Global Maker-Breaker games on sparse graphs

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Abstract

In this paper we consider Maker-Breaker games, played on the edges of *sparse* graphs. For a given graph property $\mathcal{P}$ we seek a graph (board of the game) with the smallest number of edges on which Maker can build a subgraph that satisfies $\mathcal{P}$. In this paper we focus on *global* properties. We prove the following results: 1) for the positive minimum degree game, there is a winning board with n vertices and about $10n/7$ edges, on the other hand, at least $11n/8$ edges are required; 2) for the spanning k-connectivity game, there is a winning board with n vertices and $(1+o_k(1))kn$ edges; 3) for the Hamiltonicity game, there is a winning board of constant average degree; 4) for a tree T on n vertices of bounded maximum degree Δ, there is a graph G on n vertices and at most $f(\Delta) \cdot n$ edges, on which Maker can construct a copy of T. We also discuss biased versions on these games and argue that the picture changes quite drastically there.

1 Introduction

In this paper we investigate positional games played on edge-sets of graphs. Let $\mathcal{P} = \mathcal{P}(n) \subseteq 2^{E(K_n)}$ be a graph property of n-vertex graphs, and let G be a graph on the vertex set $V(G) = V(K_n)$. The game $(E(G), \mathcal{P})$ is played by two players, called Maker and

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Breaker, who take turns in claiming one previously unclaimed edge of G, with Breaker going first. The graph G is called the *base graph* or (with a slight abuse of terminology) the *board*. The game ends when every edge of G has been claimed by some player. Maker wins the game if the graph he builds by the end of the game satisfies property $\mathcal{P}$, otherwise Breaker wins. Thus, the graph property $\mathcal{P}$ will be sometimes referred to as the family of *winning sets* (of edges). We say that the game $(E(G), \mathcal{P})$ is a *Maker's win* if Maker has a strategy that ensures his win in this game against any strategy of Breaker, otherwise the game is a *Breaker's win*. Note that G and $\mathcal{P}$ alone determine whether the game is a Maker's win or a Breaker's win. For the purposes of this paper, $\mathcal{P}$ is assumed to be monotone increasing. Hence, Maker wins $(E(G), \mathcal{P})$ if and only if he occupies an inclusion-minimal element of $\mathcal{P}$. Whenever there is no risk of confusion, we may use $\mathcal{P}$ to denote the family of inclusion-minimal members of $\mathcal{P}$.

One of the simplest examples of a graph game is the connectivity game, where the family $\mathcal{C}_1 = \mathcal{C}_1(n)$ of winning sets consists of all spanning trees of K_n – the complete graph on n vertices. Lehman's Theorem [16] asserts that Maker is able to win this game if the base graph contains the edge disjoint union of two spanning trees. That is, Maker can win on a graph with as few as $2n - 2$ edges. Clearly this is best possible.

The game parameter we introduce and study in this paper is the following.

Definition 1.1 *For a graph property $\mathcal{P} = \mathcal{P}(n) \subseteq 2^{E(K_n)}$ of graphs on n vertices, let $\hat{m}(\mathcal{P})$ be the smallest integer $m = m(n)$ for which there exists a graph G with n vertices and m edges, such that $(E(G), \mathcal{P})$ is a Maker's win. For the sake of formality we define $\hat{m}(\mathcal{P})$ to be ∞ if $(E(K_n), \mathcal{P})$ is a Breaker's win.*

While Definition 1.1 is very general and covers a large variety of very different games, in this paper we restrict our attention to *global* properties. By the term *global property*, we mean a property of n-vertex graphs that does not ignore any vertex. That is, if $\mathcal{P}$ is a global property and $G \in \mathcal{P}$, then, in particular, the minimum degree of G is positive. Two simple examples of global properties are the property of having positive minimum degree and the property of admitting a Hamilton cycle.

It would be interesting to consider 'non-global' graph properties as well. In the game theoretic context, this is bound to make a huge difference, especially when considering games $(E(G), \mathcal{P})$ where the number of vertices of G is arbitrarily large (or even infinite), but the inclusion-minimal winning sets of $\mathcal{P}$ have constant size. For example, when the target property $\mathcal{P}$ is (the containment of) a triangle, it is easy to see that $\hat{m}(\mathcal{P})$ is constant. This is true regardless of the number of vertices in the base graph. Clearly, for a global property $\mathcal{P}(n)$, the number of edges in any winning set is at least $n/2$; in particular, it grows with n. An intermediate case is that of properties $\mathcal{P} = \mathcal{P}(n)$ for which the size of every winning set does grow with n and yet it is possible that $G \in \mathcal{P}$ even though there are isolated vertices in G. Natural examples of such properties are the property of admitting a giant component and the property of admitting an almost spanning tree. While such properties are global in some sense, we do not consider them in this paper. We discuss this issue in more detail in Section 7.

Lehman's Theorem states that $\hat{m}(\mathcal{C}_1) = 2n - 2$. In many contexts connectivity is tightly related to the weaker property of having positive minimum degree, that is, of containing no isolated vertex, see [6]. The corresponding family $\mathcal{D}_1 = \mathcal{D}_1(n)$ consists of the edge sets of all graphs on n vertices which have minimum degree at least 1. The next theorem shows that connectivity and positive minimum degree behave differently in our context.

Theorem 1.2

$$(i) \quad \hat{m}(\mathcal{D}_1) \leq \frac{10}{7}n + 4 \textit{ for all } n \geq 49;$$

$$(ii) \quad \hat{m}(\mathcal{D}_1) \geq \frac{11}{8}n \textit{ for all } n.$$

It follows from Lehman's Theorem [16] that Maker can build a *k-edge-connected* spanning subgraph, when playing on the edge set of any graph that admits $2k$ pairwise edge disjoint spanning trees. Hence, for every positive integer k and sufficiently large n, there exists a graph with n vertices and $2k(n-1)$ edges, on which Maker can build a k-edge-connected spanning subgraph. This is tight for $k = 1$ by Lehman's Theorem. In our next theorem we improve this upper bound for every $k \geq 2$, even for the stronger property of being k-*vertex*-connected.

Let k be a positive integer. The family $\mathcal{C}_k = \mathcal{C}_k(n) \subseteq 2^{E(K_n)}$ consists of the edge sets of all k-vertex-connected graphs on n vertices. Since Breaker can claim at least half of the edges incident with some fixed vertex, and since the minimum degree of any k-vertex-connected graph is at least k, it follows that $\hat{m}(\mathcal{C}_k) \geq kn$. We prove that this is essentially tight for large k.

Theorem 1.3 (i) *For every positive integer k and every $n \geq 3 \cdot 2^{k+1}$, we have*

$$\hat{m}(\mathcal{C}_k) \leq \left(\frac{3}{2}k + 1\right) n.$$

(ii) *For every positive integer k, and for sufficiently large n, we have*

$$\hat{m}(\mathcal{C}_k) \leq (1 + o_k(1))kn.$$

In the Hamiltonicity game, the family of winning sets $\mathcal{H} = \mathcal{H}(n) \subseteq 2^{E(K_n)}$ consists of the edge sets of all Hamilton cycles on n vertices. Since, by the end of the game, Maker claims at most half of the board elements, and since there are n edges in a Hamilton cycle of a graph on n vertices, it is evident that $\hat{m}(\mathcal{H}) \geq 2n$. On the other hand, it was proved in [13] that Maker can almost surely build a Hamiltonian cycle, in the game played on the edge set of a random graph $G(n,p)$, where $p = (1 + o(1)) \ln n / n$. It follows that, $\hat{m}(\mathcal{H}) \leq (1/2 + o(1)) n \ln n$. We improve the aforementioned trivial lower bound, and prove an upper bound which is only a multiplicative constant factor away.

Theorem 1.4

$$(i)\quad \hat{m}(\mathcal{H}) \geq 2.5n \textit{ for all } n;$$
$$(ii)\quad \hat{m}(\mathcal{H}) \leq 21n \textit{ for all } n \geq 1600.$$

Let T be a fixed tree on n vertices. In the tree construction game $\mathcal{G}_T = \mathcal{G}_T(n)$, Maker's goal is to create a copy of T, that is, the winning sets of $\mathcal{G}_T$ are the edge sets of all graphs on n vertices that admit a copy of T. An obvious lower bound for $\hat{m}(\mathcal{G}_T)$ is $2n - 2$. We prove that if T has bounded degree, then there exists a base graph G with a linear number of edges, on which Maker wins $\mathcal{G}_T$.

