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AN ELEMENTARY PROOF OF A THEOREM CONCERNING THE DIVISION OF A REGION INTO TWO

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ABSTRACT. Intuitively obvious theorems which are hard to prove are nothing new in topology. The most celebrated case is certainly the Jordan curve theorem. For pedagogical reasons elementary proofs of such theorems never become obsolete. During their work with students of mathematics the following problem has forced itself on the authors of the paper: to prove in a reasonably elementary fashion that an open Jordan curve with its endpoints on a closed Jordan curve $\mathcal{K}$, but otherwise located in the bounded part, divides the closure of the bounded part into two parts. In this paper we take the Jordan curve theorem (JCT) for granted and then prove, in a careful, elementary way, the related fact. Unfortunately, it seems that, even given the JCT, there is still a whole lot of work to do. But there are shortcuts. For instance, we do not need to consider the problem of approximating a general curve by polygons, or the delicate limit questions arising when going back from the easy polygon case to the general case.

1. Introduction. The Jordan curve theorem (JCT) claims that a simple closed curve in a plane divides the plane (excluding the points of the curve $\mathcal{K}$ itself) into two regions in the sense that any broken line (curve consisting of connected line segments) connecting two points from different regions intersects the curve, and for any two points from the same region there exists a broken line connecting them which does not intersect the curve. Exactly one of these regions is bounded and called the *interior*; the other one is called the *exterior* of the curve. A (bounded) figure Φ determined by a simple closed curve $\mathcal{K}$ is usually defined as the union of the curve $\mathcal{K}$ and its interior. Using the fact that the curve $\mathcal{K}$ is the boundary of each of its regions, proved in [**1**], one readily obtains that the interior and the boundary of Φ, denoted by $\mathrm{In}(\Phi)$ and $\mathrm{Bd}(\Phi)$, coincide with the interior of $\mathcal{K}$ and the curve $\mathcal{K}$ itself, respectively.

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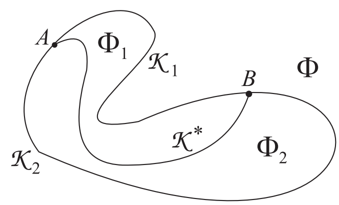

FIGURE 1. The simple open curve $\mathcal{K}^*$ divides the figure Φ into the figures Φ_1 and Φ_2.

If we denote by A and B two points on the curve $\mathcal{K}$ and by $\mathcal{K}^*$ a simple open curve connecting points A and B which is, excluding its end-points, entirely situated within the interior of $\mathcal{K}$, then we obtain two new simple closed curves, each of them formed by $\mathcal{K}^*$ and one of the two arcs of $\mathcal{K}$ with end-points in A and B, Figure 1. Therefore, we obtain two new figures, say Φ_1 and Φ_2. The fact that the figure Φ is divided into figures Φ_1 and Φ_2, so that $\Phi_1 \cup \Phi_2 = \Phi$ and $\Phi_1 \cap \Phi_2 = \mathcal{K}^*$, is intuitively obvious, but it is not a trivial corollary of the JCT.

The Jordan curve theorem itself (and its generalizations) can be proved using homology theory [**2, 3**], but the proof can also be carried out in a reasonably elementary fashion [**1, 4, 5**]. The theorem described above, can be derived using homology theory [**2**], too. The aim of this work is to present a direct and elementary proof of this statement. Here is a simple and elementary argument in the spirit of the Filippov proof [**1**].

Before we proceed any further, it would be well to adopt some definitions and notations.

Definition 1. Let m be a ray (or a line) and $\mathcal{L}$ a broken line (open or closed) in the plane intersecting m. A point or a side of the broken line $\mathcal{L}$ lying on m is said to be a *proper* point (side) of the intersection in each of the following cases:

1) it is a point belonging to a segment of $\mathcal{L}$ whose end-points lie on opposite sides of m, or

2) it is a vertex of $\mathcal{L}$ whose neighboring vertices on $\mathcal{L}$ lie on opposite sides of m, or

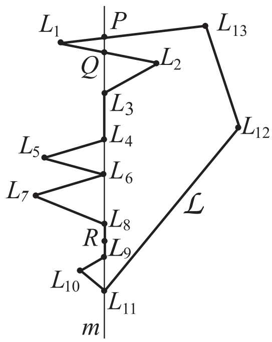

FIGURE 2. The proper points and proper sides of the intersection of $\mathcal{L}$ and m are P, Q, L_{11} and $[L_3L_4]$.

