LIMIT SHAPE OF OPTIMAL CONVEX LATTICE POLYGONS IN THE SENSE OF DIFFERENT METRICS

Miloš Stojaković[1,2]
Institute of Theoretical Computer Science, ETH Zurich
ETH Zentrum, CH-8092 Zürich, Switzerland
e-mail: MilosSt@EUnet.yu

Abstract

Classes of convex lattice polygons which have minimal l_p–perimeter with respect to the number of their vertices are said to be optimal in the sense of l_p metric.

The purpose of this paper is to prove the existence and explicitly find the limit shape of the sequence of these optimal convex lattice polygons as the number of their vertices tends to infinity.

It is proved that if p is arbitrary integer or ∞, the limit shape of the south–east arc of optimal convex lattice polygons in sense of l_p metric is a curve given parametrically by $\left(\frac{C_x^p(\alpha)}{I_p}, \frac{C_y^p(\alpha)}{I_p}\right)$, $0 < \alpha < \infty$, where

$$C_x^p(\alpha) = \frac{\alpha}{2}\left(-\frac{1}{3}(\alpha^p+1)^{-3/p} + \sum_{k=0}^{\infty}\binom{-\frac{3}{p}-1}{k}\frac{\alpha^{pk}}{pk+1}\right),$$

$$C_y^p(\alpha) = \alpha^2\left(-\frac{1}{3}(\alpha^p+1)^{-3/p} + \sum_{k=0}^{\infty}\binom{-\frac{3}{p}-1}{k}\frac{\alpha^{pk}}{pk+2}\right),$$

$$I_p = \int_0^1 (\sqrt[p]{1-l^p})^2\, dl.$$

Some applications of the limit shape in calculating asymptotic expressions for area of the optimal convex lattice polygons are presented.

Keywords: convex lattice polygon, limit shape

[1] The author is also with Institute of Mathematics, University of Novi Sad, Yugoslavia

[2] Supported by the joint Berlin/Zürich graduate program Combinatorics, Geometry, and Computation, financed by ETH Zürich and the German Science Foundation (DFG).

1 Introduction

Convex lattice polygon is a polygon whose vertices are points on the integer lattice and whose interior angles are strictly less then π radians (no three vertices are collinear). A convex lattice polygon with n vertices is called an n–gon.

A convex lattice n–gon is said to be optimal in the sense of l_p metric if it has minimal l_p–perimeter with respect to the number of its vertices. Therefore, if a convex lattice n–gon has that property, its l_p–perimeter is equal to

$$\min\left\{ \sum_{e \text{ is edge of } Q} l_p \text{ length of } e \,\middle|\, Q \text{ is a convex lattice } n\text{–gon} \right\},$$

and we denote it by $Q_p(n)$. This polygon is not necessarily unique for every given integer n. Moreover, the explicit construction of a polygon $Q_p(2k+1)$, where $p > 1$ and $2k+1$ is an arbitrary odd integer, is an open problem.

The purpose of this paper is to prove the existence and to explicitly find the limit shape of the sequence of optimal convex lattice polygons $Q_p(n)$ as n tends to infinity.

More precisely, we are going to show that the sequence of south–east arcs of normalized (to fit the unit square) polygons $\frac{1}{\mathrm{diam}_\infty(Q_p(n))} Q_p(n)$ tends to the curve γ_p as $n \to \infty$. Curve γ_p is given parametrically by $\left(\frac{C_x^p(\alpha)}{I_p}, \frac{C_y^p(\alpha)}{I_p}\right)$, $0 < s < \infty$, where

$$C_x^p(\alpha) = \frac{\alpha}{2}\left(-\frac{1}{3}(\alpha^p+1)^{-3/p} + \sum_{k=0}^{\infty}\binom{-\frac{3}{p}-1}{k}\frac{\alpha^{pk}}{pk+1}\right),$$

$$C_y^p(\alpha) = \alpha^2\left(-\frac{1}{3}(\alpha^p+1)^{-3/p} + \sum_{k=0}^{\infty}\binom{-\frac{3}{p}-1}{k}\frac{\alpha^{pk}}{pk+2}\right),$$

$$I_p = \int_0^1 (\sqrt[p]{1-l^p})^2\,dl.$$

In the cases $p \in \{1, 2, \infty\}$ this curve can be given explicitly. Limit shapes of the other three arcs of optimal convex lattice polygons are the same curves (γ_p) rotated for $\pi/2$, π and $3\pi/2$ radians and translated to form a closed curve.

Once this curve is known, it can be used for finding the asymptotic expressions for quantities (such as perimeters, diameters in different metrics, area) describing the polygons. The limit shape of n–gons is a curve obtained uniquely for all integers n. That means that asymptotic expressions representing some of the mentioned quantities for even n are the same for odd n. Since the construction of the polygons in the odd case is not known, proving this directly would usually be much more complicated then in the even case.

The limit shape considered in this paper is a "pointwise" limit shape, roughly speaking. This concept was also used in [11], where the limit shape of convex lattice polygons with minimal l_∞–diameter was obtained.

Some research has been conducted recently dealing with the "statistical" limit shapes. The initial question, formulated in ([9]), was whether and when a limit shape of some set of convex lattice polygons exists? It was proved in [2] that as $n \to \infty$, almost all convex $\frac{1}{n} \cdot \mathbf{Z}^2$-lattice polygons lying in the square $[-1, 1]^2$ are very close to a fixed curve.

Convex lattice polygons and generally the extremal problems on the integer lattice are a frequent object of interest in the research in many different fields of applied mathematics, like image processing and pattern recognition. We give a few important results related to the ones presented in this paper.

