The game of Toucher and Isolator ⋆

Chris Dowden[1], Mihyun Kang[1]⋆⋆ Mirjana Mikalački[2]⋆⋆⋆, and Miloš Stojaković[2]⋆⋆⋆

[1] Institute of Discrete Mathematics, Graz University of Technology, Graz, Austria
{dowden,kang}@math.tugraz.at
[2] Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Novi Sad, Serbia
{mirjana.mikalacki,milos.stojakovic}@dmi.uns.ac.rs

Abstract. We introduce a new positional game called 'Toucher-Isolator', which is a quantitative version of a Maker-Breaker type game. The playing board is the set of edges of a given graph G, and the two players, Toucher and Isolator, claim edges alternately. The aim of Toucher is to 'touch' as many vertices as possible (i.e. to maximise the number of vertices that are incident to at least one of her chosen edges), and the aim of Isolator is to minimise the number of vertices that are so touched. We analyse the number of untouched vertices $u(G)$ at the end of the game when both Toucher and Isolator play optimally, obtaining results both for general graphs and for particularly interesting classes of graphs, such as cycles, paths, trees, and k-regular graphs.

Keywords: positional games, Maker-Breaker, graphs

1 Introduction

One of the most fundamental and enjoyable mathematical activities is to play and analyse games, ranging from simple examples, such as snakes and ladders or noughts and crosses, to much more complex games like chess and bridge.

Many of the most natural and interesting games to play involve pure skill, perfect information, and a sequential order of play. These are known formally as 'combinatorial' games, see e.g. [4], and popular examples include Connect Four, Hex, noughts and crosses, draughts, chess, and go.

Often, a combinatorial game might consist of two players alternately 'claiming' elements of the playing board (e.g. noughts and crosses, but not chess) with the intention of forming specific winning sets, and such games are called 'positional' combinatorial games (for a comprehensive study, see [3] or [9]). In particular, much recent research has involved positional games in which the

⋆ The full version of this extended abstract is published as *The Toucher-Isolator game* in the Electronic Journal of Combinatorics 26(4) (2019), #P4.6

⋆⋆ Supported by Austrian Science Fund (FWF): W1230.

⋆⋆⋆ Partly supported by Ministry of Education, Science and Technological Development, Republic of Serbia, Grant nr. 174019.

board is the set of edges of a graph, and where the aim is to claim edges in order to form subgraphs with particular properties.

A pioneering paper in this area was that of Chvátal and Erdős [5], in which the primary target was to form a spanning tree. Subsequent work has then also involved other standard graph structures and properties, such as cliques [2, 7], perfect matchings [14, 11], Hamilton cycles [11, 13], planarity [10], and given minimum degree [8]. Part of the appeal of these games is that there are several different versions. Sometimes, in the so-called strong games, both players aim to be the first to form a winning set (c.f. three-in-a-row in a game of noughts and crosses). In others, only Player 1 tries to obtain such a set, and Player 2 simply seeks to prevent her from doing so.

This latter class of games are known as 'Maker-Breaker' positional games. A notable result here is the Erdős-Selfridge Theorem [6], which establishes a simple but general condition for the existence of a winning strategy for Breaker in a wide class of such problems. A quantitative generalisation of this format then involves games in which Player 1 aims to form as many winnning sets as possible, and Player 2 tries to prevent this (i.e. Player 2 seeks to minimise the number of winning sets formed by Player 1).

In this paper, we introduce a new quantitative version of a Maker-Breaker style positional game, which we call the 'Toucher-Isolator' game. Here, the playing board is the set of edges of a given graph, the two players claim edges alternately, the aim of Player 1 (Toucher) is to 'touch' as many vertices as possible (i.e. to maximise the number of vertices that are incident to at least one of her edges), and the aim of Player 2 (Isolator) is to minimise the number of vertices that are touched by Toucher (i.e. to claim all edges incident to a vertex, and do so for as many vertices as possible).

This problem is thus simple to formulate and seems very natural, with connections to other interesting games, such as claiming spanning subgraphs, matchings, etc. In particular, we note that it is related to the well-studied Maker-Breaker vertex isolation game (introduced by Chvátal and Erdős [5]), where Maker's goal is to claim all edges incident to a vertex, and it is hence also related to the positive min-degree game (see [1, 9, 12]), where Maker's goal is to claim at least one edge of every vertex.

Our Toucher-Isolator game can be thought of as a quantitative version of these games, where Toucher now wants to claim at least one edge on *as many* vertices as possible, while Isolator aims to isolate *as many* vertices as possible. However, the game has never previously been investigated, and so there is a vast amount of unexplored territory here, with many exciting questions. What are the best strategies for Toucher and Isolator? How do the results differ depending on the type of graph chosen? Which graphs provide the most interesting examples?

