Enumeration of Hamiltonian Cycles in Some Grid Graphs

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Abstract

In polymer science, Hamiltonian paths and Hamiltonian circuits can serve as excellent simple models for dense packed globular proteins. Generation and enumeration of Hamiltonian paths and Hamiltonian circuits (compact conformations of a chain) are needed to investigate thermodynamics of protein folding. Hamiltonian circuits are a mathematical idealization of polymer melts, too. The number of Hamiltonian cycles on a graph corresponds to the entropy of a polymer system. In this paper, we present new characterizations of the Hamiltonian cycles in a labeled rectangular grid graph $P_m \times P_n$ and in a labeled thin cylinder grid graph $C_m \times P_n$. We proved that for any fixed m, the numbers of Hamiltonian cycles in these grid graphs, as sequences with counter n, are determined by linear recurrences. The computational method outlined here for finding these difference equations together with the initial terms of the sequences has been implemented. The generating functions of the sequences are given explicitly for some values of m. The obtained data are consistent with data obtained in the works by Kloczkowski and Jernigan, and Schmalz et al.

1. Introduction

In polymer science, the functional properties of proteins depend upon their three-dimensional structures, which arise because particular sequences of amino acids in polypeptide chains fold to generate, from linear chains, compact domains with specific structures. For solving so-called *the protein folding problem* one needs: a) to develop a model of proteins which contains the essentials of the system but which, at the same time, is sufficiently simple so as to allow the calculation of the variety of dynamic and static properties characterizing the system; b) to recognize the fact that one is simply forced to simplify the forces acting among the amino acids [2]. In a simplified lattice model, the protein is represented by a string of beads that is arranged on a cubic lattice. The study of Hamiltonian paths (compact conformations of a chain) has been advocated as a first approximation for understanding qualitatively the excluded-volume mechanisms at work behind protein folding [10, 19].

A Hamiltonian path of an undirected graph is a path that visits each vertex exactly once. Closed Hamiltonian path is called Hamiltonian cycle (or Hamiltonian circuit).

The radius of gyration R of a polymer of length $l \gg 1$ is expected to scale like

$$R \sim l^{\nu},$$

where ν is a standard critical exponent [6, 10]. The ratios of the number C_1 of Hamiltonian open paths and the number C_0 of Hamiltonian cycles in L^d hypercube in d dimensions

$$\frac{C_1}{C_0} \sim l^{\gamma+1} \sim L^{(\gamma+1)d},$$

where γ is another critical exponent [6, 10].
In addition, the probability that two ends of an open polymer join so as to form a ring polymer [10] is

$$p_{adj} = \frac{C_0 L^d}{C_1}.$$

On the other hand, Hamiltonian circuits are a mathematical idealization of polymer melts. The number of Hamiltonian cycles on a graph corresponds to the entropy of a polymer system. The entropy per site is

$$\frac{S}{N} = \frac{1}{N} \ln\, C_{N,P},$$

where $C_{N,P}$ is the number of Hamiltonian circuits in a N-point lattice with periphery P [4, 18].

From these reasons, articles [10, 12, 13, 15, 18] and [12, 14] are devoted, among other things, to the problem of enumeration of Hamiltonian cycles in two- and three-dimensional lattices by using transfer matrix approach [20]. The problem of enumeration of Hamiltonian cycles on $m \times n$ tubes, i.e. $C_m \times P_n$ is discussed in [18] and [14] (note that the $2 \times 2 \times n$ lattice and $4 \times n$ tubes are isomorphic).

Mathematicians have dealt with this counting problem for two-dimensional lattices, i.e. $m \times n$ grid graph ($m, n \in N$), too. The enumeration of Hamiltonian cycles, abbreviated HC, for small values of m, fixing m and letting n grow, was studied by *ad hoc* methods [9, 16, 17, 22]. An algorithm which allows us to systematically compute generating functions for these sequences (with counter n) for any m was for the first time described in [1] (less known) and somewhat later, independently in [21]. It has been noticed that one should examine this problem on other grid graphs such as grid cylinders (where the left and right, or top and bottom, boundaries of the rectangular grid are wrapped around and connected to each other) or grid-tori (where the left edge of the rectangular grid is connected to the right and the top edge is connected to the bottom one) [7, 11]. With this article we are trying to partially fill this gap.

