$ game. Obviously, Maker needs to make at least $\lfloor \frac{n}{2} \rfloor$ moves, as this is the size of a member of $\mathcal{M}_n$. We show that if n is odd, then he does not need more moves, whereas if n is even, then he needs just one more move. A similar result, showing the tightness of the obvious lower bound for the minimum degree game $\mathcal{D}_n$, easily follows.

Theorem 1.1 *(i)*

$$\tau_M(\mathcal{M}_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \text{ is odd} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even} \end{cases}$$

(ii)

$$\tau_M(\mathcal{D}_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

As mentioned earlier, Chvátal and Erdős [5] proved that Maker can win the $(1, 1)$ Hamilton cycle game on K_n within $2n$ rounds. Here we show that for sufficiently large n, Maker can win the $(1, 1)$ Hamilton cycle game within $n + 2$ rounds. This bound is now only 1 away from the obvious lower bound.

Theorem 1.2 *For sufficiently large n,*

$$n + 1 \leq \tau_M(\mathcal{H}_n) \leq n + 2.$$

A corollary of the proof of the previous theorem is that Maker can win the "Hamilton path" game within $n - 1$ moves, which is clearly best possible.

Theorem 1.3 *For sufficiently large n,*

$$\tau_M(\mathcal{HP}_n) = n - 1,$$

where $\mathcal{HP}_n$ is the hypergraph whose hyperedges are all Hamilton paths of K_n.

Let $\mathcal{V}_n^k$ be the hypergraph whose hyperedges are all spanning k-vertex-connected subgraphs of K_n. The classical theorem of Lehman [9] asserts that Maker can build a 1-connected spanning graph in $n-1$ moves. From Theorem 1.2 it follows that Maker can build a 2-vertex-connected spanning graph for the price of spending just 3 more (that is, in $n+2$) moves.

In the following, we obtain a generalization of the latter fact for every $k \geq 3$. As every k-connected graph has minimum degree at least k, Maker needs at least $kn/2$ moves just to build a member of $\mathcal{V}_n^k$ (even if Breaker does not play at all). The next theorem shows that this trivial lower bound is asymptotically tight, that is, there is a strategy for Maker to build a k-vertex-connected graph in $kn/2 + o_k(n)$ moves.

Theorem 1.4 *For every fixed $k \geq 3$ and sufficiently large n,*

$$kn/2 \leq \tau_M(\mathcal{V}_n^k) \leq kn/2 + (k+4)(\sqrt{n} + 2n^{2/3} \log n).$$

An easy consequence of Theorems 1.1, 1.2 and 1.4, is that for every fixed $k \geq 1$ Maker can build a graph with minimum degree at least k within $(1+o(1))kn/2$ moves. This is clearly asymptotically optimal.

1.2 Slow strategies for Avoider and fast strategies for Enforcer

In the Avoider-Enforcer non-planarity game, Avoider loses the game as soon as his graph becomes non-planar. Clearly, Enforcer can win this game within $3n-5$ moves no matter how he plays; that is, $\tau_E(\mathcal{NP}_n) \leq 3n-5$, where $\mathcal{NP}_n$ is the hypergraph whose hyperedges are all non-planar subgraphs of K_n. On the other hand, Avoider can keep from losing for $\frac{3}{2}n-3$ moves by simply fixing any triangulation and claiming its edges arbitrarily for as long as possible.

The following theorem asserts that the trivial upper bound is essentially tight, that is, Avoider can refrain from building a non-planar graph for at least $(3-o(1))n$ moves. More precisely,

Theorem 1.5

$$\tau_E(\mathcal{NP}_n) > 3n - 28\sqrt{n}.$$

In the Avoider-Enforcer non-k-coloring game $\mathcal{NC}_n^k$, Avoider loses the game as soon as his graph becomes non-k-colorable. Avoider can play for at least $(1-o(1))\frac{(k-1)n^2}{4k}$ moves without losing by simply fixing a copy of the k-partite Turán-graph and claiming half of its edges. On the other hand, it is not hard to see that the game is an Enforcer's win if it is played until the end (see [8]), so Avoider will lose after at most $\frac{1}{2}\binom{n}{2} \approx \frac{n^2}{4}$ moves. In our next theorem we essentially close the gap between the two bounds for the case $k=2$ (the "non-bipartite game"). We also improve the trivial lower bound and establish the order of magnitude of the second order term of $\tau_E(\mathcal{NC}_n^2)$.

Theorem 1.6

$$\frac{n^2}{8} + \frac{n-2}{12} \leq \tau_E(\mathcal{NC}^2_n) \leq \frac{n^2}{8} + \frac{n}{2} + 1.$$

Next, we look at two Avoider-Enforcer games that turn out to be of similar behavior. In the game $\mathcal{D}_n$ Enforcer wins as soon as the minimum degree in Avoider's graph becomes positive, and in the game $\mathcal{T}_n$ Enforcer wins as soon as Avoider's graph becomes connected and spanning. Enforcer wins both games (see [7]), entailing $\tau_E(\mathcal{D}_n), \tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n}{2}$. On the other hand, Avoider can choose an arbitrary vertex v, and, for as long as possible, claim only edges which are not incident with v, implying $\tau_E(\mathcal{D}_n), \tau_E(\mathcal{T}_n) > \frac{1}{2}\binom{n-1}{2}$. This determines the first order term for both parameters. In the following theorem we determine the second order term and the order of magnitude of the third.

Theorem 1.7

$$\frac{1}{2}\binom{n-1}{2} + \left(\frac{1}{4} - o(1)\right)\log n < \tau_E(\mathcal{D}_n) \leq \tau_E(\mathcal{T}_n) \leq \frac{1}{2}\binom{n-1}{2} + 2\log_2 n + 1.$$

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