3) it is a side of $\mathcal{L}$ whose adjacent sides (excluding the common points) lie on opposite sides of m.

In Figure 2 the points P, Q and L_{11} are the proper points of intersection of $\mathcal{L} = L_1L_2 \cdots L_{13}L_1$ and the line m, but the points L_3, L_4, L_6, L_8, R and L_9 are not. Also, the segment $[L_3L_4]$ is a proper side of this intersection, but the segment $[L_8L_9]$ is not.

Besides the notation $\mathrm{In}\,(\Phi)$ and $\mathrm{Bd}\,(\Phi)$ for the interior and the boundary of a given set Φ, we use also the notation $\mathrm{Ex}\,(\Phi)$ and $\mathrm{Cp}\,(\Phi)$ for the exterior and the complement of Φ. The neighborhood of a point A and radius δ is $\mathcal{N}(A,\delta) \stackrel{\text{def}}{=} \{X \mid d(A,X) < \delta\}$, where $d(A,X)$ denotes the Euclidean distance between two points. The distance from a point X to a set φ (or between two sets φ_1 and φ_2) is defined by $d(X,\varphi) \stackrel{\text{def}}{=} \inf\{d(X,Y) \mid Y \in \varphi\}$ ($d(\varphi_1,\varphi_2) \stackrel{\text{def}}{=} \inf\{d(X,Y) \mid X \in \varphi_1 \wedge Y \in \varphi_2\}$).

For a given coordinate system Oxy, we denote by $r^+(Z)$ and $r^-(Z)$ the half-lines extending from Z in the positive and negative directions of the y-axis, respectively. Let $\mathcal{L}$ be a broken line and Z a point which does not belong to $\mathcal{L}$. Then, $n(Z,\mathcal{L})$ denotes the number of all proper points and proper sides of the intersection of $\mathcal{L}$ and $r^+(Z)$. In the case of self-crossing of the broken line $\mathcal{L}$, i.e., if the broken line is not simple, we are counting each proper point (side) of intersection according to its multiplicity.

Now we can define a function, with domain in the set of all points of the plane excluding the points of $\mathcal{L}$, by

$$N(Z,\mathcal{L}) = \begin{cases} 0 & \text{for } n(Z,\mathcal{L}) \text{ even }, \\ 1 & \text{for } n(Z,\mathcal{L}) \text{ odd.} \end{cases}$$

We now invoke a lemma used in the Filippov proof [**1**].

Lemma 1. *Let $\mathcal{L} : L_1L_2\cdots L_n$ and $\mathcal{M} : M_1M_2\cdots M_m$ be two disjoint broken lines, not necessarily simple, in the plane Oxy. If the broken line $\mathcal{L}$ is closed or $\mathcal{M}$ lies entirely between the vertical lines (lines parallel to the y-axis) through the points L_1 and L_n, respectively, then $N(Z,\mathcal{L})$ is constant, for all $Z \in \mathcal{M}$.*