A classical paper of Jarnik [5] is dealing with $Q_2(n)$ polygons. He has constructed a subsequence of such optimal polygons in order to solve the following problem: *What is the maximal number of points from $\mathbf{Z}^2$ which lie on a continuous strictly convex curve γ of length s, when s tends to infinity?* It turned out that such number is $\frac{3}{\sqrt[3]{2\pi}} s^{2/3} + \mathcal{O}(s^{1/3})$. The exponent and constant in the leading term are the best possible.

In [8] it is shown that the exponent $2/3$ can be decreased by imposing suitable smoothness condition on γ. In particular, if γ has a continuous third derivative with a sensible bound, the best possible value of the exponent lies in $[1/2, 3/5]$. Since the function $f(x) = \sqrt{x}$ defined on $[0, n]$ is in $C^\infty([0, n])$ and the number of integral points on the curve $y = f(x)$ is $\lfloor n^{1/2} \rfloor$, obviously $1/2$ is the lower bound for the mentioned exponent.

The asymptotic expression for the l_q–perimeter of optimal (in the sense of l_p metric) convex lattice polygons $Q_p(n)$ as a function of the number of its vertices n (where p and q are any integers or ∞) is derived in [7].

Some problems considering convex polygons determined by lattice points on strictly convex curves cutting the maximal number of lattice points, with respect to the length of the curve were studied in [4].

The rest of the paper is organized as follows. The basic definitions, denotations and previously known results used in the main part are given in Section 2. In Section 3 the limit shape is found for some special values of n (a subsequence of natural numbers), and in Section 4 the general case (arbitrary n) is solved. In Section 5 the expression for the area of the optimal n–gons is given as an application of the limit shape.

2 Preliminaries

If a and b are integers, $a \perp b$ means that the greatest common divisor for a and b is 1. Also, we shall say that $1 \perp 0$.

By $\mu(n)$ we shall denote Möbius function, defined as:

$$\mu(1) = 1;$$

if $n > 1$ and $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime decomposition of n, then

$$\mu(n) = \begin{cases} (-1)^k, & \text{if } a_1 = \ldots = a_k = 1 \\ 0, & \text{otherwise} \end{cases} .$$

Let $e = ((x_1, y_1), (x_2, y_2))$ be an edge of a convex lattice polygon. The l_p–distance length of e is defined as

$$l_p(e) = \sqrt[p]{|x_2 - x_1|^p + |y_2 - y_1|^p}, \quad p \geq 1.$$

We shall denote the differences $|x_2 - x_1|$ and $|y_2 - y_1|$ by $x(e)$ and $y(e)$, respectively. The quotient of these differences $y(e)/x(e)$ is defined to be the slope of e.

The perimeter in sense of l_p metric of a convex lattice polygon Q is defined by

$$\mathrm{per}_p(Q) = \sum_{e \text{ is edge of } Q} l_p(e).$$

The diameter in sense of l_∞ metric of a convex lattice polygon Q is defined by

$$\mathrm{diam}_\infty(Q) = \max\Big\{ l_\infty\Big((x_1, y_1), (x_2, y_2)\Big) \,\Big|\, (x_1, y_1),\ (x_2, y_2) \text{ are vertices of } Q \Big\}.$$

For $n > 1$, $U_p(n)$ represents the partition function which counts the number of the positive solutions of the equation $n = x^p + y^p$, where x, y are relatively prime integers. If $n = 1$, we define $U_p(1) = 1$ for $p = 1, 2, \ldots, \infty$ (we take $x = 1$, $y = 0$ as a solution).

In the following theorem the number of lattice points inside domains bounded by a Lamé's curve is estimated.

Theorem 1. *[6] The number of lattice points belonging to the area $|x|^\beta + |y|^\beta = u$ with fixed $\beta \geq 2$ is*

$$C_\beta u^{\frac{2}{\beta}} + \mathcal{O}(u^{\omega_\beta}),$$

where C_β is the area inside the curve $|x|^\beta + |y|^\beta = u$, and

$$\omega_\beta = \begin{cases} \frac{2}{3\beta} & 2 \leq \beta \leq 3 \\ \frac{1}{\beta} - \frac{1}{\beta^2} & \beta > 3 \end{cases}$$

In [10], the following sequence of integers is introduced

$$n_p(t) = 4 \sum_{i=1}^{t} U_p(i), \qquad t = 1, 2, 3, \ldots$$

First, we shall consider optimal lattice polygons with $n_p(t)$ $(t = 1, 2, \ldots)$ vertices. It is shown in [10] by explicit construction that the optimal convex lattice polygon $Q_p(n_p(t))$ is determined uniquely. For each integer t, $Q_p(n_p(t))$ is constructed as follows, using "greedy algorithm".

The polygon consists of four isometric arcs, whose edge slopes coincide with the set

$$S_p(t) = \Big\{ \frac{k}{l} \;\Big|\; k, l \text{ are integers}, \; k^p + l^p \leq t, \; k \perp l \Big\}.$$

We shall denote the vertices of $Q_p(n_p(t))$ by

$$A_0 = (x_0, y_0), \quad A_1 = (x_1, y_1), \quad \ldots \quad , A_n = (x_{n_p(t)}, y_{n_p(t)}) = A_0$$

in counterclockwise order.

Let e_1, e_2, $\ldots e_{n_p(t)}$ be the edges determined by consecutive points from the previous sequence. Then, the edges e_1, e_2, $\ldots$, $e_{n_p(t)}$ can be arranged into four arcs. If the angle between the positively oriented x–axis and the edge $A_{i-1}A_i$ is observed, then the south–east arc contains the edges whose angles belong to $[0, \frac{\pi}{2})$. North–east, north–west and south–west arcs are defined analogously.