Theorem 1.5 *For every Δ, there is $A = A(\Delta)$, such that for all sufficiently large n,*

$$\hat{m}(\mathcal{G}_T) \leq An$$

holds for every tree T on n vertices with maximum degree at most Δ.

Finally, we make an observation concerning *biased games* – a widely studied generalization, suggested by Chvátal and Erdős [8]. In a biased $(a : b)$ game, Maker claims a board elements in each round, whereas Breaker claims b board elements in each round. The games we have studied so far are thus $(1 : 1)$ games. For a property $\mathcal{P} = \mathcal{P}(n) \subseteq 2^{E(K_n)}$ and a positive integer q, let $\hat{m}(\mathcal{P}; q)$ be the smallest integer m for which there exists a graph G with n vertices and m edges, such that Maker can build a graph with property $\mathcal{P}$, when playing a $(1 : q)$ game on $E(G)$ (again $\hat{m}(\mathcal{P}; q)$ is defined to be ∞ if the $(1 : q)$ game $(E(K_n), \mathcal{P})$ is a Breaker's win).

Though Lehman's Theorem [16] is a "perfect theorem" for the $(1 : 1)$ connectivity game, it fails to provide any implications when Breaker plays with a bias larger than 1. In fact, for all of the games studied in our paper, the parameter $\hat{m}$ undergoes a "phase transition" as Breaker's bias changes from 1 to 2. Indeed, as all previous theorems indicate, $\hat{m}(\mathcal{P}) = \Theta(n)$ whenever $\mathcal{P} \in \{\mathcal{D}_1(n), \mathcal{C}_k(n), \mathcal{H}(n), \mathcal{G}_T(n)\}$. Our next theorem shows that when Breaker's bias is at least 2, he can isolate a vertex of the base graph G as long as $e(G) < cn \ln n$, where $c > 0$ is an appropriate constant. It follows that $\hat{m}(\mathcal{P}; q) = \omega(n)$ whenever $q \geq 2$ and $\mathcal{P} \in \{\mathcal{D}_1(n), \mathcal{C}_k(n), \mathcal{H}(n), \mathcal{G}_T(n)\}$. For the connectivity game $\mathcal{C}_1$, we obtain a fairly sharp bound.

Theorem 1.6 *Let $\varepsilon > 0$, let $n = n(\varepsilon)$ be sufficiently large, and let $q = q(n)$ be an integer.*

$$(i)\quad \textit{If } q \leq (\ln 2 - \varepsilon)n/\ln n, \textit{ then } \hat{m}(\mathcal{C}_1, q) \leq (1/2 + \varepsilon)qn \log_2 n;$$
$$(ii)\quad \textit{If } q \geq 2, \textit{ then } \hat{m}(\mathcal{D}_1, q) \geq (1/2 - \varepsilon)(q-1)n \ln n.$$

Note that there is no upper bound on q in Part 2 of Theorem 1.6. Hence, the asserted lower bound on $\hat{m}(\mathcal{D}_1, q)$ might exceed $\binom{n}{2}$. This is fine, as $\hat{m}$ is defined to be ∞ in this case.

For the sake of simplicity and clarity of presentation, we do not make a particular effort to optimize the constants obtained in theorems we prove. We also omit floor and ceiling signs whenever these are not crucial. Many of our results are asymptotic in nature and, whenever necessary, we assume that n is sufficiently large. Throughout the paper, ln stands for the natural logarithm, and $\log_2$ for binary logarithm. Our graph-theoretic notation is standard and follows that of [20]. In particular, we use the following. For a graph G, let $V(G)$ and $E(G)$ denote its sets of vertices and edges respectively, and let $v(G) = |V(G)|$ and $e(G) = |E(G)|$. For a set $A \subseteq V(G)$, let $G[A]$ denote the subgraph of G induced on the vertex set A. For a set $A \subseteq V(G)$, let $E_G(A)$ denote the set of edges of G with both endpoints in A and let $e_G(A) = |E_G(A)|$. For disjoint sets $A, B \subseteq V(G)$, let $E_G(A, B)$ denote the set of edges of G with one endpoint in A and one endpoint in B, and let $e_G(A, B) = |E_G(A, B)|$. Sometimes, if there is no risk of confusion, we discard the subscript G in the above notation. For a vertex $v \in V(G)$ and $A \subseteq V(G)$, let $N_A(v)$ denote the set of all vertices of A that are adjacent to v in G, and let $d_A(v) = |N_A(v)|$. We abbreviate $d_{V(G)}(v)$ to $d(v)$. Let $\bar{d}(G)$ denote the average degree of G.

The rest of this paper is organized as follows: in Section 2 we prove Theorem 1.2; in Section 3 we prove Theorem 1.3; in Section 4 we prove Theorem 1.4; in Section 5 we prove Theorem 1.5 and in Section 6 we prove Theorem 1.6. Finally, in Section 7, we present some open problems.

2 The positive min-degree game

Proof of Theorem 1.2. Starting with the upper bound, let D_7 be a *double diamond*, that is, $V(D_7) = \{v_1, v_2, \dots, v_7\}$ and $E(D_7) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4, v_4v_5, v_4v_6, v_5v_6, v_5v_7, v_6v_7\}$. Note that D_7 has seven vertices and ten edges.

Lemma 2.1 *Playing on $E(D_7)$, Maker can win the $(1:1)$ positive minimum degree game, as the first or second player.*

The proof of Lemma 2.1 is by case analysis. For the sake of completeness we include it here.

Proof of Lemma 2.1. Clearly, it suffices to prove that Maker can win the game as the second player. We will refer to the subgraphs of D_7 induced on vertex sets $\{v_1, v_2, v_3, v_4\}$ and $\{v_4, v_5, v_6, v_7\}$ as *the upper diamond* and *lower diamond*, respectively.

During the course of play, Maker will occasionally *pair up* two unclaimed edges, or mark some unclaimed edges with a "$\star$". Throughout the game, whenever Breaker claims one of the two paired edges, Maker responds by immediately claiming the other one.

We will describe Maker's strategy for the upper diamond, and the strategy for the lower diamond will be identical, just "mirrored", swapping the vertices v_5, v_6, v_7 and v_2, v_3, v_1, respectively, in the strategy description.

Without loss of generality assume that Breaker makes his first move in the upper diamond. Breaker can play his first move in essentially three different ways.

1. If he claims v_2v_3, Maker responds by claiming v_1v_2, pairs up v_1v_3 and v_3v_4, and marks v_2v_4 with a "$\star$".
2. If Breaker claims v_1v_2, Maker claims v_1v_3, pairs up v_2v_3 and v_2v_4, and marks v_3v_4 with a "$\star$".
3. If Breaker claims v_3v_4, Maker claims v_1v_3, pairs up v_1v_2 and v_2v_3, and marks v_2v_4 with a "$\star$".

If Breaker claims the edge marked with a star in the upper diamond, Maker will claim an edge in the lower diamond. We distinguish between two cases.

(i) Up to that point, no player has played in the lower diamond. Maker claims v_5v_6 and pairs up v_4v_5 with v_4v_6, and also v_5v_7 with v_6v_7.

(ii) Some edges are already claimed in the lower diamond. Maker claims the edge of the lower diamond which is marked with a "$\star$".

If Breaker claims the edge marked with a star in the lower diamond, then Maker responds immediately by claiming the edge marked with a star in the upper diamond (note that it must be free).

It is easy to see that, following this strategy, at the end of the game Maker will have two edges in the diamond in which Breaker has claimed the edge marked with a "$\star$", and three edges in the other diamond. Then, Maker's graph will have positive degree in all four vertices of the diamond in which he has three edges, and in all vertices but v_4, of the diamond in which he has two edges. This concludes the proof of the lemma. $\square$

Let $n = 7k + r$, where $0 \leq r \leq 6$ and $k \geq 7$. Let G be a graph consisting of $2r$ vertex disjoint copies of K_4 and $k - r$ vertex disjoint copies of D_7. Note that $v(G) = n$ and $e(G) = 12r + 10(k - r) = 10k + 2r \leq \frac{10}{7}n + 4$. Moreover, it is clear that Maker wins the $(1 : 1)$ positive minimum degree game played on $E(G)$, as he can play $k+r$ separate games in parallel – one game on each connected component; whenever Breaker claims an edge of some connected component, Maker responds by playing in the same connected component. Lemma 2.1 implies that he is able to win on each copy of D_7. Moreover, he can win on each copy of K_4 by Lehman's Theorem [16], as K_4 admits two edge disjoint spanning trees.