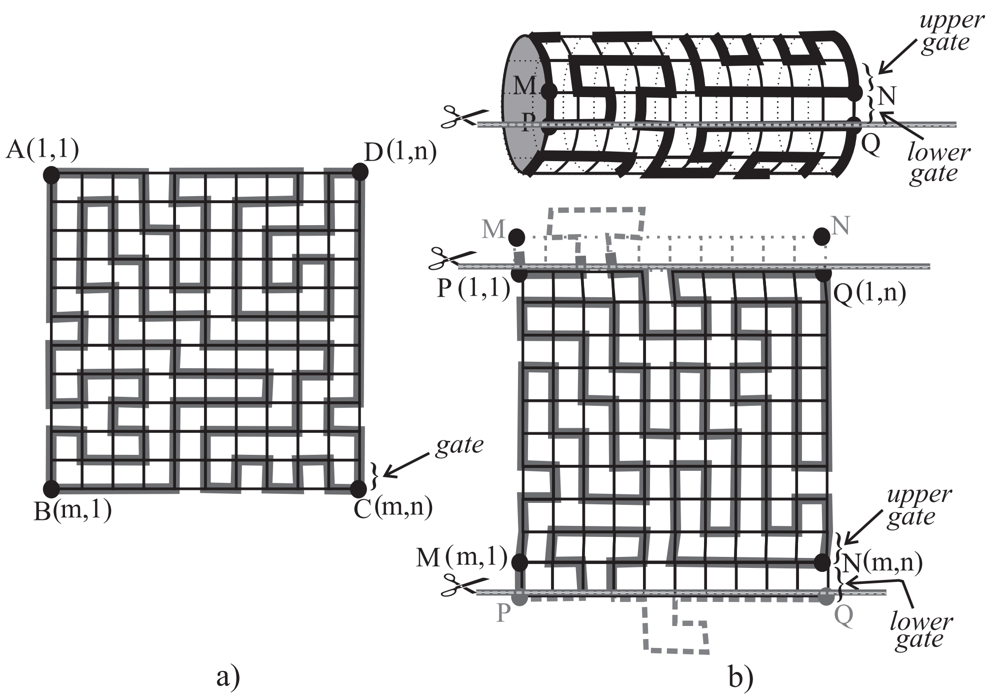

Figure 1: a) Rectangular grid graph with a HC; b) Thin cylinder grid graph with a HC

This article is dedicated to enumeration of HC's in a labeled rectangular grid graph $P_m \times P_n$ and a labeled thin cylinder grid graph $C_m \times P_n$ (Figure 1), where symbol "$\times$" labels cartesian graph product [8], and P_k and C_k denote an open path and a cycle with k vertices, respectively. The main goal of this article is the construction of a new characterization of Hamiltonian cycles in $P_m \times P_n$ which enables us to come, through adapting it, to a characterization of Hamiltonian cycles in $C_m \times P_n$. In contrast to [1], where squares (regions) were encoded, here we approach the coding vertices themselves.

The obtained characterizations of HC's in observed labeled graphs $P_m \times P_n$ and $C_m \times P_n$, where m is fixed, enable us to reduce the problem of enumeration of all the HC's in each of these graphs to the problem of enumeration of all oriented walks of fixed length $(n-1)$ in a special digraph with initial and final vertices in special sets. We call the initial vertices *emphasized* vertices of the digraph.

We denote by $r_m(n)$ $(m \geq 1)$ and $c_m(n)$ $(m \geq 2)$ the numbers of HC's in $P_m \times P_n$ and $C_m \times P_n$, respectively. The algorithms outlined here have been implemented. By programs, written in Pascal language for finding these digraphs and by using well-known technique we have got the generating functions of the sequences $r_m = (r_m(1), r_m(2), r_m(3), \ldots)$ and $c_m = (c_m(1), c_m(2), c_m(3), \ldots)$ explicitly for some values of m again.

2. Characterization of the HC's

It is easy to prove that $r_m(n) > 0$ *iff* *the number of vertices* $(m \cdot n)$ *is even* and $c_m(n) > 0$ *for all* $n \in N$ $(m \geq 2)$. In addition, $r_m(1) = 0$ for $m \geq 1$; $r_1(n) = 0$ and $r_2(n) = 1$ for $n \geq 1$; $c_m(1) = 1$ for $m \geq 2$; $c_2(n) = 4$ for $n \geq 2$. Therefore, below we take the assumption that $m \geq 3$ and $n \geq 2$, and simultaneously consider both graph $P_m \times P_n$ and graph $C_m \times P_n$.