Let A_0 be the vertex having the minimal x–coordinate of all vertices having the minimal y–coordinate (the "left of the lowest" point). Then the vertex $A_{\frac{1}{4}n_p(t)}$ will be the one having the minimal y–coordinate of all vertices having the maximal x–coordinate (the "lowest of the outermost right" point). For convenience and without loss of generality, let us assume $A_0 = (0, 0)$. Since the slope of the edge e_i is equal to $y(e_i)/x(e_i)$ it follows that the vertices of the south–east arc of the polygon $Q_p(n_p(t))$ are:

$$\begin{aligned}
A_0 &= (0, 0),\\
A_1 &= (x(e_1),\ y(e_1)),\\
A_2 &= (x(e_1) + x(e_2),\ y(e_1) + y(e_2)),\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
A_{\frac{1}{4}\cdot n_p(t)} &= (x(e_1) + x(e_2) + \ldots + x(e_{\frac{1}{4}n_p(t)}),\\
&\qquad\quad y(e_1) + y(e_2) + \ldots + y(e_{\frac{1}{4}n_p(t)})).
\end{aligned}$$

The slopes belonging to the south–east arc have to be arranged in the increasing order

$$\frac{0}{1} = \frac{y(e_1)}{x(e_1)} < \frac{y(e_2)}{x(e_2)} < \ldots < \frac{y(e_{\frac{1}{4}n_p(t)})}{x(e_{\frac{1}{4}n_p(t)})},$$

and

$$S_p(t) = \left\{ \frac{y(e_1)}{x(e_1)}, \frac{y(e_2)}{x(e_2)}, \ \ldots, \ \frac{y(e_{\frac{1}{4}\cdot n_p(t)})}{x(e_{\frac{1}{4}\cdot n_p(t)})} \right\}.$$

The remaining three arcs are obtained by the rotations by $\frac{\pi}{2}, \pi$ and $\frac{3\pi}{2}$ radians around the point $(0, y(e_1) + y(e_2) + \ldots + y(e_{\frac{1}{4}n_p(t)}))$.

It is proved in [10] that a polygon constructed in this way is a unique convex lattice polygon with $n_p(t)$ vertices whose l_p–perimeter is minimal.

Thus, we have a sequence of integers representing numbers of vertices of optimal convex lattice polygons (in sense of l_p metric) that can be explicitly constructed.

The following theorem gives the asymptotic expression for $n_p(t)$.

Theorem 2. [10] *The function $n_p(t)$ can be estimated by*

$$n_p(t) = \frac{6A_p}{\pi^2} t^{2/p} + \mathcal{O}(t^{1/p}),$$

where A_p equals the area of the planar shape $|x|^p + |y|^p \leq 1$.

Similar method is used to construct $Q_p(2k)$. For every even integer $2k$, there exists an integer t such that $n_p(t-1) \leq 2k < n_p(t)$. Polygon $Q_p(2k)$ is constructed by adding edges to $Q_p(n_p(t-1))$. More precisely, $(2k - n_p(t-1))/2$ edges having the length $\sqrt[p]{t}$ are added to the south–east arc of $Q_p(n_p(t-1))$, and $(2k - n_p(t-1))/2$ edges with the same slopes are added to the north–west arc of $Q_p(n_p(t-1))$, i.e. for each edge e added to the south–east arc, there is an edge e' added to the north–west arc such that $y(e')/x(e')=y(e)/x(e)$ ($x(e) \perp y(e)$ and $x(e') \perp y(e')$ are satisfied). Now it is easy to check that the $2k$–gon obtained by this construction is optimal in sense of l_p metric.

The explicit construction of $Q_p(2k+1)$, where $2k+1$ is an arbitrary odd integer, is an open problem (for all $p > 1$).

Each polygon $Q_p(n)$ has no more then 4 edges with the same slope, and that gives the lower bound for l_p–perimeter of $Q_p(n)$. If $n_p(t-1) \leq n < n_p(t)$, then

$$(n - n_p(t-1)) \sqrt[p]{t} + 4 \sum_{i=1}^{t-1} \sqrt[p]{i}\, U_p(i) \leq \mathrm{per}_p(n)$$

This lower bound will be called the greedy lower bound, denoted by $\mathrm{glb}_p(n)$.

3 Limit shape of $Q_p(n_p(t))$

The next lemma gives the asymptotic expressions of the coordinates of vertices of $Q_p(n_p(t))$.

Lemma 1. *If $(x_p(n_p(t), \alpha), y_p(n_p(t), \alpha))$ is the end point of the edge with slope α of the south-east arc of $Q_p(n_p(t))$, then the following asymptotic* *expressions hold:*

$$x_p(n_p(t),\alpha) = \frac{6}{\pi^2} C_x^p(\alpha) t^{3/p} + \mathcal{O}(t^{2/p}\cdot \log t),$$

$$y_p(n_p(t),\alpha) = \frac{6}{\pi^2} C_y^p(\alpha) t^{3/p} + \mathcal{O}(t^{2/p}\cdot \log t).$$

Proof. From the construction of $Q_p(n_p(t))$ we have that

$$x_p(n_p(t),\alpha) = \sum_{\substack{k\perp l,\\ k^p+l^p\le t,\\ k/l\le\alpha}} l,$$

$$y_p(n_p(t),\alpha) = \sum_{\substack{k\perp l,\\ k^p+l^p\le t,\\ k/l\le\alpha}} k.$$

Let $D(v)$ be the number of lattice points (a,b,c) satisfying $a\perp b$ which belong to the 3–dimensional body

$$\mathcal{B}(v) \;=\; \Big\{(x,y,z)\,\Big|\; x>0,\, y>0,\, x^p+y^p\le v,\, \frac{y}{x}\le\alpha,\, 0<z\le x\Big(\frac{t}{v}\Big)^{1/p}\Big\},$$

and let $B(v)$ be the number of all lattice points (a,b,c) which belong to $\mathcal{B}(v)$ ($a\perp b$ not required), where v is any positive number. Note that if we take $v=t$, $D(t)$ equals $x_p(n_p(t),\alpha)$.