We now prove the lower bound. Let G be a graph with average degree α and with n vertices, such that Maker has a winning strategy for the game $\mathcal{D}_1$ on $E(G)$. We will prove that $\alpha \geq 11/4$. If G had a vertex of degree less than 2, the game would obviously be a Breaker's win. Also, if two vertices of degree 2 are neighbors, then Breaker can win the game in two moves. Hence, from now on we can assume that every vertex of G has degree at least 2, and no two vertices of degree 2 are adjacent; in particular, $\alpha > 2$.

We denote by R the set of vertices of G that have degree exactly 2, and let $L = V(G) \setminus R$, $r = |R|$, $\ell = |L|$, and

$$k = \sum_{v \in L} (d_L(v) - 2), \tag{1}$$

$$s = \sum_{v \in L} (d(v) - 3). \tag{2}$$

Our first goal is to prove that

$$\text{every vertex } v \in L \text{ has } d_L(v) \geq 2. \tag{3}$$

Assume for the sake of contradiction that there exists a vertex $v \in L$ such that $d_L(v) \leq 1$. Based on this assumption, we will devise a winning strategy for Breaker. He plays as follows.

In his first move, Breaker claims an arbitrary edge connecting v with some vertex $u \in R$. Maker is then forced to claim the only remaining unclaimed edge incident with u, as otherwise he would lose immediately. As long as there are unclaimed edges connecting v with R, Breaker keeps claiming them, forcing the response of Maker in each of these moves. When all such edges are claimed by Breaker, there will be at most one unclaimed edge incident with v. Breaker can claim it and thus win. This contradicts our assumption that Maker wins the game.

It follows that the following two equalities hold:

$$\alpha(\ell + r) = \alpha n = \sum_{v \in V(G)} d(v) = 2r + 3\ell + s, \tag{4}$$

$$\alpha(\ell + r) = \alpha n = \sum_{v \in V(G)} d(v) = 4r + 2\ell + k. \tag{5}$$

The first equality is obtained by simply summing the degrees of the vertices of R and L. The second equality is obtained by considering separately the contribution of the edges between R and L and the edges with both end points in L, to the sum of degrees.

For every vertex $v \in L$, we say that v is *satisfied* if one of the two following conditions holds:

1. $d_L(v) \geq 3$ or $d(v) \geq 4$.

   In this case we say that v is satisfied by itself.

2. The first condition does not hold, and there exists $w \in N_L(v)$ such that $d_L(w) \geq 3$.

   Here, we say that v is satisfied by w.

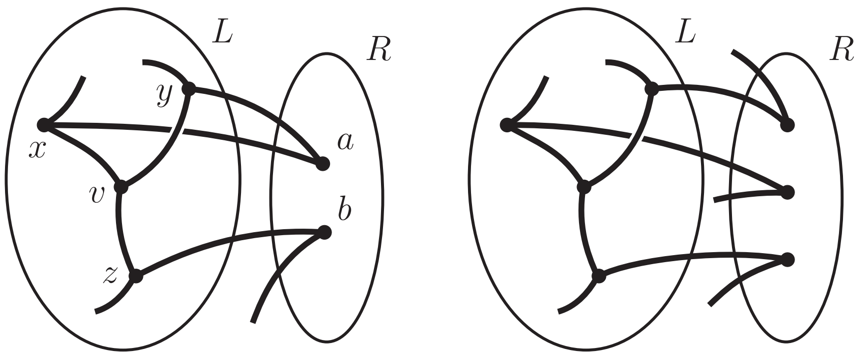

Figure 1: Two possible scenarios.

We will show that every vertex in L must be satisfied. Assume for the sake of contradiction that $v \in L$ is not satisfied. Then, neither of the conditions above holds. Since the first condition is not satisfied we have $d_L(v) = 2$ and $d(v) = 3$, which means that v has two neighbors, w_1 and w_2, in L, and one neighbor v' in R. Since the second condition is not satisfied, we have $d_L(w_i) = 2$, for every $i \in \{1, 2\}$. Since $d(v') = 2$ and v is its neighbor, there has to be $i_0 \in \{1, 2\}$ such that w_{i_0} is not a neighbor of v'. If Breaker starts the game by claiming the edge vw_{i_0} in his first move, Maker has to leave one of the vertices v and w_{i_0} untouched after his first move. Then, Breaker can routinely isolate that untouched vertex, by claiming incident edges one by one, leaving the only edge inside L for the last move. This contradicts our assumption that Maker wins the game.

We proceed by showing that

$$\forall v \in L \text{ such that } d(v) = d_L(v) = 3,\ \exists w \in N_L(v) \text{ that satisfies itself.} \tag{6}$$

Assume for the sake of contradiction that there exists a vertex $v \in L$ such that $d(v) = d_L(v) = 3$, and, for every $w \in N_L(v)$, we have $d_L(w) = 2$ and $d(w) = 3$. Then, some part of G resembles one of the two scenarios shown in Figure 1. Assume first that the situation is as depicted on the left. Vertex z cannot be adjacent to both x and y; assume without loss of generality that x and z are not adjacent. Breaker can win the game by claiming the edges zb, zv, ax, xv in his first 4 moves. Note that each of these moves forces some counter move of Maker by creating an immediate threat of Breaker's win. Moreover, Breaker's 4th move creates a double threat, at x and v. Since Maker cannot claim an edge which is incident with both x and v in his next move, Breaker can isolate one of them in Maker's graph and thus win. In the other possible scenario, Breaker can win similarly. Either way, the game is a Breaker's win contrary to our assumption.

It follows that every vertex $v \in L$ such that $d(v) = d_L(v) = 3$ satisfies at most 3 vertices (including itself).

Now, we claim that

$$3k + s \geq \ell. \tag{7}$$

To see this we go over every vertex in L and count the number of vertices it satisfies. As noted above, every vertex $v \in L$ with $d(v) = d_L(v) = 3$ satisfies at most 3 vertices. Moreover, its contribution to the sum in (1) is $d_L(v) - 2 = 1$ and its contribution to the sum in (2) is $d(v) - 3 = 0$. Hence, its total contribution to the left hand side of (7) is 3. Every other vertex $u \in L$ satisfies at most $d_L(u) + 1$ vertices. On the other hand its contribution to the left hand side of (7) is

$$3(d_L(u) - 2) + (d(u) - 3) \geq d_L(u) + 1,$$

where this inequality holds whenever $d_L(u) \geq 3$ and $d(u) \geq 4$. Claim (7) now readily follows since every vertex of L is satisfied.

Adding equality (4) to equality (5) multiplied by 3, and applying (7), we get

$$\alpha(\ell + r) \geq \frac{7}{2}r + \frac{5}{2}\ell. \tag{8}$$

Since $\alpha > 2$, it follows by (4) and the definition of L that

$$\begin{aligned} \alpha(\ell + r) &\geq 2r + 3\ell \implies \\ r(\alpha - 2) &\geq \ell(3 - \alpha) \implies \\ \frac{r}{\ell} &\geq \frac{3 - \alpha}{\alpha - 2}. \end{aligned} \tag{9}$$

If $\alpha \geq 7/2$, then we are done. Otherwise, it follows from (8) that

$$\begin{aligned} \ell\left(\alpha - \frac{5}{2}\right) &\geq r\left(\frac{7}{2} - \alpha\right) \implies \\ \frac{\alpha - \frac{5}{2}}{\frac{7}{2} - \alpha} &\geq \frac{r}{\ell}. \end{aligned} \tag{10}$$

Combining (9) and (10), we get

$$\frac{\alpha - \frac{5}{2}}{\frac{7}{2} - \alpha} \geq \frac{3 - \alpha}{\alpha - 2},$$

entailing $\alpha \geq 11/4$ as claimed. $\square$

3 The k-connectivity game

Proof of Theorem 1.3.

Proof of Part (i) We will make use of the following simple lemma.

Lemma 3.1 *Playing on the edge set of $K_{3,3}$, Maker, as the first or second player, has a winning strategy for the positive minimum degree game.*

The proof of Lemma 3.1 is by case analysis. For the sake of completeness we include it here.

Proof of Lemma 3.1. Clearly it suffices to prove that Maker can win this game as the second player. Let us denote the vertices of the two partite sets of $K_{3,3}$ by u_1, u_2, u_3, and v_1, v_2, v_3, respectively. Assume without loss of generality that Breaker claims the edge u_1v_1 in his first move. In his first move Maker responds by claiming the edge u_1v_2; in his next move he claims one of the edges u_2v_1, u_3v_1. This is possible regardless of Breaker's strategy. Assume without loss of generality that Maker claims u_2v_1 in his second move.