Proof. Denote by $p_1, p_2, \dots, p_k$ $(k \le n+m)$ all possible vertical lines passing through the vertices of $\mathcal{L}\cup\mathcal{M}$ in such a way that p_i lies between p_{i-1} and p_{i+1} $(2 \le i \le k-1)$. On every segment of the broken line $\mathcal{M}$ between two adjacent vertical lines, including the end points, the function $N(Z,\mathcal{L})$ is constant. To prove it, consider one such segment with end-points Z_0 and Z_1 $(Z_0 \in p_i,\quad Z_1 \in p_{i+1})$ and an arbitrary point Z between Z_0 and Z_1. For every proper point (or proper side) of the intersection of $\mathcal{L}$ and the ray $r^+(Z_0)$, there is exactly one segment of $\mathcal{L}$ entirely or in part situated within the strip between the lines p_i and p_{i+1} which contains (adjoins) this proper point (proper side) of the intersection of $\mathcal{L}$ and $r^+(Z_0)$. The intersection of this segment and the ray $r^+(Z)$ is a proper point of intersection (first three pictures in Figure 3), too. Similarly, for every point (or side) of $\mathcal{L}$ on the ray $r^+(Z_0)$ which is not a proper point (side) of the intersection, there is no segment or there are exactly two segments entirely or in part situated within the strip between the lines p_i and p_{i+1} and incident on this point (or this segment) (the last two pictures in Figure 3). In the second case, the ray $r^+(Z)$ crosses these two segments of $\mathcal{L}$ in two proper points of intersection. From above, we see that the numbers $n(Z_0,\mathcal{L})$ and $n(Z,\mathcal{L})$ are of the same parity, i.e., $N(Z_0,\mathcal{L}) = N(Z,\mathcal{L})$. Similarly, we can obtain that $N(Z_1,\mathcal{L}) = N(Z,\mathcal{L})$ for every $Z \in [Z_0Z_1]$. Since the function $N(Z,\mathcal{L})$ is constant on any two adjacent segments of $\mathcal{M}$, it is constant for all $Z \in \mathcal{M}$. □

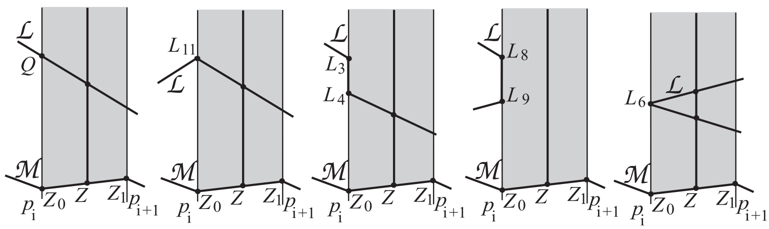

FIGURE 3. The numbers $n(Z_0, \mathcal{L})$ and $n(Z, \mathcal{L})$ are of the same parity.

Besides Lemma 1, the Filippov proof of the JCT [**1**] consists of the following steps:

I. The proof that a simple closed curve $\mathcal{K}$ in a plane divides the plane into at least two regions. (The author chooses a y-axis not intersecting the curve $\mathcal{K}$ and constructs a point $C \notin \mathcal{K}$ which cannot be connected to the origin O (from the y-axis) by any broken line without intersecting the curve $\mathcal{K}$.)

II. The proof that a simple open curve in a plane cannot divide the plane (i.e any two points not belonging to the curve can be connected by a broken line without intersecting that curve).

III. The proof that the simple closed curve $\mathcal{K}$ is a boundary of each of its regions.

IV. The proof that a simple closed curve $\mathcal{K}$ in a plane divides the plane into exactly two regions. (From the assumption that points B, C belong to the exterior of an arbitrary region of $\mathcal{K}$ the author proves that the points B and C can be connected by a broken line without intersecting the curve $\mathcal{K}$.)

For the purpose of completing the proof of the theorem in question we invoke the proof of step III in the Filippov proof of the JCT.

Lemma 2. *A simple closed curve $\mathcal{K}$ is the boundary of each of its regions.*

Proof. Let $\mathcal{O}_1, \mathcal{O}_2, \ldots$ be the regions determined by $\mathcal{K}$. Regions are open sets and $\mathrm{In}\,(\mathcal{O}_i) = \mathcal{O}_i \subseteq \mathrm{Ex}\,(\mathcal{O}_j)$, $j \neq i$; $i, j = 1, 2, \ldots$. Consequently, $\cup_{j=1,2,\ldots}\mathcal{O}_j \subseteq \mathrm{In}\,(\mathcal{O}_i) \cup \mathrm{Ex}\,(\mathcal{O}_i)$, which implies $\mathrm{Bd}\,(\mathcal{O}_i) \subseteq$