Due to the definition of $\mathcal{B}(v)$ we have that $B(v) = \text{volume}(\mathcal{B}(v)) + \mathcal{O}(\text{area}(\mathcal{B}(v)))$, and we can derive the asymptotic expression for $B(v)$. Condition $x^p+y^p\le v$ implies that $x\le v^{1/p}$, $y\le v^{1/p}$ which together with the upper bound for z gives $\mathcal{O}(\text{area}(\mathcal{B}(v))) = \mathcal{O}(t^{1/p}v^{1/p})$.

On the other hand, we have

$$\begin{aligned}
\text{volume}(\mathcal{B}(v)) &= \iint_{\substack{x^p+y^p\le v\\ x,y>0,y\le\alpha x}} x\Big(\frac{t}{v}\Big)^{1/p}\,dx\,dy\\
&= \int_0^{\alpha\sqrt[p]{\frac{v}{\alpha^p+1}}}\int_{\frac{y}{\alpha}}^{\sqrt[p]{v-y^p}} x\Big(\frac{t}{v}\Big)^{1/p}\,dx\,dy\\
&= t^{1/p}v^{2/p}\int_0^{\frac{\alpha}{\sqrt[p]{\alpha^p+1}}}\int_{\frac{n}{\alpha}}^{\sqrt[p]{1-n^p}} m\,dm\,dn\\
&= t^{1/p}v^{2/p}\,\frac{1}{2}\int_0^{\frac{\alpha}{\sqrt[p]{\alpha^p+1}}}\Big((\sqrt[p]{1-n^p})^2-\frac{n^2}{\alpha^2}\Big)dn.
\end{aligned}$$

We introduce $C_x^p(\alpha)$, $\alpha > 0$ by

$$C_x^p(\alpha) = \frac{1}{2}\int_0^{\frac{\alpha}{\sqrt[p]{\alpha^p+1}}} ((\sqrt[p]{1-n^p})^2 - \frac{n^2}{\alpha^2})\, dn.$$

This function can be represented by series, in the following way

$$C_x^p(\alpha) = \frac{\alpha}{2}\left(-\frac{1}{3}(\alpha^p+1)^{-3/p} + \sum_{k=0}^{\infty}\binom{-\frac{3}{p}-1}{k}\frac{\alpha^{pk}}{pk+1}\right),$$

and now we have

$$B(v) = C_x^p(\alpha)t^{1/p}v^{2/p} + \mathcal{O}(t^{1/p}v^{1/p}).$$

From the definitions of $B(v)$ and $D(v)$, we have that the following equalities hold

$$B(t) = \sum_{m=1}^{\infty} D\Big(\frac{t}{m^p}\Big),$$

$$B\Big(\frac{t}{a^p}\Big) = \sum_{m=1}^{\infty} D\Big(\frac{t}{(ma)^p}\Big).$$

In the following derivation we shall use the two well–known equalities ([1]):

$$\sum_{a|l}\mu(a) = \begin{cases} 1, & l=1 \\ 0, & l>1 \end{cases};$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} \ \Big(\zeta \text{ denotes Riemann zeta function}\Big).$$

Also, we shall use the following inequality

$$\left|\sum_{n=[\sqrt[p]{t}]+1}^{\infty}\frac{\mu(n)}{n^2}\right| \le \sum_{n=[\sqrt[p]{t}]+1}^{\infty}\frac{1}{n^2} = \mathcal{O}\Big(\frac{1}{\sqrt[p]{t}}\Big).$$

Thus, we have

$$D(t) \;=\; \sum_{l=1}^{\infty} D\Big(\frac{t}{l^p}\Big)\Big(\sum_{a|l}\mu(a)\Big)$$

$$
\begin{aligned}
&= \sum_{n=1}^{[\sqrt[p]{t}]} \mu(n)\Big(\sum_{m=1}^{\infty} D\Big(\frac{t}{n^p m^p}\Big)\Big) \\
&= \sum_{n=1}^{[\sqrt[p]{t}]} \mu(n) B\Big(\frac{t}{n^p}\Big) \\
&= \sum_{n=1}^{[\sqrt[p]{t}]} \mu(n)\bigg(t^{1/p}\Big(\frac{t}{n^p}\Big)^{2/p} C_x^p(\alpha) + \mathcal{O}\Big(t^{1/p}\frac{t^{1/p}}{n}\Big)\bigg) \\
&= \sum_{n=1}^{[\sqrt[p]{t}]} \mu(n)\frac{t^{3/p}}{n^2} C_x^p(\alpha) + \mathcal{O}\Big(\sum_{n=1}^{[\sqrt[p]{t}]} \big|\mu(n)\big|\frac{t^{2/p}}{n}\Big) \\
&= C_x^p(\alpha) t^{3/p}\Big(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2} - \sum_{n=[\sqrt[p]{t}]+1}^{\infty}\frac{\mu(n)}{n^2}\Big) + \mathcal{O}(t^{2/p}\cdot\log t) \\
&= C_x^p(\alpha) t^{3/p}\frac{6}{\pi^2} + \mathcal{O}(t^{2/p}\cdot\log t).
\end{aligned}
$$