Note that after Breaker has played his third move, both u_3 and v_3 are incident with at most two of Breaker's edges, where at least one of these two inequalities is strict. Assume without loss of generality that v_3 is incident with at most one edge of Breaker. In his third move Maker claims a free edge which is incident with u_3. If this edge is u_3v_3, then Maker has already won. Otherwise, in his fourth move he claims a free edge which is incident with v_3. This concludes the proof of the lemma. $\square$

We are now ready to prove the theorem, by induction on k. Actually, we are going to prove a slightly stronger statement, by constructing a graph G_n^k on n vertices with average degree at most $3k+2-5/n$, such that Maker has a winning strategy for the k-vertex-connectivity game, played on the edge-set of G_n^k. For $k=1$, take G_n^1 to be any graph which is the edge disjoint union of two spanning trees. The fact that the connectivity game on $E(G_n^1)$ is a Maker's win, follows from Lehman's Theorem [16]. Note that such a graph on n vertices exists for every $n \geq 4$, and its average degree is $4-4/n \leq 5-5/n$.

Next, assume that for every $n \geq 3 \cdot 2^{k+1}$, there exists a graph G_n^k on n vertices with average degree at most $3k+2-5/n$, such that Maker can win the k-vertex-connectivity game on $E(G_n^k)$. Given $n_0 \geq 3 \cdot 2^{k+2}$, we want to construct the graph $G_{n_0}^{k+1}$.

Let $n_0 = 6t+s$ for some $t \in \mathbb{N}$ and $0 \leq s \leq 5$, and let $n_1 = \lceil \frac{n_0}{2} \rceil = 3t + \lceil \frac{s}{2} \rceil$ and $n_2 = \lfloor \frac{n_0}{2} \rfloor = 3t + \lfloor \frac{s}{2} \rfloor$. We will construct $G_{n_0}^{k+1}$ by taking a copy $G_1 = (V_1, E_1)$ of $G_{n_1}^k$, a (disjoint) copy $G_2 = (V_2, E_2)$ of $G_{n_2}^k$, and then adding some edges between V_1 and V_2.

Let $V_1 = \{v_1, \ldots, v_{n_1}\}$ and let $V_2 = \{u_1, \ldots, u_{n_2}\}$. Connect the vertices of G_1 to the vertices of G_2 by t vertex disjoint copies of $K_{3,3}$ and by additional s pairs of edges, that is, $G_{n_0}^{k+1} := (V_{n_0}^{k+1}, E_{n_0}^{k+1})$, where $V_{n_0}^{k+1} = V_1 \cup V_2$ and $E_{n_0}^{k+1} = E_1 \cup E_2 \cup \{v_{3r-i}u_{3r-j} : 1 \leq r \leq t,\ 0 \leq i,j \leq 2\} \cup \{v_{3t+i}u_j : 1 \leq i \leq \lceil \frac{s}{2} \rceil, 1 \leq j \leq 2\} \cup \{u_{3t+i}v_j : 1 \leq i \leq \lfloor \frac{s}{2} \rfloor, 1 \leq j \leq 2\}$. Since the average degree of G_1 and G_2 is at most $3k+2-5/n_1$ and $3k+2-5/n_2$, respectively, we conclude after a straightforward calculation that $\bar{d}(G_{n_0}^{k+1}) \leq 3(k+1)+2-5/n_0$.

It remains to prove that Maker can win the $(k+1)$-vertex-connectivity game, played on the edges of $G_{n_0}^{k+1}$. His strategy is the following. Whenever Breaker claims some edge of G_i, where $i = 1, 2$, Maker claims an edge of G_i as well, playing according to the strategy whose existence is guaranteed by the induction hypothesis. Similarly, whenever Breaker claims an edge of some copy of $K_{3,3}$ that connects V_1 and V_2, Maker plays in this same copy of $K_{3,3}$ according to the strategy whose existence is guaranteed by Lemma 3.1. Whenever Breaker claims an edge $v_{3t+i}u_j$ for some $1 \leq i \leq \lceil \frac{s}{2} \rceil$ and $1 \leq j \leq 2$, Maker claims $v_{3t+i}u_\ell$, where $\ell \in \{1, 2\} \setminus \{j\}$. Similarly, whenever Breaker claims an edge $u_{3t+i}v_j$ for some $1 \leq i \leq \lfloor \frac{s}{2} \rfloor$ and $1 \leq j \leq 2$, Maker claims $u_{3t+i}v_\ell$, where $\ell \in \{1, 2\} \setminus \{j\}$. Whenever Maker cannot play according to this rule, he claims some arbitrary free edge.

It follows that the subgraph of G_i, $i = 1, 2$, that Maker will claim by the end of the game will be k-vertex-connected. Moreover, in Maker's graph, every vertex of V_1 has at least one neighbor in V_2 and vice versa.

Let G_M denote the graph that Maker has built by the end of the game. We will prove that G_M is $(k+1)$-vertex-connected. Let $S \subseteq V_1 \cup V_2$ be any set of size at most k. Assume first that $|S \cap V_1| \leq k-1$ and $|S \cap V_2| \leq k-1$. By the induction hypothesis, both $(G_M \cap G_1) \setminus S$ and $(G_M \cap G_2) \setminus S$ are connected. Moreover, there is at least one edge of Maker between $(G_M \cap G_1) \setminus S$ and $(G_M \cap G_2) \setminus S$ as, by the choice of n_0, there are $t \geq k+1$ vertex disjoint copies of $K_{3,3}$ connecting V_1 and V_2, and Maker has claimed at least one edge in each of them.

Next, assume without loss of generality that $S \subseteq V_1$ is of size k. Let $u \in V_1 \setminus S$. By Maker's strategy, there is at least one edge of G_M connecting u with some vertex x of V_2. Since $S \cap V_2 = \emptyset$ and since $G_2 \cap G_M$ is connected by the induction hypothesis, it follows that $(G_2 \cap G_M) \setminus S$ is connected as well. It follows that $G_M \setminus S$ is connected.

Proof of Part (ii) We will make use of the following result from [10].

Theorem 3.2 *If n is sufficiently large, then Maker has a winning strategy for the $(n/2 - 3\sqrt{n \ln n})$-vertex-connectivity game, played on the edge-set of K_n.*

Let k_0 be the smallest integer for which Theorem 3.2 holds with $n = k_0$. For $k < k_0/2$ the assertion of the theorem holds by Part 1 of this theorem. Let $k \geq k_0/2$, and let $\varepsilon = \varepsilon(k)$ be the real number which satisfies the equation

$$\frac{(2+\varepsilon)k}{2} - 3\sqrt{(2+\varepsilon)k \ln((2+\varepsilon)k)} = k.$$

Clearly, such an ε exists, it is unique, and it tends to 0 as k tends to infinity. Let G consist of $m \geq (2+\varepsilon)k$ cliques $Q_1, \ldots, Q_m$, each on either $(2+\varepsilon)k$ vertices, or one more vertex, where Q_i is connected to Q_{i+1} by a matching of size $2k$, for every $1 \leq i \leq m-1$. Note that, by choosing appropriate values for m and $|Q_i|$, for every $1 \leq i \leq m$, we can choose $V(G)$ to be any integer, as long as it is larger than $((2+\varepsilon)k)^2$. Clearly, $e(G) \leq m\binom{(2+\varepsilon)k+1}{2} + 2k(m-1) = (1+o_k(1))kv(G)$.