$\mathcal{K}$, for all $i = 1, 2, \ldots$. To prove the opposite inclusion $\mathcal{K} \subseteq \mathrm{Bd}(\mathcal{O}_i)$, for all $i = 1, 2, \ldots$, let us assume that there exist a region $\mathcal{O}_i$ and a point $A \in \mathcal{K} \setminus \mathrm{Bd}(\mathcal{O}_i)$. There is a neighborhood $N(A, \epsilon)$ ($\epsilon > 0$) of the point A for which $N(A, \epsilon) \cap \mathrm{Bd}(\mathcal{O}_i) = \varnothing$. (The boundary of an arbitrary set is closed.) If we remove an arc of $\mathcal{K}$ lying in the neighborhood and containing the point A, we obtain a simple open curve which contains the whole set $\mathrm{Bd}(\mathcal{O}_i)$. Now, we can find points $X \in \mathrm{In}(\mathcal{O}_i)$ and $Y \in \mathrm{Ex}(\mathcal{O}_i)$ which can be connected by a broken line without intersecting that open curve. Contradiction. □

2. Proof of the theorem. Now, we can state the main theorem:

Theorem 1. *Let A and B be two points from the boundary of a bounded plane figure Φ determined by a simple closed curve $\mathcal{K} = \mathrm{Bd}(\Phi)$, and let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the arcs of $\mathcal{K}$ determined by the points A and B. Let $\mathcal{K}^*$ be a simple open curve connecting A and B which is, excluding these two points, situated entirely in the interior of $\mathcal{K}$. If we denote by Φ_1 and Φ_2 the figures determined by the simple closed curves $\mathcal{K}' = \mathcal{K}_1 \cup \mathcal{K}^*$ and $\mathcal{K}'' = \mathcal{K}_2 \cup \mathcal{K}^*$, respectively, then the following equations hold*

$$\Phi = \Phi_1 \cup \Phi_2 \tag{2}$$

$$\Phi_1 \cap \Phi_2 = \mathcal{K}^*. \tag{3}$$

Proof. This proof is involved, since we find it necessary to divide it into more cases as follows. We begin by choosing a coordinate system with the y-axis (and thus the origin O) outside a disk covering the curve $\mathcal{K}$. This disk also covers the region $\mathrm{In}(\Phi)$, and consequently, curves $\mathcal{K}'$ and $\mathcal{K}''$, and their interiors $\mathrm{In}(\Phi_1)$ and $\mathrm{In}(\Phi_2)$. Thus, $O \in \mathrm{Ex}(\Phi) \cap \mathrm{Ex}(\Phi_1) \cap \mathrm{Ex}(\Phi_2)$.

Step I. Instead of proving that $\Phi_1 \cup \Phi_2 \subseteq \Phi$, we will prove its contrapositive

$$\mathrm{Ex}(\Phi) \subseteq \mathrm{Ex}(\Phi_1) \cap \mathrm{Ex}(\Phi_2). \tag{4}$$

Consider an arbitrary point X from the set $\mathrm{Ex}(\Phi)$ and the broken line connecting X and O which does not intersect the curve $\mathcal{K}$. Since this broken line lies entirely in the set $\mathrm{Ex}(\Phi)$, it cannot meet the curve $\mathcal{K}^*$ ($\mathcal{K}^* \subseteq \Phi$). Consequently, X belongs to the same region as O with respect to both the curve $\mathcal{K}'$ and the curve $\mathcal{K}''$, i.e., it is in both $\mathrm{Ex}(\Phi_1)$ and $Ex(\Phi_2)$.

Step II. To prove that $\Phi_1 \cap \Phi_2 = \mathcal{K}^*$, it is sufficient to prove that

$$\mathrm{Bd}(\Phi_2) \cap \mathrm{In}(\Phi_1) = \varnothing, \tag{5}$$

(and analogously that $\mathrm{Bd}(\Phi_1) \cap \mathrm{In}(\Phi_2) = \varnothing$, as well), and that