2.1. Code matrix

First, "cut and develop in plane" the cylindrical surface of given graph $C_m \times P_n$ as shown in Figure 1b), i.e. observe the corresponding rectangular grid graph whereby the identification of each vertex in the row below the m-th one (segment MN) with the corresponding vertex in the first row (segment PQ) has been performed. In this way, we can use the words: *left, right, upper* and *lower* to mark the position of an adjacent vertex of a vertex. For both graphs $P_m \times P_n$ and $C_m \times P_n$ we label each vertex by ordered pair (i, j) $(1 \leq i \leq m,\ 1 \leq j \leq n)$ where i represents the ordinal number of the row viewed from up to down, while j represents the ordinal number of the column, viewed from left to right. For the purposes hereinafter, we call the vertical edge $(m-1, n)(m, n)$ in $P_m \times P_n$ the *gate*, and the vertical edges $(m-1, n)(m, n)$ and $(m, n)(1, n)$ in $C_m \times P_n$ the *upper and lower gate*, respectively (Figure 1).

Let us observe an arbitrary HC in one of our graphs (we simultaneously analyze both the graph $P_m \times P_n$ and $C_m \times P_n$). One of six possible situations shown in Figure 2 is assigned to each vertex (i,j) $(1 \leq i \leq m,\ 1 \leq j \leq n)$. We call the label of this situation *alpha-letter* of the vertex and denote it by $\alpha_{i,j}$.

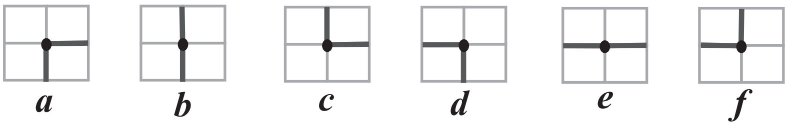

Figure 2: Six possible situations for given HC in any vertex

Note that if we know the alpha-letter of a vertex (i,j), then the alpha-letter of its adjacent vertex can not be any letter from set $\{a,b,c,d,e,f\}$, but is determined with the digraphs $\mathcal{D}_{lr}$ and $\mathcal{D}_{ud}$, as shown in Figure 3, depending on the mutual position of the two adjacent vertices. For example, if $\alpha_{i,j} = a$, then $\alpha_{i,j+1} \in \{d,e,f\}$ and $\alpha_{i+1,j} \in \{b,c,f\}$. In Figure 1 a) $\alpha_{2,2} = a$, $\alpha_{2,3} = d$ and $\alpha_{3,2} = b$.

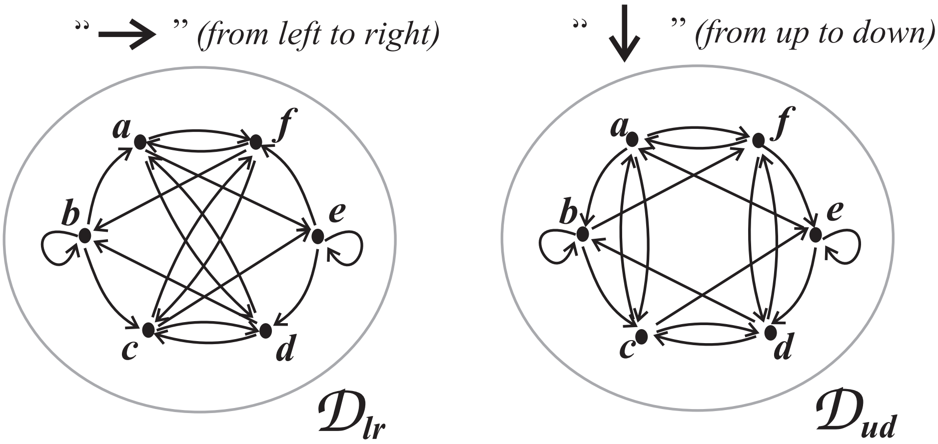

Figure 3: Left-right and up-down adjacency of two vertices

Note that for graph $P_m \times P_n$ alpha-letters for corner vertices (A,B,C and D) have to be $\alpha_{1,1} = a$, $\alpha_{m,1} = c$, $\alpha_{1,n} = d$ and $\alpha_{m,n} = f$, respectively, which is not so for the graph $C_m \times P_n$ since its “corner” vertices P, Q, M and N have valency 3. The word $\alpha_{1,j}\alpha_{2,j}\alpha_{3,j}\ldots\alpha_{m,j}$, where $1 \leq j \leq n$ is called *alpha-word* for j-th column. Note that, in the case of a rectangular grid graph, it can be obtained by concatenating some words of the following types: $ab^tc, ab^tf, db^tc, db^tf$ and e, where $t \geq 0$. We call each such word (subword of the alpha-word) a *fragment* of HC in the j-th column. For the HC depicted in Figure 4, the numbers of fragments in the first, fourth, fifth and last column are 3, 6, 5 and 2, respectively.