Since $D(t)$ equals $x_p(n_p(t),\alpha)$, we have

$$
x_p(n_p(t),\alpha) = \frac{6}{\pi^2} C_x^p(\alpha) t^{3/p} + \mathcal{O}(t^{2/p}\cdot\log t).
$$

The proof for $y_p(n_p(t),\alpha)$ is analogous. If we consider slightly different 3–dimensional body

$$
\mathcal{B}'(v) = \Big\{(x,y,z)\ \Big|\ x>0,\ y>0,\ x^p+y^p\le v,\ \frac{y}{x}\le\alpha,\ 0<z\le y\Big(\frac{t}{v}\Big)^{1/p}\Big\},
$$

following the same course of proof we obtain

$$
y_p(n_p(t),\alpha) = \frac{6}{\pi^2} C_y^p(\alpha) t^{3/p} + \mathcal{O}(t^{2/p}\cdot\log t),
$$

where $C_y^p(\alpha)$ is given by

$$
C_y^p(\alpha) = \int_0^{\frac{\alpha}{\sqrt[p]{\alpha^p+1}}} \big(n\sqrt[p]{1-n^p} - \frac{n^2}{\alpha}\big)\, dn.
$$

This function can also be represented by series, in the following way

$$
C_y^p(\alpha) = \alpha^2\left(-\frac{1}{3}(\alpha^p+1)^{-3/p} + \sum_{k=0}^{\infty}\binom{-\frac{3}{p}-1}{k}\frac{\alpha^{pk}}{pk+2}\right). \qquad \diamondsuit
$$

The following lemma gives asymptotic expressions satisfied by coordinates of the vertices of normalized polygons $\dfrac{1}{\mathrm{diam}_\infty(Q_p(n_p(t)))}Q_p(n_p(t))$.

Lemma 2. *If $(\tilde{x}_p(n_p(t),\alpha),\tilde{y}_p(n_p(t),\alpha))$ is the end point of the edge with slope α of the south-east arc of the polygon $\dfrac{1}{diam_\infty(Q_p(n_p(t)))}Q_p(n_p(t))$, then the following asymptotic expressions hold:*

$$\tilde{x}_p(n_p(t),\alpha) = \frac{C_x^p(\alpha)}{I_p} + \mathcal{O}\Big(\frac{\log t}{t^{1/p}}\Big),$$

$$\tilde{y}_p(n_p(t),\alpha) = \frac{C_y^p(\alpha)}{I_p} + \mathcal{O}\Big(\frac{\log t}{t^{1/p}}\Big),$$

where

$$I_p = \int_0^1 (\sqrt[p]{1-l^p})^2 \, dl.$$

Proof. Obviously, diameter in sense of l_∞ metric of $Q_p(n_p(t))$ is equal to $2 \cdot x_p(n_p(t),\alpha)$ when α tends to infinity. Using Lemma 1. we have

$$\begin{aligned}
\mathrm{diam}_\infty(Q_p(n_p(t))) &= 2 \cdot x(n_p(t),\infty) \\
&= \frac{6t^{3/p}}{\pi^2}\int_0^1 (\sqrt[p]{1-l^p})^2 \, dl + \mathcal{O}(t^{2/p} \cdot \log t) \\
&= \frac{6}{\pi^2} I_p t^{3/p} + \mathcal{O}(t^{2/p} \cdot \log t),
\end{aligned}$$

and therefore we have

$$\tilde{x}_p(n_p(t),\alpha) = \frac{x(n_p(t),\alpha)}{\mathrm{diam}_\infty(Q_p(n_p(t)))} = \frac{C_x^p(\alpha)}{I_p} + \mathcal{O}\Big(\frac{\log t}{t^{1/p}}\Big). \qquad \diamondsuit$$

Theorem 3. *The limit shape of the south–east arc of sequence of polygons $Q_p(n_p(t))$ is the curve γ_p given parametrically by $\left(\dfrac{C_x^p(s)}{I_p}, \dfrac{C_y^p(s)}{I_p}\right)$, $0 < s < \infty$.*

Proof. We denote the normalized optimal convex lattice polygons by

$$\tilde{Q}_p(n_p(t)) = \frac{1}{\mathrm{diam}_\infty(Q_p(n_p(t)))} Q_p(n_p(t)).$$

Firstly, we are going to show that for fixed α the sequence of vertices $\{(\tilde{x}_p(n_p(t),\alpha),\tilde{y}_p(n_p(t),\alpha))\}_{t \geq t_0}$ converges to a point on the curve γ_p. Note that if $\tilde{Q}_p(n_p(t_0))$ has an edge with slope α then $\tilde{Q}_p(n_p(t))$ also has an edge with slope α, for all $t > t_0$. Now, using the asymptotic expressions from the previous lemma, we have

$$\lim_{t\to\infty} (\tilde{x}_p(n_p(t),\alpha), \tilde{y}_p(n_p(t),\alpha)) = \left(\frac{C_x^p(\alpha)}{I_p}, \frac{C_y^p(\alpha)}{I_p}\right).$$

Secondly, we are going to prove that for each point on the curve γ_p there exists a sequence of vertices of polygons $\tilde{Q}_p(n_p(t))$ which converges to that point.

The beginning and end vertices of south-east arc of $\tilde{Q}_p(n_p(t))$ converge to $(0,0)$ and $(1/2,1/2)$ respectively, and those two points are also the beginning and end points of the curve γ_p (for $s = 0, \infty$). If s is positive rational number, there exists a sequence of vertices ($\alpha = s$) which converges to $\gamma_p(s)$.