2 General graphs

Given a graph $G = (V(G), E(G))$, we use $u(G)$ to denote the number of untouched vertices at the end of the game when both Toucher and Isolator play optimally. We obtain both upper and lower bounds on $u(G)$, some of which are applicable to all graphs and some of which are specific to particular classes of graphs (e.g. cycles or trees).

Clearly, one of the key parameters in our game will be the degrees of the vertices (although, as we shall observe later, the degree sequence alone does not fully determine the value of $u(G)$). In our bounds for general G, perhaps the most significant is the upper bound of Theorem 1. Here, we find that it suffices just to consider the vertices with degree at most three (we again re-iterate that all our bounds are tight).

Theorem 1. *For any graph G, we have*

$$d_0 + \frac{1}{2}d_1 - 1 \leq u(G) \leq d_0 + \frac{3}{4}d_1 + \frac{1}{2}d_2 + \frac{1}{4}d_3,$$

where d_i denotes the number of vertices with degree exactly i.

Proof (Sketch). **Upper bound:** Toucher uses a pairing strategy to touch enough vertices for the statement to hold. We define a collection of disjoint pairs of edges, and Toucher's strategy will be to wait (and play arbitrarily) until Isolator claims an edge within a pair, and then immediately respond by claiming the other edge (unless she happens to have already claimed it with one of her previous arbitrary moves, in which case she can again play arbitrarily). This way, Toucher will certainly claim at least one edge in every pair. To create a pairing, we add an auxiliary vertex and connect it to all odd degree vertices of G. The graph created in this way is even, and each of its components has an Eulerian tour. For each of these Eulerian tours, we then arbitrarily choose one of two orientations. Removing the auxiliary vertex leaves an orientation of G. Now, for each vertex that has at least 2 incoming edges, we take two such arbitrary edges and pair them. Some degree 3 vertices and all vertices of degree at least 4 will be covered by such pairing, and now we should consider the vertices of degree 1 and 2 and the remaining vertices of degree 3. We deal with this in the following way: we collect them and pair them arbitrarily. If their number is odd, Toucher takes one of these edges in the very first move (before Isolator claimed anything).
Lower bound: Lower bound is obtained by the fact that Isolator can claim at least half of all edges whose at least one endpoint has degree 1, including at least half of the edges whose both endpoints have degree 1. □

For certain degree sequences, the bounds given in Theorem 1 can be improved by our next result.

Theorem 2. *For any graph G, we have*

$$\sum_{v\in V(G)} 2^{-d(v)} - \frac{|E(G)|+7}{8} \leq u(G) \leq \sum_{v\in V(G)} 2^{-d(v)},$$

where $d(v)$ denotes the degree of vertex v.

Equivalently, we have

$$\sum_{i\geq 0} 2^{-i} d_i - \frac{|E(G)|+7}{8} \leq u(G) \leq \sum_{i\geq 0} 2^{-i} d_i,$$

where d_i again denotes the number of vertices with degree exactly i.

Note also that $|E(G)|$ will be small if the degrees are small, and so Theorem 2 then provides a fairly narrow interval for the value of $u(G)$ (observe that Theorem 1 already provides a narrow interval if the degrees are large).

Proof (Sketch). The proof relies on the approach of Erdős and Selfridge [6] and their “danger” function, defined as follows:
A vertex touched by Toucher has the danger value 0, while a vertex untouched by Toucher incident with k free edges (edges unclaimed by anyone) has danger value 2^{-k}.
The total danger of the graph is the sum of the danger values for all vertices. When the game is over, the total danger of the graph is precisely the number of untouched vertices.
When Isolator claims an edge, the total danger increases by the sum of the dangers of the endpoints of that edge. On the other hand, when Toucher claims an edge, the total danger decreases by the sum of the dangers of the endpoints of that edge.
Upper bound: The upper bound is obtained by adding the strategy of Toucher to all the aforementioned. Toucher will always choose the edge that maximises the sum of danger values of the two vertices that are touched. Therefore, after two consequent moves of Toucher and Isolator, the total danger never increases throughout the game. The given upper bound follows.
Lower bound: For the lower bound one has to carefully track change in total danger value after one round in the game, i.e. the consecutive moves of Isolator and of Toucher, given that Isolator plays in such a way to maximise the sum of danger of the endpoints of the claimed edge. The total danger value decreases by at most $\frac{1}{4}$ after one round, and also, after the first move of Toucher, the total danger decreases by at most one. Noting that there are $\left\lfloor \frac{|E(G)|-1}{2} \right\rfloor$ rounds after first Toucher's move, the given lower bound follows.

□

Remark 1. Note that the upper bound of Theorem 2 will be better than the upper bound of Theorem 1 if

$$\sum_{i\geq 4} 2^{3-i} d_i < 2d_1 + 2d_2 + d_3.$$

Remark 2. Note that the lower bound of Theorem 2 will be better than the lower bound of Theorem 1 if

$$|E(G)| < 1 + \sum_{i\geq 2} 2^{3-i} d_i.$$

As $2|E(G)| = \sum_{i\geq 1} i d_i$, this will occur if d_2 is sufficiently large (e.g. consider a path or a cycle, in which case the lower bound of Theorem 1 is ineffective).