$$\mathrm{In}(\Phi_1) \cap \mathrm{In}(\Phi_2) = \varnothing. \tag{6}$$

II a) Suppose that the relation (5) is not true, i.e., that there exists a point Y on the curve $\mathcal{K}_2$, different from both A and B, which belongs to the set $\mathrm{In}(\Phi_1)$. Since this point does not belong to the curve $\mathcal{K}' = \mathcal{K}_1 \cup \mathcal{K}^*$, there is a neighborhood of Y, say $\mathcal{N}(Y, \delta)$ ($\delta > 0$), which is disjoint from $\mathcal{K}'$ (δ can be chosen to be less than the distance of the point Y from $\mathcal{K}'$). Observe that all points of $\mathcal{N}(Y, \delta)$ are in the same region with respect to $\mathcal{K}'$, i.e., in $\mathrm{In}(\Phi_1)$. But, since Y is a boundary point of Φ, every neighborhood of Y, in particular the neighborhood $\mathcal{N}(Y, \delta)$, intersects $\mathrm{Cp}(\Phi) = \mathrm{Ex}(\Phi)$ (Lemma 2). From (4) we obtain that $\mathcal{N}(Y, \delta) \cap \mathrm{Ex}(\Phi_1) \neq \varnothing$. The contradiction just reached shows that our assumption must be false and hence that the relation (5) holds.

II b) Suppose that the relation (6) is not true, i.e., that there exists a point $Z \in \mathrm{In}(\Phi_1) \cap \mathrm{In}(\Phi_2)$. Consider an arbitrary point S on the curve $\mathcal{K}_1$ distinct from both A and B. Since the point S belongs to $\mathrm{Ex}(\Phi_2)$ (just proved in II a), there exists a neighborhood $\mathcal{N}(S, \delta)$ which lies entirely within $\mathrm{Ex}(\Phi_2)$. In this neighborhood we can find a point T from $\mathrm{In}(\Phi_1)$ because $S \in \mathrm{Bd}(\Phi_1)$ (Lemma 2). Now, we can connect the points Z and T, two points from the set $\mathrm{In}(\Phi_1)$, with a broken line which will lie entirely in $\mathrm{In}(\Phi_1)$ (Figure 4). Observe that this broken line can intersect none of the curves $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}^*$ because $\mathcal{K}_1 \cup \mathcal{K}^* = \mathrm{Bd}(\Phi_1)$ and $\mathcal{K}_2 \cap \mathrm{In}(\Phi_1) = \varnothing$. Thus, the points Z and T belong to the same region with respect to the figure Φ_2, too. But $Z \in \mathrm{In}(\Phi_2)$ and $T \in \mathcal{N}(S, \delta) \subseteq \mathrm{Ex}(\Phi_2)$, a contradiction.

The last contradiction completes the proof for (3).

Step III. In the remainder of the proof we shall establish the relation $\Phi \subseteq \Phi_1 \cup \Phi_2$ by proving its contrapositive $\mathrm{Ex}(\Phi_1) \cap \mathrm{Ex}(\Phi_2) \subseteq$

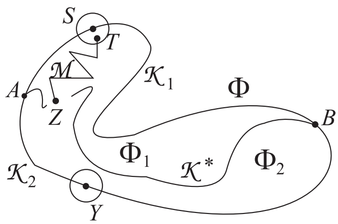

FIGURE 4. The intersection of Φ_1 and Φ_2 is just the curve $\mathcal{K}^*$.

Ex (Φ). For this purpose it suffices to prove that the intersection Ex $(\Phi_1) \cap$ Ex $(\Phi_2) \cap$ In (Φ) is the empty set. Again, assume the opposite. Let there exist a point X such that

$$X \in \mathrm{Ex}\,(\Phi_1) \cap \mathrm{Ex}\,(\Phi_2) \cap \mathrm{In}\,(\Phi). \tag{7}$$

Let L and R be points of the curve $\mathcal{K}$ with the smallest and the largest first coordinates, respectively. (If there exists more than one point with the property of having the smallest (or the largest) first coordinate, we can choose any of them.) Further, let l and r be the vertical lines passing through the points L and R, respectively, and m a line parallel to them and situated between them (in the strip determined by l and r). The points L and R divide $\mathcal{K}$ into two curves $\mathcal{K}_3$ and $\mathcal{K}_4$. Both of them intersect the line m because the points L and R lie on opposite sides of m. Let C be a point of $\mathcal{K} \cap m$ with the highest value of the second coordinate. Without loss of generality, assume that the point C belongs to $\mathcal{K}_3$, Figure 5. Denote by D the point of intersection $\mathcal{K}_3 \cap m$ with the lowest second coordinate (the case of coincidence of the points C and D is not excluded) and by $\mathcal{K}_5$ the arc of $\mathcal{K}_3$ determined by points C and D. Now, choose a point E on the ray $r^-(D)$, different from D, situated within a neighborhood of D which is disjoint with $\mathcal{K}_4$.