Also, the maximal distance of two consecutive vertices A_{k-1} and A_k (where slope of $[A_{k-1}, A_k]$ is m/n, $(m,n) = 1$) of $\tilde{Q}_p(n_p(t))$ tends to zero as t tends to infinity:

$$\begin{aligned} l_2([A_{k-1}A_k]) &= \frac{\sqrt{m^2+n^2}}{\operatorname{diam}_\infty(Q_p(n_p(t)))} \\ &\leq \frac{\sqrt{2}t^{1/p}}{\frac{6}{\pi^2}I_p t^{3/p} + \mathcal{O}(t^{2/p}\log t)} \to 0, \text{ when } t \to 0. \end{aligned}$$

Therefore, γ_p is the limit shape of the south–east arcs of the polygon sequence $\tilde{Q}_p(n_p(t))$. $\diamondsuit$

4 Limit shape of $Q_p(n)$

In this section we are going to find the limit shape of the sequence of polygons $Q_p(n)$ (where n is arbitrary integer). Note that in some cases polygon $Q_p(n)$ is not uniquely determined for fixed n, and that for $n = 2k+1$, $p > 1$ the construction of $Q_p(n)$ is not known.

First we are going to find the limit shape for polygons $\tilde{Q}_p(2k)$, where $2k$ is an arbitrary even integer.

Theorem 4. *The limit shape of the south-east arc of sequence of polygons $Q_p(2k)$ is the curve γ_p given parametrically by* $\left(\frac{C_x^p(s)}{I_p}, \frac{C_y^p(s)}{I_p}\right)$, $0 < s < \infty$.

Proof. For each even number $2k$ we can find an integer t such that $n_p(t-1) \leq 2k < n_p(t)$. From the construction of $Q_p(2k)$ we have that the set of edge slopes of $Q_p(n_p(t-1))$ is a subset of the set of edge slopes of $Q_p(2k)$, and the set of edge slopes of $Q_p(2k)$ is a subset of the set of edge slopes of $Q_p(n_p(t))$.

Therefore we have the following asymptotic inequalities

$$x_p(n_p(t-1), \alpha) \leq x_p(2k, \alpha) \leq x_p(n_p(t), \alpha),$$

$$\mathrm{diam}_\infty(Q_p(n_p(t-1))) \leq \mathrm{diam}_\infty(Q_p(2k)) \leq \mathrm{diam}_\infty(Q_p(n_p(t))).$$

Since we know that the asymptotic expression for both first and last part of each of the last two inequalities is the same (Lemma 1.):

$$x_p(n_p(t-1), \alpha) = x_p(n_p(t), \alpha) = \frac{6}{\pi^2} C_x^p(\alpha) t^{3/p} + \mathcal{O}(t^{2/p} \cdot \log t),$$

$$\mathrm{diam}_\infty(Q_p(n_p(t-1))) = \mathrm{diam}_\infty(Q_p(n_p(t))) = \frac{6}{\pi^2} I_p t^{3/p} + \mathcal{O}(t^{2/p} \cdot \log t),$$

so we can derive

$$\tilde{x}_p(2k, \alpha) = \frac{C_x^p(\alpha)}{I_p} + \mathcal{O}\left(\frac{\log t}{t^{1/p}}\right),$$

$$\tilde{y}_p(2k, \alpha) = \frac{C_y^p(\alpha)}{I_p} + \mathcal{O}\left(\frac{\log t}{t^{1/p}}\right).$$

The rest of the proof is analogous to the proof of Theorem 3. $\diamondsuit$

The explicit construction of $Q_p(2k+1)$, where $2k+1$ is an arbitrary odd integer, is an open problem (for all $p > 1$). Therefore, the following lemma proven by Stojaković in [7] is of great importance to us.

Lemma 3. [7] *If $2k+1$ is an odd number, and t is an integer such that*

$$n_p(t-1) < 2k+1 < n_p(t),$$

then the number of edges of the polygon $Q_p(2k+1)$ longer (in sense of l_p–metric) then $t^{1/p}$ is upper bounded by $\mathcal{O}(t^{1/p+\varepsilon})$, for arbitrary $\varepsilon > 0$.

Using the last lemma, we can give the asymptotic position of the vertices of $Q_p(2k+1)$.

Lemma 4. *If $2k+1$ is an odd number, and t is an integer such that $n_p(t-1) < 2k+1 < n_p(t)$, then the coordinates of the vertices of $Q_p(2k+1)$ satisfy the following asymptotic expressions*

$$x_p(2k+1,\alpha) = \frac{6}{\pi^2} C_x^p(\alpha) t^{3/p} + \mathcal{O}(t^{2/p+\varepsilon}),$$

$$y_p(2k+1,\alpha) = \frac{6}{\pi^2} C_y^p(\alpha) t^{3/p} + \mathcal{O}(t^{2/p+\varepsilon}).$$

Proof. First we shall prove that the length (in sense of l_p–metric) of an edge of polygon $Q_p(2k+1)$ is upper bounded. If we assume that there is an edge of $Q_p(2k+1)$ longer than $2t^{1/p}$, then we have

$$\mathrm{per}_p(Q_p(2k+1)) > \mathrm{glb}_p(2k+1) + t^{1/p} = \mathrm{glb}_p(2k+2) = \mathrm{per}_p(Q_p(2k+2)),$$

which is a contradiction.