Let us now describe Maker's strategy. Whenever Breaker claims an edge which connects some vertex of Q_i, for some $1 \leq i \leq m-1$, with some vertex of Q_{i+1}, Maker claims another such edge. By applying a straightforward pairing strategy, he can make sure he claims k edges with one end point in Q_i and the other in Q_{i+1}, for every $1 \leq i \leq m-1$. Whenever Breaker claims an edge of Q_i, for some $1 \leq i \leq m$, Maker responds by claiming an edge in Q_i, and by Theorem 3.2, which is applicable since $(2+\varepsilon)k \geq k_0$, Maker can build a k-vertex-connected subgraph of Q_i, for every $1 \leq i \leq m$. It is easy to verify that the graph obtained by connecting two vertex disjoint k-connected graphs by a matching of size k is k-connected. This ensures that the graph Maker builds by the end of the game will be k-vertex-connected. □

4 The Hamilton cycle game

Proof of Theorem 1.4. First, we prove the lower bound. Let $G = (V, E)$ be a graph on n vertices, on which Maker wins the Hamilton cycle game. We will prove that $e(G) \geq 2.5n$. It is clear that the minimum degree of G is at least 4. Indeed, if $x \in V$ is of degree at most 3, then, since Breaker is the first player, he can force the degree of x in Maker's graph to be at most 1. Similarly, if $x, y \in V$ are two adjacent vertices of degree 4, then Breaker can win by claiming (x, y) in his first move and then forcing the degree of either x or y in Maker's graph to be at most 1. Hence, the vertices of degree 4 in G form an independent set (note the the empty set is considered to be independent). Since Maker wins the game, G must be Hamiltonian; let $C = (v_0, \ldots, v_{n-1}, v_0)$ be a Hamilton cycle of G. Let $\{v_{i_j} \in V : 0 \leq j \leq r-1\}$ denote the vertices of degree 4 in G. We will prove that there are at least r vertices of degree at least 6; note that this entails $e(G) \geq 2.5n$. First, assume that $r \geq 2$. In order to prove our claim, it suffices to prove that there is a vertex of degree at least 6 between any two consecutive vertices of degree 4, that is, for every $0 \leq j \leq r-1$ there exists an index $i_j < t < i_{j+1}$, where $j+1$ is reduced modulo r, and t and i_{j+1} are reduced modulo n, such that $d(v_t) \geq 6$. Assume for the sake of contradiction that $d(v_s) = 5$, for every $i_j < s < i_{j+1}$. We will provide Breaker with a winning strategy, contrary to our assumption that the game is a Maker's win. Let G_j denote the subgraph of G, induced by the vertices $\{v_s : i_j \leq s \leq i_{j+1}\}$. Let $P : v_{i_j}, w_1, \ldots, w_\ell, v_{i_{j+1}}$ be a shortest path between v_{i_j} and $v_{i_{j+1}}$ in G_j. Breaker plays as follows. In his first move, he claims $v_{i_j}w_1$. If Maker does not respond by claiming an edge which is incident with v_{i_j}, then Breaker can claim two more edges which are incident with it and thus win. Hence, assume that on his first move Maker claims an edge $v_{i_j}x$, for some $x \in V$. Note that $x \notin P$, as P is a shortest path. Similarly, on his second move, Breaker claims w_1w_2. Maker is forced to claim some edge w_1y for some $y \in V \setminus P$, as otherwise Breaker will force the degree of w_1 in Maker's graph to be at most 1. Breaker continues playing in this fashion until he either wins or claims $w_\ell v_{i_{j+1}}$. At this point, in order to avoid losing, Maker must claim some edge which is incident with w_ℓ as well as some edge which is incident with $v_{i_{j+1}}$. Clearly, this is impossible, and thus Breaker wins in two moves. Finally, assume for the sake of contradiction that $r = 1$ and the maximum degree of G is 5. Again we will prove that Breaker wins the game in this case. Assume without loss of generality that $d(v_0) = 4$. In his first move Breaker claims v_0v_1. We can assume that Maker responds by claiming v_0v_{n-1} as otherwise he loses as before (with $v_{i_j} = v_0 = v_{i_{j+1}}$). Since G is Hamiltonian and $d(v_0) > 2$, there is a path between v_0 and v_1 in $G \setminus \{v_0v_1, v_0v_{n-1}\}$. Let P' be a shortest path between v_0 and v_1 in $G \setminus \{v_0v_1, v_0v_{n-1}\}$. Breaker plays as in the previous case, with $P = P'$, and thus wins. It follows that $e(G) \geq 2.5n$ as claimed.

Next, we prove the upper bound. We will make use of the following result of [14].

Proposition 4.1 *Maker (as the first or second player) has a winning strategy for the Hamilton cycle game, played on the edges of K_n, provided $n \geq 38$.*

Let $m \geq 40$ be an integer and let $G_n = (V, E)$ be a graph on $n = m(d+1)+r$ vertices where $d = 38$ and $0 \leq r \leq d$. The graph G_n consists of m cliques $K_0, \ldots, K_{m-1}$ and m additional vertices $u_0, \ldots, u_{m-1}$, where $v(K_i) = d + 1$, for every $0 \leq i \leq r - 1$, and $v(K_i) = d$, for every $r \leq i \leq m - 1$. For every $0 \leq i \leq m - 1$, there is an edge in G_n between u_i and every vertex of both K_i and K_{i+1} (the indices are reduced modulo m). Clearly, the number of edges in G_n is at most $m\binom{d+1}{2} + 2m(d + 1) \leq \left(\frac{d}{2} + 2\right) n = 21n$. For every $1 \leq i \leq m$, the subgraph of G_n, induced on the vertices of $\{u_{i-1}, u_i\} \cup V(K_i)$ (the indices are taken modulo m), will be called the *i-th part of G_n*.

Now we provide Maker with a winning strategy for the Hamilton cycle game, played on the edge set of G_n. Maker plays m separate games in parallel, that is, whenever Breaker claims some edge of the i-th part of G_n, Maker claims an edge in the same part (this is always possible, except for maybe once for each part, if Breaker claims the last edge of this part; whenever this happens Maker claims an arbitrary free edge). Maker's strategy for the i-th part of G_n is as follows. He subdivides this game further into two separate games played in parallel, one on the edges of K_i and the other on the board $\hat{E}_i := \{(u_{i-1}, v) : v \in V(K_i)\} \cup \{(u_i, v) : v \in V(K_i)\}$. Again, in each of the moves he responds by claiming an edge in the same subgame as Breaker.

Playing on $E(K_i)$, Maker will build a spanning cycle; this is possible by Proposition 4.1. On $\hat{E}_i$, in his first two moves, Maker claims edges u_iw_1 and $u_{i-1}w_2$, for some $w_1, w_2 \in V(K_i)$, $w_1 \neq w_2$. Then he proceeds by claiming arbitrary edges, just making sure that he does not claim *both* u_iw_2 and $u_{i-1}w_1$.

It remains to prove that, if Maker plays according to this strategy, then, for every $1 \leq i \leq m$, his graph will contain a path between u_{i-1} and u_i (the indices are reduced modulo m) spanning the i-th part of G_n. Fix some $1 \leq i \leq m$. The strategy presented above ensures that Maker's graph will contain a spanning cycle $C = (v_1, v_2, \ldots, v_t, v_1)$ of K_i, and moreover, it will satisfy $d_{V(K_i)}(u_{i-1}) \geq 1$, $d_{V(K_i)}(u_i) \geq 1$, $d_{V(K_i)}(u_{i-1}) + d_{V(K_i)}(u_i) \geq t$, and

$N_{V(K_i)}(u_{i-1}) \neq N_{V(K_i)}(u_i)$. Let $\Gamma(u_{i-1}) := \{w \in V(K_i) : \exists u \in N_{V(K_i)}(u_{i-1}), uw \in E(C)\}$. Obviously, $|\Gamma(u_{i-1})| \geq d_{V(K_i)}(u_{i-1})$, implying $|\Gamma(u_{i-1})| + d_{V(K_i)}(u_i) \geq t$. To prove our claim, it suffices to show that $\Gamma(u_{i-1}) \cap N_{V(K_i)}(u_i) \neq \emptyset$.

If $|\Gamma(u_{i-1})| + d_{V(K_i)}(u_i) > t$, our claim follows directly. But for $|\Gamma(u_{i-1})| + d_{V(K_i)}(u_i) = t$ we must have both $|\Gamma(u_{i-1})| = d_{V(K_i)}(u_{i-1})$ and $d_{V(K_i)}(u_{i-1}) + d_{V(K_i)}(u_i) = t$. It is not hard to see that $|\Gamma(u_{i-1})| = d_{V(K_i)}(u_{i-1})$ can hold only if $N_{V(K_i)}(u_{i-1})$ contains either every, or every other vertex on C. In the former case we immediately get a contradiction with $d_{V(K_i)}(u_i) \geq 1$, and in the latter case we get $d_{V(K_i)}(u_{i-1}) = d_{V(K_i)}(u_i) = t/2$, and then $N_{V(K_i)}(u_{i-1}) \neq N_{V(K_i)}(u_i)$ implies $\Gamma(u_{i-1}) \cap N_{V(K_i)}(u_i) \neq \emptyset$. This concludes the proof of the theorem. □

5 The tree construction game

In this section we will prove Theorem 1.5. We start by introducing several tools that will be used in the proof. For the sake of uniformity of presentation, we label all of them as lemmas, though they are of quite different nature and depth.