3 Specific graphs

Moving on from these general bounds, it is already interesting to play on relatively small graphs (such as cycles, paths and 2-regular graphs), and to try to determine the optimal strategies and the proportion of untouched vertices. We again obtain tight upper and lower bounds, both for C_n and for the closely related game on P_n (the path on n vertices).

Theorem 3. *For all n, we have*

$$\frac{3}{16}(n-3) \leq u(C_n) \leq \frac{n}{4}.$$

Theorem 4. *For all n, we have*

$$\frac{3}{16}(n-2) \leq u(P_n) \leq \frac{n+1}{4}.$$

We also extend the game to general 2-regular graphs (i.e. unions of disjoint cycles). Our main achievement here is to obtain a *tight* lower bound of $u(G) \geq \frac{n-3}{6}$, which (by a comparison with the lower bound of Theorem 3) also demonstrates that $u(G)$ is not solely determined by the degree sequence.

Theorem 5. *For any 2-regular graph G with n vertices, we have*

$$\frac{n-3}{6} \leq u(G) \leq \frac{n}{4}.$$

An interesting and natural extension of the game on paths is obtained by considering general trees, although this additional freedom in the structure can make the problem significantly more challenging. Here, we derive the following tight bounds, and provide the examples of graphs satisfying these bounds exactly.

Theorem 6. *For any tree T with $n > 2$ vertices, we have*

$$\frac{n+2}{8} \leq u(T) \leq \frac{n-1}{2}.$$

It follows from Theorem 1 that there will be no untouched vertices in k-regular graphs if $k \geq 4$, and it is natural to consider what happens in the 3-regular case.

A direct consequence of Theorem 2 is the following.

Corollary 1. *For any 3-regular graph G with n vertices, we have*

$$u(G) \leq \frac{n}{8}.$$

We observe that there are 3-regular graphs for which $u(G) = 0$, and one might expect that this could be true for all such graphs. However, we in fact manage to construct a class of examples for which a constant proportion of vertices remain out of Toucher's reach.

Theorem 7. *For all even $n \geq 4$, there exists a 3-regular graph G with n vertices satisfying*

$$u(G) \geq \left\lfloor \frac{n}{24} \right\rfloor.$$

4 Concluding remarks

We cannot hope to obtain exact results just by looking at the degree sequence of the graph. Hence, we are curious to know if any other properties or parameters of the graph can be utilised to give more precise bounds.

Finally, what is the largest possible proportion of untouched vertices for a 3-regular graph? By Theorem 7 and Corollary 1, we know that this is between $\frac{1}{24}$ and $\frac{1}{8}$.

References

1. Balogh, J., Pluhár, A.: The positive minimum degree game on sparse graphs, Electron. J. Combin. 19 Paper 22 (2012)
2. Balogh ,J., Samotij, W.: On the Chvátal-Erdős triangle game, Electron. J. Combin. 18 Paper 72 (2011)
3. Beck ,J.: Combinatorial Games: Tic-Tac-Toe Theory, Cambridge University Press (2008)
4. Berlekamp, E., Conway, J., Guy, R.: Winning Ways For Your Mathematical Plays, AK Peters (2001)
5. Chvátal, V., Erdős, P.: Biased positional games, Ann. Discrete Math. 2 221–228 (1978)
6. Erdős, P.,Selfridge, J.: On a combinatorial game, J. Combin. Theory Ser. A 14 298–301 (1973)
7. Gebauer, H.: On the Clique-Game, European J. Combin. 33 8–19 (2012)
8. Gebauer, H., Szabó, T.: Asymptotic random graph intuition for the biased connectivity game, Random Structures Algorithms 35 431–443 (2009)
9. Hefetz, D.,Krivelevich, M., Stojaković, M., Szabó, T.: Positional Games, Springer (2014)
10. Hefetz, D., Krivelevich, M., Stojaković, M., Szabó, T.: Planarity, colorability and minor games, SIAM J. Discrete Math. 22 194–212 (2008)
11. Hefetz, D., Krivelevich, M., Stojaković, M., Szabó, T.: Fast winning strategies in Maker-Breaker games, J. Combin. Theory Ser. B 99 39–47 (2009)
12. Hefetz, D., Krivelevich, M., Stojaković, M., Szabó, T.: Global Maker-Breaker games on sparse graphs, European J. Combin. 32 162–177 (2011)
13. Krivelevich, M.: The critical bias for the Hamiltonicity game is $(1 + o(1))n/\ln n$, J. Amer. Math. Soc. 24 125–131 (2011)
14. Mikalački, M., Stojaković, M.: Fast strategies in biased Maker-Breaker games, Discrete Math. Theor. Comput. Sci. 20 Paper 6 (2018)