Therefore, for every edge e of $Q_p(2k+1)$ we have

$$l_p(e) = \sqrt[p]{x(e)^p + y(e)^p} \le 2t^{1/p} \Rightarrow$$

$$x(e) \le 2t^{1/p}, \quad y(e) \le 2t^{1/p}.$$

If we denote the number of edges of the polygon $Q_p(2k+1)$ longer (in sense of l_p–metric) then $t^{1/p}$ by δ, we can give the following upper bound for $|x_p(n_p(t),\alpha) - x_p(2k+1,\alpha)|$ (using Lemma 3. and Theorem 2.):

$$\begin{aligned} |x_p(n_p(t),\alpha) - x_p(2k+1,\alpha)| &\le \max\Big(2t^{1/p}\cdot\delta,\ t^{1/p}\cdot(n_p(t)-2k+1)\Big) \\ &= \mathcal{O}\Big(\max(t^{2/p+\varepsilon}, t^{2/p})\Big) = \mathcal{O}(t^{2/p+\varepsilon}), \end{aligned}$$

As a consequence of this inequality and Lemma 1. we can obtain the asymptotic expression for $x_p(2k+1,\alpha)$:

$$x_p(2k+1,\alpha) = \frac{6}{\pi^2} C_x^p(\alpha) t^{3/p} + \mathcal{O}(t^{2/p+\varepsilon}).$$

The course of proof for $y_p(2k+1,\alpha)$ is analogous. $\diamondsuit$

Theorem 5. *The limit shape of the south–east arc of sequence of polygons $Q_p(2k+1)$ is the curve γ_p given parametrically by $\left(\frac{C_x^p(s)}{I_p}, \frac{C_y^p(s)}{I_p}\right)$, $0 < s < \infty$.*

Proof. Firstly, we are going to show that that for fixed α sequence of vertices $\{(\tilde{x}_p(n_p(t),\alpha),\tilde{y}_p(n_p(t),\alpha))\}_{t\geq t_0}$ converges to a point on the curve γ_p.

The last lemma (for $\alpha=\infty$) gives us the expression for $\mathrm{diam}_\infty Q_p(2k+1)$:

$$\mathrm{diam}_\infty Q_p(2k+1) = 2\cdot x_p(2k+1,\infty) = \frac{6}{\pi^2}I_p + \mathcal{O}(t^{2/p+\varepsilon}).$$

Now using this and the asymptotic expressions from lemma 4., we have

$$\lim_{t\to\infty}(\tilde{x}_p(2k+1,\alpha),\tilde{y}_p(2k+1,\alpha)) = \left(\frac{C_x^p(\alpha)}{I_p},\frac{C_y^p(\alpha)}{I_p}\right).$$

Secondly, we are going to prove that for each point on the curve γ_p there exists a sequence of vertices of polygons $\tilde{Q}_p(n_p(t))$ which converges to that point.

The beginning and end vertices of south-east arcs of $\tilde{Q}_p(2k+1)$ converge to the beginning and end points of the curve γ_p. If s is positive rational number, there exists a sequence of vertices ($\alpha = s$) which converges to $\gamma_p(s)$.

Also, the maximal distance of between consecutive vertices A_{l-1}, A_l (slope of $[A_{l-1},A_l]$ is m/n, $(m,n)=1$) of $\tilde{Q}_p(2k+1)$ tends to zero as t tends to infinity:

$$\begin{aligned} l_2([A_{l-1}A_l]) &= \frac{\sqrt{m^2+n^2}}{\mathrm{diam}_\infty(Q_p(n_p(t)))} \\ &\leq \frac{\sqrt{2}t^{1/p}}{\frac{6}{\pi^2}I_p t^{3/p}+\mathcal{O}(t^{2/p}\log t)} \to 0,\ \text{when } t\to 0. \end{aligned}$$

$$(l_p(e)\leq 2t^{1/p} \Rightarrow l_2(e)\leq 2\sqrt{2}t^{1/p} \text{ is used})$$

Therefore, γ_p is the limit shape of the south–east arcs of the polygon sequence $\tilde{Q}_p(2k+1)$. $\diamondsuit$

Finally, we present the main theorem of this paper, in which the limit shape for the sequence of polygons $Q_p(n)$ (where n is arbitrary integer) is given.

Theorem 6. *The limit shape of the south-east arc of sequence of optimal convex lattice polygons $Q_p(n)$ is the curve γ_p given parametrically by* $\left(\frac{C_x^p(s)}{I_p},\frac{C_y^p(s)}{I_p}\right)$, $0<s<\infty$.

Limit shapes of the north–east, north–west and south–west arc are the same curves (γ_p) rotated for $\pi/2$, π and $3\pi/2$ radians (respectively) and translated to form a closed curve.

Proof. This theorem is a direct consequence of Theorems 4. and 5., and the symmetry of the construction of optimal convex lattice polygons. $\diamondsuit$

Note 1. The problem of finding the limit shape in case $p = \infty$ can be solved similarly, and all theorems proved above hold in that case too.

The limit shape γ_∞ of south–east arc of the sequence of polygons $Q_\infty(n)$ is also obtained as a parametrically given curve $\left(C_x^\infty(s), C_y^\infty(s)\right)$, $0 < s < \infty$, where

$$C_x^\infty(\alpha) = \begin{cases} \dfrac{\alpha}{3}, & \alpha \le 1 \\ \dfrac{1}{2} - \dfrac{1}{6\alpha^2}, & \alpha > 1 \end{cases},$$

$$C_y^\infty(\alpha) = \begin{cases} \dfrac{\alpha^2}{6}, & \alpha \le 1 \\ \dfrac{1}{2} - \dfrac{1}{3\alpha}, & \alpha > 1 \end{cases}.$$

The explicit form of γ_∞ is

$$y(x) = \begin{cases} \dfrac{3}{2}x^2, & x \in (0, 1/3] \\ \dfrac{1}{2} - \sqrt{\dfrac{1-2x}{3}}, & x \in (1/3, 1/2) \end{cases}.$$

Note 2. In case $p = 2$ the limit shape of the sequence of optimal convex lattice polygons $Q_2(n)$ is a circle.