Lemma 5.1 *Let $\Delta, K \geq 2$ be integers. Let $T = (V, E)$ be a tree on $v(T) \geq K$ vertices, with maximum degree at most Δ. Then, there exists a decomposition $V = V_1 \cup \ldots \cup V_t$ of the vertex set of T such that:*

1. *$K \leq |V_i| \leq (\Delta + 1)K$, for every $1 \leq i \leq t$;*
2. *$T[V_i]$ is connected, for every $1 \leq i \leq t$.*

Proof of Lemma 5.1. If $v(T) \leq (\Delta + 1)K$, we take the whole of V to be V_1. Otherwise we choose an arbitrary vertex v of T, and root T at v. For every vertex $w \in V$, let $D(w)$ be the vertex set of the subtree of T rooted at w. We claim that there exists a vertex $w \in V$ such that $K \leq |D(w)| \leq \Delta K$. Indeed, assume for the sake of contradiction that no vertex of V satisfies the inequality above. Let $w \in V$ be such that $|D(w)| \geq K$ is minimal. It follows by our assumption that $|D(w)| \geq \Delta K + 1$. Let $u_1, \ldots, u_s$ be the children of w in T. Then clearly $s \leq \Delta$ and $|D(w)| = 1 + \sum_{i=1}^{s} |D(u_i)|$. Hence, for some u_i we have $|D(u_i)| \geq \Delta K / s \geq K$. This contradicts the minimality of $|D(w)|$, as clearly $|D(u_i)| < |D(w)|$. Hence, let $w \in V$ be some vertex satisfying $K \leq |D(w)| \leq \Delta K$. Define $V_1 = D(w)$ and remove w and its descendants from T. Note that $T[V_1]$ is connected. Moreover, $T' := T \setminus V_1$ is also connected and is therefore a tree. Since $v(T') \geq (\Delta + 1)K - \Delta K = K$, we can apply induction to T'. This yields the desired partition, and concludes the proof of the lemma. □

In order to state our next lemma, we need the following definition.

Definition 5.2 *A graph $G = (V, E)$ is called* (p, ε)-regular *if:*

1. $|d(v) - p|V|| \leq \varepsilon|V|$ *for every* $v \in V$*;*
2. $|e_G(S, T) - p|S||T|| \leq \varepsilon|S||T|$ *for every pair of disjoint subsets* $S, T \subset V$ *with* $|S|, |T| \geq \varepsilon|V|$.

Lemma 5.3 *There exists a constant k_1 such that for every $n \geq k_1$, Maker can build a $(1/2, n^{-0.1})$-regular graph in a $(1 : 1)$ Maker-Breaker game, played on the edge set of the complete graph K_n.*

Proof of Lemma 5.3. This was proved in [10] (see also [3] for an alternative proof). □

Lemma 5.4 *For every Δ there exists $k_2 = k_2(\Delta)$, such that for every $n \geq k_2$, the following holds. Let T be a tree on n vertices, with maximum degree at most Δ, rooted at r. Let G be a $(1/2, n^{-0.1})$-regular graph on n vertices, and let v be an arbitrary vertex of G. Then G contains a copy of T, rooted at v.*

Proof of Lemma 5.4. This is a particular instance of the famous Blow-Up Lemma, proved by Komlós, Sárközy and Szemerédi [15]. □

One should note that the Blow-Up Lemma is usually stated without the additional requirement that the embedding is such that a particular vertex r of the embedded graph T is to be mapped into a specified vertex v of the host graph G. However, a study of the proof of the Blow-Up Lemma, certainly of the version given by Rödl and Ruciński in [18], readily reveals that this extra condition can be met too.

Lemma 5.5 *There exists a constant k_3 such that for every $n_1, n_2 \geq k_3$, Maker can build a graph of positive minimum degree in a $(1 : 1)$ Maker-Breaker game, played on the edge-set of the complete bipartite graph K_{n_1,n_2}.*

Proof of Lemma 5.5. It follows by Lemma 10 in [13], that Maker can build a subgraph of K_{n_1,n_2} with minimum degree at least $\min\{\lfloor n_1/4 \rfloor, \lfloor n_2/4 \rfloor\}$. The lemma now follows by choosing $k_3 \geq 4$. Note that one can also prove this lemma by a straightforward adaptation of the proof of Lemma 3.1. □

Proof of Theorem 1.5:

Let T be a tree with vertex set $\{1, \ldots, n\}$, and with maximum degree at most Δ. Let $K = \max\{k_1, k_2, k_3\}$, where k_1, k_2, and k_3 are the constants whose existence is guaranteed by Lemmas 5.3 , 5.4, and 5.5, respectively. Apply Lemma 5.1 to T with Δ, K as defined above. Let $(V_1, \ldots, V_t)$ be the obtained decomposition of $V(T)$; clearly $t = \Theta(n)$.

Before defining the board of the game, we define an auxiliary graph S with vertex set $\{v_1, \ldots, v_t\}$. The vertices of S are associated with the parts V_i of the aforementioned decomposition of $V(T)$ in a straightforward way, namely, v_i corresponds to V_i for every $1 \leq i \leq t$. For every $1 \leq i < j \leq t$, there is an edge of S between v_i and v_j if and only if there is an edge of T connecting some vertex of V_i and some vertex of V_j. It is easy to see that S is in fact a tree. Moreover, since the degrees in T do not exceed Δ and each part V_i has at most $(\Delta+1)K$ vertices, the maximum degree of S is at most $(\Delta+1)\Delta K$.

The board of the tree construction game is the edge set of a simple graph G on the vertex set $\{1, \ldots, n\}$. The edge set of G is

$$\bigcup_{i=1}^{t} \{(x,y) : \{x,y\} \subseteq V_i\} \cup \bigcup_{(v_i,v_j) \in E(S)} \{(x,y) : x \in V_i, y \in V_j\},$$

that is, we put a complete graph inside each V_i, and connect V_i and V_j by a complete bipartite graph whenever they are connected by an edge in T.

Observe that the maximum degree of G can be bounded from above as follows:

$$\Delta(G) \leq (\max |V_i|) \cdot (1 + (\Delta+1)\Delta K) = O(1)\,.$$

It follows that G has $\Theta(n)$ edges.

We now prove that Maker can build a copy of T, while playing against Breaker in a $(1:1)$ game on the edge-set of G. Maker's strategy is as follows: he plays $2t-1$ separate games in parallel. That is, whenever Breaker claims an edge with both end points in V_i (for some $1 \leq i \leq t$), Maker responds by claiming a free edge with both end points in V_i and whenever Breaker claims an edge with one end point in V_i and the other in V_j (for some $1 \leq i < j \leq t$), Maker responds by claiming a free edge with one end point in V_i and the other in V_j. Whenever this is not possible (at most $2t-1$ times – once per game), Maker claims an arbitrary free edge. Maker's strategy for each separate game is as follows:

- For every $1 \leq i \leq t$, when playing on $E(G[V_i])$ Maker creates a $(1/2, |V_i|^{-0.1})$-regular graph. This is possible due to Lemma 5.3.
- For every $1 \leq i < j \leq t$ for which $(v_i, v_j) \in E(S)$, when playing on the edges of the complete bipartite graph between V_i and V_j, Maker builds a graph of positive minimum degree, thus connecting every vertex of V_i to V_j and every vertex of V_j to V_i. This is possible due to Lemma 5.5.

Let M denote the graph built by Maker by the end of the game. We claim that M admits a copy of T. We will construct such a copy by embedding it in pieces, following some search order (say, BFS) on the auxiliary tree S. Assume without loss of generality that the order in which we wish to embed the $T[V_i]$'s in M is $T[V_1], T[V_2], \ldots, T[V_t]$. Assume we have already embedded $T[V_1], \ldots, T[V_i]$ and the edges of T that connect them, and now wish to embed $T[V_{i+1}]$. Note that (unless $i = 0$) some vertex $r_{i+1} \in V_{i+1}$ is connected in T to some (already embedded) vertex $u \in V_j$, for some $1 \leq j \leq i$. Let u' denote the image of u in the embedding and let $v' \in V_{i+1}$ be an arbitrary neighbor of u' in M. Note that such a neighbor v' exists by Maker's strategy for the game on the complete bipartite graph connecting V_j and V_{i+1}. We embed the edge ur_{i+1} into the edge $u'v'$. Then we embed $T[V_{i+1}]$ into $M[V_{i+1}]$ such that r_{i+1} serves as the root of $T[V_{i+1}]$ and is mapped into v'. Since $M[V_{i+1}]$ is $(1/2, |V_{i+1}|^{-0.1})$-regular, such a rooted embedding is possible due to Lemma 5.4. □

6 The biased positive minimum degree and connectivity games

Proof of Theorem 1.6.