From the fact that the points X and O both belong to the sets Ex (Φ_1) and Ex (Φ_2), we can introduce two broken lines $\mathcal{M}_1$ and $\mathcal{M}_2$ connecting these points and for which

$$\mathcal{M}_1 \cap \mathcal{K}' = \mathcal{M}_1 \cap (\mathcal{K}_1 \cup \mathcal{K}^*) = \varnothing, \tag{8}$$

$$\mathcal{M}_2 \cap \mathcal{K}'' = \mathcal{M}_2 \cap (\mathcal{K}_2 \cup \mathcal{K}^*) = \varnothing. \tag{9}$$

Note that the point E belongs neither to $\mathcal{K}_3$ nor $\mathcal{K}_4$. Since $\mathcal{K}_3 \cup \mathcal{K}_4 = \mathcal{K}$, we obtain that E belongs either to In (Φ) or Ex (Φ). In both cases

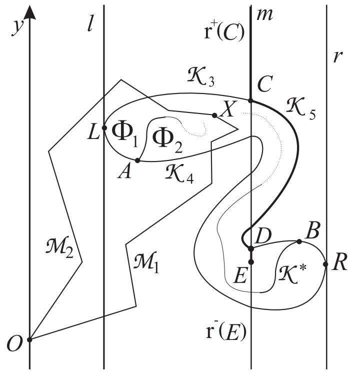

Figure 5. There is no point from the set $\mathrm{Ex}\,(\Phi_1) \cap \mathrm{Ex}\,(\Phi_2) \cap \mathrm{In}\,(\Phi)$.

we will achieve the contradiction simultaneously. Namely, in the case $E \in \mathrm{In}\,(\Phi)$ we denote by $\mathcal{M}$ a broken line connecting E and X ($E, X \in \mathrm{In}\,(\Phi)$) which does not intersect $\mathcal{K}$; in the case $E \in \mathrm{Ex}\,(\Phi)$, we denote by $\mathcal{M}$ a broken line connecting O and E ($O, E \in \mathrm{Ex}\,(\Phi)$) which does not intersect $\mathcal{K}$.

Since the y-axis does not cross the curve $\mathcal{K}$, the distance $d(r^+(O), \mathcal{K})$ is a positive number, i.e.,

$$\varepsilon_0 \stackrel{\text{def}}{=} d(r^+(O), \mathcal{K}) > 0. \tag{10}$$

From (8) and (9) we have

$$\varepsilon_1 \stackrel{\text{def}}{=} d(\mathcal{M}_1, \mathcal{K}') = d(\mathcal{M}_1, \mathcal{K}_1 \cup \mathcal{K}^*) > 0 \tag{11}$$

and

$$\varepsilon_2 \stackrel{\text{def}}{=} d(\mathcal{M}_2, \mathcal{K}'') = d(\mathcal{M}_2, \mathcal{K}_2 \cup \mathcal{K}^*) > 0. \tag{12}$$

From the choice of points D and E, we get

$$\varepsilon_3 \stackrel{\text{def}}{=} d(r^-(E), \mathcal{K}_3) > 0 \tag{13}$$

and

$$\varepsilon_4 \stackrel{\text{def}}{=} d([ED] \cup \mathcal{K}_5 \cup r^+(C), \mathcal{K}_4) > 0. \tag{14}$$

The broken line $\mathcal{M}$ and the curve $\mathcal{K}$ are disjoint (for both the cases $E \in \mathrm{In}\,(\Phi)$ and $E \in \mathrm{Ex}\,(\Phi)$), and so we have

$$\varepsilon_5 \stackrel{\text{def}}{=} d(\mathcal{M}, \mathcal{K}) > 0. \tag{15}$$

We introduce now a closed broken line $\mathcal{L}$ with vertices on the curve $\mathcal{K}$, including the points A, B, L, R, C and D, and with segments (sides) of lengths less than

$$\varepsilon^* \stackrel{\text{def}}{=} \min(\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5).$$