In the figure 1. limit shape curves are shown for $p = 1$ (dotted curve),$2, 3, 5$ and ∞ (graphs from bottom to top respectively).

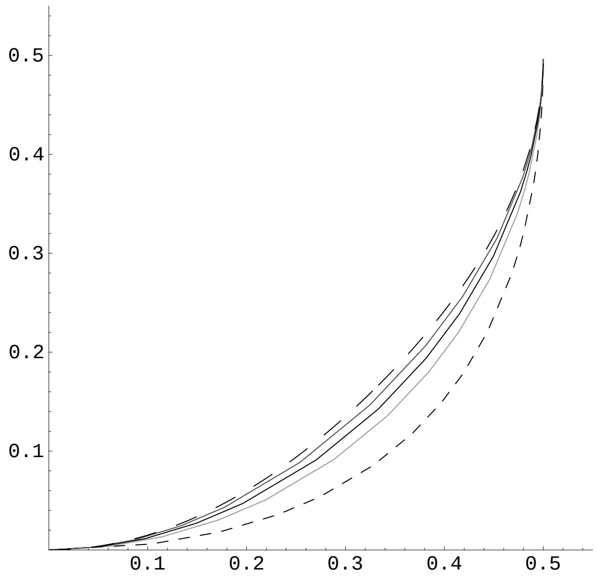

Figure 1. Limit shapes for $p = 1$ (dotted curve), 2, 3, 5 and ∞

5 Area of $Q_p(n_p(t))$

Knowing the limit shape of classes of optimal convex lattice polygons makes the further work on asymptotic problems dealing with them easier. Although we do not know much on optimal convex lattice polygons with odd number of vertices, the existence of a common limit shape enables us to treat the whole sequence of classes of optimal polygons at once.

One of the important questions is: "What is the the minimal area $A(n)$ of a convex lattice polygon with n vertices?" Analysing Jarnik's curve, Žunić in [12] obtained an upper bound $1/54n^3$ for $A(n)$. Recently, Bárány and Tokushige showed in [3] that the limit $\lim A(n)/n^3$ is equal to the minimum of a set of real values which is believed to be 0.0185067.

Using the limit shape we can obtain the area of optimal convex lattice polygons in any metric. In other words, we can prove the following:

- In case $p = 1$, we have that the area of $Q_1(n)$ is

$$P(Q_1(n)) = \frac{5\pi^2}{2592}n^3 + \mathcal{O}(n^{5/2+\varepsilon}) \qquad (\frac{5\pi^2}{2592} \approx 0.0190386).$$

- In case $p = 2$, we have that the area of $Q_2(n)$ is

$$P(Q_2(n)) = \frac{1}{54}n^3 + \mathcal{O}(n^{5/2+\varepsilon}) \qquad (\frac{1}{54} \approx 0.0185185)!$$

- In case $p = 3$, we have that the area of $Q_3(n)$ is

$$P(Q_3(n)) \approx 0.0190335n^3 + \mathcal{O}(n^{5/2+\varepsilon}).$$

....

- In case $p = \infty$, we have that the area of $Q_\infty(n)$ is

$$P(Q_\infty(n)) = \frac{5\pi^2}{2592}n^3 + \mathcal{O}(n^{5/2+\varepsilon}) \qquad (\frac{5\pi^2}{2592} \approx 0.0190386).$$

Note that all the leading coefficients (in the area expressions) are considerably close. The optimal polygons in the sense of Euclidean metric $Q_2(n)$ have the smallest area (as a function of the number of their vertices) of all optimal polygons considered in this paper.

References

[1] T. M. Apostol, "Introduction to Analytic Number Theory", Springer-Verlag, 1976.

[2] I. Bárány, The Limit Shape of Convex Lattice Polygons, *Discrete & Comput. Geom.*, 13, (1995), 279–295.

[3] I. Bárány, N. Tokushige, The Minimum Area of Convex Lattice n–gons, *Workshop Extremal Combinatorics 2002*, ETH Zürich.

[4] E. Bombieri and J. Pila, The Number of Integral Points on Arcs and Ovals, *Duke Mathematical Journal*, 59, (1989), 337–357.

[5] V. Jarnik, Über die Gitterpunkte auf konvexen Kurven, *Math. Zeitschrift*, 24, (1925), 500–518.

[6] E. Krätzel, Lattice Points (VEB Deutscher Verlag der Wissentschaften), Berlin, 1988.

[7] M. Stojaković, Perimeter of Optimal Convex Lattice Polygons in the Sense of Different Metrics, *Bull. Austral. Math. Soc.*, 63, (2001), 229–242.

[8] H. P. F. Swinerton–Dyer, The Number of Lattice Points on a Convex Curve, *J. Number Theory,* 6, (1974), 128–135.

[9] A. M. Vershik, The limit shape of Convex Lattice Polygons and Related Topics, *Funksional Anal. Appl.*, 28, (1994), 16–25.

[10] J. Žunić, Extremal Problems On Convex Lattice Polygons In Sense Of l_p–metrics, *Discrete Mathematics*, 259, (2002), 237–250.

[11] J. Žunić, Limit Shape of Convex Lattice Polygons Having the Minimal l_∞ Diameter w.r.t. the Number of Their Vertices, *Discrete Mathematics*, 187, (1998), 245–254.

[12] J. Žunić, Notes on Optimal Convex Lattice Polygons, *Bull. London Math. Soc.*, 30, (1998), 377–385.