Proof of Part (i) Fix $\varepsilon > 0$, sufficiently large $n = n(\varepsilon)$, and $q = q(n) \leq (\ln 2 - \varepsilon)n/\ln n$. We first prove that there exists a graph $G = (V, E)$ with n vertices and $dn(1/2 + o(1))$ edges, where $d = (1+\varepsilon)q \log_2 n$, which satisfies the following property:

$$e(A, V \setminus A) \geq (1 - \varepsilon/2)d|A||V \setminus A|/n, \text{ for every } A \subseteq V. \tag{11}$$

If $d \geq c\sqrt{n}$ for some $c > 0$, then almost surely a binomial random graph $G \in G(n, d/n)$ satisfies (11); this follows from standard bounds on the tails of the binomial distribution. For smaller values of d, we prove the following claim.

Claim 6.1 *Let $\varepsilon > 0$ and let $G = (V, E)$ be a random n-vertex d-regular graph, where $\ln n \leq d = o(\sqrt{n})$. Then almost surely $e(A, V \setminus A) \geq (1 - \varepsilon/2)d|A||V \setminus A|/n$ for every $A \subseteq V$.*

Proof of Claim 6.1. In the proof we will use the following two results from [7] regarding the distribution of edges in the probability space $\mathcal{G}_{n,d}$ of random n-vertex d-regular graphs.

Theorem 6.2 *[7] If $d = o(\sqrt{n})$, then almost surely every subset U of the vertices of a graph, drawn uniformly at random from $\mathcal{G}_{n,d}$, satisfies*

$$\left| e(U) - \binom{|U|}{2}\frac{d}{n} \right| = O(|U|\sqrt{d}).$$

Theorem 6.3 *[7] If $d = o(\sqrt{n})$, then almost surely every pair of disjoint subsets U, W of the vertices of a graph, drawn uniformly at random from $\mathcal{G}_{n,d}$ satisfies*

$$\left| e(U,W) - \frac{|U||W|d}{n} \right| = O(\sqrt{|U||W|d}).$$

Let $A \subseteq V$ be any subset of size $1 \leq a \leq n/\ln\ln n$. According to Theorem 6.2, the number of edges of G with both endpoints in A is almost surely at most $\frac{da^2}{2n} + O(a\sqrt{d}) = o(da)$. Since G is d-regular, it follows that $e(A, V \setminus A) \geq (1 - o(1))da(n-a)/n$. Next, let $A \subseteq V$ be any subset of size $n/\ln\ln n \leq a \leq n/2$. According to Theorem 6.3, the number of edges with one endpoint in A and the other in $V \setminus A$ is almost surely at least $\frac{da(n-a)}{n} - O(\sqrt{da(n-a)}) = (1 - o(1))da(n-a)/n$. □

Now, assume that G is a graph with n vertices and $dn(1/2 + o(1))$ edges, which satisfies Property (11) (if $d = o(\sqrt{n})$ and dn is odd, then take $G' \in \mathcal{G}_{n-1,d}$, add a new vertex v, and connect it to arbitrary d vertices of G'). In order to build a spanning connected subgraph of G (and thus win) Maker will claim an edge in every cut of G, that is, he will assume the role of Cut-Breaker in the Cut game on G. Let $\mathcal{F}$ be the hypergraph whose vertices are the edges of G and whose hyperedges are the edge sets of all bipartite spanning induced subgraphs of G. Note that due to (11) we have

$$\begin{aligned}
\sum_{A \in \mathcal{F}} 2^{-|A|/q} &\leq \sum_{r=1}^{n/2} \binom{n}{r} 2^{-(1-\varepsilon/2)dr(n-r)/(qn)} \\
&\leq \sum_{r=1}^{n/2} \left[\frac{en}{r} 2^{-(1-\varepsilon/2)(1+\varepsilon)(1-r/n)\log_2 n} \right]^r \\
&\leq \sum_{r=1}^{\sqrt{n}} \left[en^{1-(1+\varepsilon')} \right]^r + \sum_{r=\sqrt{n}}^{n/2} \left[en^{1/2-(1/2+\varepsilon'')} \right]^r \\
&= o(1).
\end{aligned}$$

It follows that we can apply Beck's criterion for Breaker's win in biased positional games [5], to conclude that Cut-Breaker has a winning strategy for the $(q:1)$ Cut game on G, and thus Maker has a winning strategy for the $(1:q)$ connectivity game on G.

Proof of Part (ii) Our argument is a generalization of the argument applied by Chvátal and Erdős [8] to provide a winning strategy for Breaker in the biased Maker-Breaker connectivity game played on the edge set of the complete graph K_n. Let $G = (V, E)$ be a graph with n vertices and $m \leq (1/2 - \varepsilon)(q-1)n\ln n$ edges. Let $d = 2m/n$ denote the average degree of G.

Set

$$s = n^{1-\varepsilon/2}, \quad d_1 = d + \frac{d}{\ln \ln n} .$$

Breaker's strategy is divided into two phases. In the first phase, Breaker builds a graph $G_B \subseteq G$, such that there exists a subset $U \subseteq V(G)$ with the following properties:

(a) $|U| = s$.

(b) $d(v) \le d_1$ for every $v \in U$.

(c) Every edge of $G[U]$ is claimed by Breaker, that is, $E(G[U]) \subseteq E(G_B)$.

(d) Maker has not claimed any edge which is incident with U.

In the second phase, Breaker isolates one of the vertices of U in Maker's graph.

Let V_0 be the set of vertices of G of degree at most d_1, and let $x = |V_0|$. Clearly,

$$(n-x)d_1 \le \sum_{v \in V \setminus V_0} d(v) \le \sum_{v \in V} d(v) = dn ,$$

implying

$$x \ge \frac{n}{1 + \ln \ln n} .$$

In the first phase, Breaker plays as follows. If $q = O(1)$, then Breaker, even before the game has started, can choose $U \subseteq V_0$ to be any independent set of G of size s (such a set exists as $G[V_0]$ has x vertices and its maximum degree is at most d_1; therefore its independence number is at least $x/(d_1+1) \gg n^{1-\varepsilon/2}$.)

Assume then, that $q = \omega(1)$. Breaker builds the required graph G_B in at most s rounds. Assume that just after Breaker's ith move, where $0 \le i < s$, Breaker has built a graph G^i, such that there exists a set $U_i \subseteq V(G)$ of size i that satisfies properties (b), (c), and (d) (where G_B is replaced by G^i). Let $M_i \subseteq V(G)$ denote the set of vertices of degree 0 in Maker's current graph, and let $R_i = M_i \cap (V_0 \setminus U_i)$. Clearly, $|R_i| \ge |V_0| - |U_i| - 2i \ge (1-o(1))n/\ln\ln n$. Since $d(v) \le d_1$ holds for every $v \in U_i$, we have $e(U_i, R_i) \le d_1|U_i| \le d_1 s$. It follows that there exist two vertices $u, v \in R_i$ such that

$$\begin{aligned} d_{U_i}(u) + d_{U_i}(v) &\le \frac{2e(U_i, R_i)}{|R_i|} \le \frac{(1+o(1))2d_1 s}{\frac{n}{\ln\ln n}} \\ &\le \frac{(1+o(1))2q \ln n \ln\ln n}{n^{\varepsilon/2}} = o(q) . \end{aligned} \tag{12}$$

In his $(i+1)$st move, Breaker claims all edges of

$$E_{i+1} := E(G) \cap (\{uv\} \cup \{ux : x \in U_i\} \cup \{vx : x \in U_i\}).$$

This is possible by (12). He then claims additional $q-|E_{i+1}|$ arbitrary free edges. Let G^{i+1} denote Breaker's graph just after his $(i+1)$st move. Let $U'_{i+1}=U_i\cup\{u,v\}$. On his next move, Maker claims some edge (x,y). Clearly, $|U'_{i+1}\cap\{x,y\}|\leq 1$. Deleting an appropriate vertex from U'_{i+1} we obtain a set $U_{i+1}\subseteq V_0$ of size $i+1$ that satisfies properties (b), (c), and (d) (where G_B is replaced by G^{i+1}). In particular, after Breaker's sth move he builds the desired graph, with $G_B:=G^s$ and $U:=U_s$. This concludes the first phase of Breaker's strategy.