Similarly, we introduce also a broken line $\mathcal{L}^*$ connecting the points A and B with vertices on the curve $\mathcal{K}^*$ and with segments of lengths less than ε^*. By $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$ and $\mathcal{L}_5$, we denote the open broken line formed by the sides of $\mathcal{L}$ whose vertices all lie on $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{K}_3$, $\mathcal{K}_4$ and $\mathcal{K}_5$, respectively. Since the distance of any point of $\mathcal{L}$ from the curve $\mathcal{K}$ is less than $\varepsilon^*/2$, and the same is true for $\mathcal{L}^*$ and $\mathcal{K}^*$, from (11) and (12) it follows that

$$\mathcal{M}_1 \cap (\mathcal{L}_1 \cup \mathcal{L}^*) = \varnothing$$

and

$$\mathcal{M}_2 \cap (\mathcal{L}_2 \cup \mathcal{L}^*) = \varnothing.$$

From the last two relations, using Lemma 1 twice, we obtain

$$N(O, \mathcal{L}_1 \cup \mathcal{L}^*) = N(X, \mathcal{L}_1 \cup \mathcal{L}^*)$$

and

$$N(O, \mathcal{L}_2 \cup \mathcal{L}^*) = N(X, \mathcal{L}_2 \cup \mathcal{L}^*).$$

Using this we obtain that

$$\begin{aligned} N(O, \mathcal{L}) &= N(O, \mathcal{L}_1 \cup \mathcal{L}_2) = N(O, \mathcal{L}_1 \cup \mathcal{L}^*) \oplus N(O, \mathcal{L}_2 \cup \mathcal{L}^*) \\ &= N(X, \mathcal{L}_1 \cup \mathcal{L}^*) \oplus N(X, \mathcal{L}_2 \cup \mathcal{L}^*) \\ &= N(X, \mathcal{L}_1 \cup \mathcal{L}_2) = N(X, \mathcal{L}), \end{aligned}$$

where $\oplus$ denotes the addition modulo 2. Now, we obtain from above and from (10)

$$N(X, \mathcal{L}) = N(O, \mathcal{L}) = 0. \tag{16}$$

Using (15) and the fact that the distance of any point of $\mathcal{L}$ from the curve $\mathcal{K}$ is less than $\varepsilon^*/2$ we achieve $\mathcal{M} \cap \mathcal{L} = \varnothing$. By applying the lemma to the broken lines $\mathcal{M}$ and $\mathcal{L}$, and taking into account (16), we obtain (in both cases $E \in \mathrm{In}\,(\Phi)$ and $E \in \mathrm{Ex}\,(\Phi)$)

$$N(E, \mathcal{L}) = 0. \tag{17}$$

By similar considerations, we obtain from (13) and (14)

$$r^-(E) \cap \mathcal{L}_3 = \varnothing \tag{18}$$

and

$$([ED] \cup \mathcal{L}_5 \cup r^+(C)) \cap \mathcal{L}_4 = \varnothing. \tag{19}$$

Since the ray $r^-(E)$ and the broken line $\mathcal{L}_3$ are disjoint, the number $n(E, \mathcal{L}_3)$ is equal to the number of all proper points and proper sides of intersection of $\mathcal{L}_3$ and the line m. This latter number is odd because the end-points of the broken line $\mathcal{L}_3$ are on opposite sides of m and in going from one end-point of $\mathcal{L}_3$ to the other one we change the side of m as many times as the number of all proper points and proper sides of intersection of $\mathcal{L}_3$ and the line m. Consequently,

$$N(E, \mathcal{L}_3) = 1. \tag{20}$$

Further, applying Lemma 1 to the broken lines $[ED] \cup \mathcal{L}_5$ and $\mathcal{L}_4$ and using (19), we obtain

$$N(E, \mathcal{L}_4) = N(C, \mathcal{L}_4) = 0. \tag{21}$$

Finally, relation (20) and (21) give us

$$N(E, \mathcal{L}) = N(E, \mathcal{L}_3 \cup \mathcal{L}_4) = N(E, \mathcal{L}_3) \oplus N(E, \mathcal{L}_4) = 1 \oplus 0 = 1,$$

which is in contradiction with (17). $\square$

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