In the second phase, Breaker isolates one of the vertices of U in Maker's graph. In order to prove that he can achieve this goal, we use the formalism of the so-called Box Game, introduced in [8]. Let $\mathcal{H}=\{A_1,A_2,\ldots,A_s\}$ be a family of pairwise disjoint sets. The $(1:q)$ Box Game $\mathcal{H}$, is played by two players, called Box-Maker and Box-Breaker, who take turns claiming elements of the board $\bigcup_{i=1}^s A_i$. Box-Breaker claims one element per move whereas Box-Maker claims q. Box-Maker wins the game if and only if he claims all elements of A_i, for some $1\leq i\leq s$. It was proved in [8] that Box-Maker (as the first or second player) wins this game if $s\cdot\max_i|A_i|\leq f(s,q)$ (in fact they only consider the case in which $||A_i|-|A_j||\leq 1$ for every $1\leq i<j\leq s$, but the more general claim follows from theirs in a straightforward way), where the function $f(s,q)$ satisfies

$$f(s,q)\geq (q-1)s\sum_{i=1}^{s-1}1/i\ . \tag{13}$$

Coming back to the $(1:q)$ connectivity game, we set $U=\{u_1,\ldots,u_s\}$, and define the family $\mathcal{H}_G=\{A_1,A_2,\ldots,A_s\}$, where $A_i=\{u_iv\in E(G):v\in V(G)\setminus U\}$, for every $1\leq i\leq s$. Breaker plays as in the Box Game, assuming the role of Box-Maker on $\mathcal{H}_G$. It is evident that, if Box-Maker can win the $(1:q)$ game $\mathcal{H}_G$, then Breaker can win the $(1:q)$ connectivity game, by isolating some vertex of U in Maker's graph.

Hence, it remains to prove that the aforementioned sufficient condition for Box-Maker's win in $\mathcal{H}_G$ is satisfied. By (13), we have

$$\begin{aligned}
f(s,q) &\geq (q-1)s\sum_{i=1}^{s-1}\frac{1}{i}\\
&\geq (q-1)s\ln(s-1)\\
&\geq (q-1)s\ln(n^{1-\varepsilon/2}-1)\\
&\geq (q-1)s(1-\varepsilon)\ln n.
\end{aligned}$$

The claim now follows since $\max_i|A_i|\leq d_1$ and $d\leq(1-2\varepsilon)(q-1)\ln n$. This concludes the proof of Theorem 1.6. □

7 Concluding remarks and open problems

Bounded degree graphs. It would be very interesting to find out if Theorem 1.5 could be extended from bounded degree trees to arbitrary bounded degree graphs.

Problem 7.1 *Does there exist a constant $B = B(\Delta)$ for which the following holds: for any n-vertex graph H of maximum degree at most Δ, there exists an n-vertex graph G with at most Bn edges, such that Maker can build a copy of H when playing on $E(G)$?*

Finiteness. Despite our efforts, the precise asymptotic form of $\hat{m}(\mathcal{D}_1)$ still eludes us. The following question might actually be of greater importance than the actual numerical value of the asymptotics.

Problem 7.2 *Is the determination of $\hat{m}(\mathcal{D}_1)$ a finite problem? That is, does there exist a constant c_0 for which there is a graph K on at most c_0 vertices such that*

$$\hat{m}(\mathcal{D}_1) = \frac{e(K)}{v(K)} \cdot n + O(1)?$$

Asking an analogous question for the perfect matching game, or more generally, for any spanning graph game in which there is a winning set consisting of disconnected pieces of constant order, is also interesting.

Various definitions of sparseness. Throughout this paper, sparseness was measured in terms of the edge number. However, other natural measures of sparseness could be used. One possibility would be to use measures involving the property itself.

For example, if Maker's goal is to build a connected spanning graph, then a natural question to ask is how large must the *connectivity* of the board be, in order to ensure Maker's win? It follows from Lehman's Theorem and from a theorem of Tutte [19] and independently Nash-Williams [17], that Maker can win the connectivity game on any 4-edge-connected graph. On the other hand, it is easy to find 3-vertex-connected graphs (for example, almost every 3-regular graph) on which Breaker wins this game.

An analogous but more challenging question concerns a coloring game, in which Maker's goal is to build a non-k-colorable graph. It is easy to see that there exists a non-k-colorable graph G, for example a complete $(k+1)$-partite graph with sufficiently large parts, such that playing on $E(G)$, Maker is able to build a graph $G_M \subseteq G$, satisfying $\chi(G_M) = \chi(G)$ and thus, in particular, G_M is non-k-colorable.

A more interesting question to ask is how large should the chromatic number of the base graph be, in order to *guarantee* that Maker wins the non-k-colorability game, played on its edges. Formally, we are interested in the smallest integer $r = r_q(k)$ such that playing a $(1:q)$ game on *any* non-r-colorable graph, Maker is able to build a non-k-colorable graph. It is not hard to show that $r_1(k) \leq k^2+2$. Indeed, let G be an arbitrary (k^2+2)-chromatic graph. Let $e \in E(G)$ denote the edge Breaker claims in his first move, and let $G' := G \setminus e$. Clearly $\chi(G') \geq \chi(G) - 1 = k^2 + 1$. It is well-known (and easy to see) that for any subgraph $H \subseteq G'$, we have $\chi(G') \leq \chi(H)\chi(G' - E(H))$. In particular, at the end of the game, at least one of the players will claim the edges of a non-k-colorable graph. Assume for the sake of contradiction that Maker does not have a winning strategy. It follows by the argument above that Breaker has a winning strategy. Since Maker starts the game on G', he can apply strategy stealing and win the game, a contradiction. This settles the existence of $r_1(k)$. For biased games however, we know hardly anything. We do not know any upper bound on $r_q(k)$, even when Breaker's bias q is as small as 2. Indeed, even the following, seemingly innocent, question is open: can Maker build a *non-bipartite* graph when playing a $(1 : 2)$ game on the edge-set of a 1000-chromatic graph? An analogous open problem for vertices appeared in the paper of Duffus, Łuczak, and Rödl [9]. Here we suggest the following problem whose resolution might prove useful in attacking the aforementioned problem of Duffus, Łuczak, and Rödl.

Problem 7.3 *Let G be an arbitrary r-chromatic graph, where $r \geq r_1(2)$ is a large constant. Find an explicit strategy for Maker to claim the edges of a non-bipartite graph, in a $(1 : 1)$ game on $E(G)$.*

Questions of similar flavor are discussed in more detail in a recent paper [1] of Alon and the first two authors of the present paper.

Biased connectivity game: Theorem 1.6 establishes an abrupt change in the number of edges required for Maker's win in the connectivity game when Breaker's bias changes from 1 to 2. It would be very interesting to determine the exact constant in the asymptotic dependence of $\hat{m}(\mathcal{C}_1, q)$ on q. We conjecture the following.

Conjecture 7.4

$$\hat{m}(\mathcal{C}_1, q) = \frac{1}{2}(q - 1 + o(1))n \ln n.$$

The validity of Conjecture 7.4 would imply that $\hat{m}(\mathcal{D}_1) = (1 + o(1))\hat{m}(\mathcal{C}_1)$ holds for every $q \geq 2$. It would thus follow that, as in many other scenarios, the properties 'connectivity' and 'positive minimum degree' are tightly connected with respect to $\hat{m}$ (in biased games). Moreover, it might indicate that the aforementioned abrupt change in $\hat{m}$, which occurs in the transition from unbiased to biased games, is due to the vanishing of the leading term in the right hand side of 7.4 when $q = 1$.

Non-global properties: As noted in the Introduction, it makes sense to study $\hat{m}$ for different properties as well. For example, for a fixed graph H one could study the smallest number of edges of a graph G on which Maker can build a copy of H. Clearly, the answer is a function of H which is independent of the number of vertices of the base graph. Indeed, it is evident (via *Strategy Stealing*) that $\hat{m}(\mathcal{P}_H) \leq \hat{r}(H)$, where $\mathcal{P}_H$ is the property of admitting a copy of H and $\hat{r}(H)$ is the so-called *size Ramsey number* of H. Somewhat similarly, one could study $\hat{m}(\mathcal{P}_T(\alpha, n, d))$, where $G \in \mathcal{P}_T(\alpha, n, d)$ if and only if $|V(G)| = n$ and G admits a copy of every tree on αn vertices with maximum degree at most d. When studying this parameter, one could use different known sufficient conditions for embedding such trees in expanding graphs (see e.g. [11, 12, 2, 4]). It seems therefore that studying $\hat{m}(\mathcal{P})$ for non-global properties $\mathcal{P}$ is of a different flavor and might require different tools. We plan to study them in a separate paper.

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