In the case of a thin cylinder grid graph alpha-word is treated as a cyclical one, i.e. the ordered pair $(\alpha_{m,j}, \alpha_{1,j})$ must be a directed edge in digraph $\mathcal{D}_{ud}$. In this case, a suffix and a prefix (in this order) of an alpha-word can belong to (and build) the same fragment of HC.

Let us now consider a HC of the graph $P_m \times P_n$ (like we do for the graph $C_m \times P_n$) and for any k, $1 \le k \le n$, its subgraph which is induced by all vertices (i, j), where $1 \le i \le m$, $1 \le j \le k$. This subgraph represents a union of (disjoint) paths. We call these paths (components) *pieces of the HC in the k-th level*, abbreviated PHC^k. For fixed k $(1 \le k \le n)$, every PHC^k is associated with its *ordinal number* in the following sense: the ordinal number of a PHC^k is determined as the ordinal number of the first its vertex viewed from above to down in the sequence consisting of all vertices from k-th column which represent the first appearances of all PHC^k's.

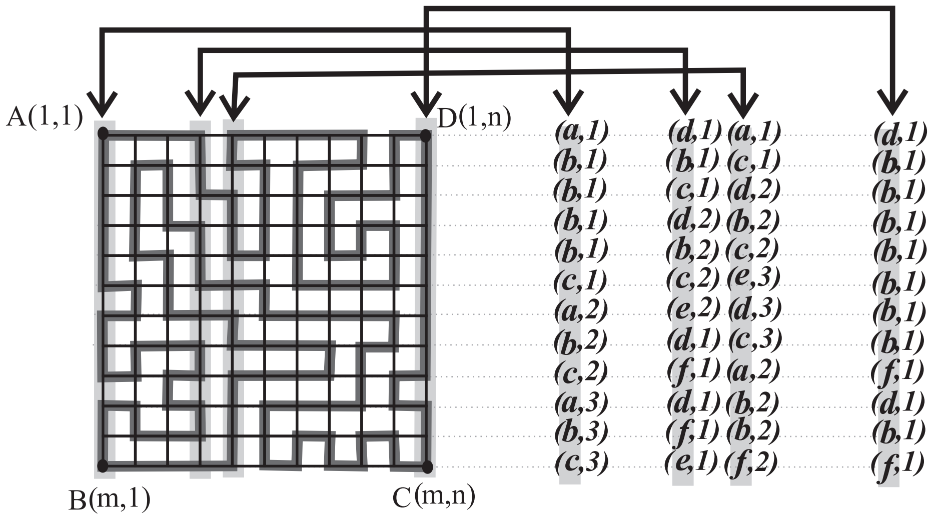

Figure 4: Coding vertices of graph $P_m \times P_n$

For the example shown in Figure 4, we have three PHC^1's, two PHC^4's, three PHC^5's and only one PHC^n. Note that the number of PHC^n is always equal to one because HC is connected graph. The sequence consisting of all vertices in the first column which represent the first appearances of all PHC^1's is $(1, 1)(7, 1)(10, 1)$; for the fourth, fifth and last (n-th) column these sequences are $(1, 4)(4, 4)$; $(1, 5)(3, 5)(6, 5)$ and $(1, 1)$, respectively. Thus, for example, the ordinal numbers of PHC^5's which are represented by vertices $(1, 5), (3, 5), (6, 5)$ are $1, 2, 3$, respectively.

Now, with each Hamiltonian cycle of the observed graph ($P_m \times P_n$ or $C_m \times P_n$) we associate the so-called *code matrix* $[(\alpha_{i,j}, n_{i,j})]_{m \times n}$ defining its elements in the following way:

1. $\alpha_{i,j}$ is the alpha-letter of the vertex (i, j) $(1 \le i \le m,\ 1 \le j \le n)$;

2. $n_{i,j}$ *is an ordinal number of the* PHC^j *which contains the vertex* (i, j).

Thus, all vertices in the same column j $(1 \leq j \leq n)$ belonging to the same PHC^j must have the same second number $n_{i,j}$, called *piece-number* for the vertex (i, j). Note that $n_{1,j} = 1$ for $1 \leq j \leq n$, and $n_{i,n} = 1$ for $1 \leq i \leq m$. The word $n_{1,j}n_{2,j}n_{3,j} \ldots n_{m,j}$, where $1 \leq j \leq n$ is called *piece-word* for j-th column. In the case of a thin cylinder grid graph this word is treated as a cyclical one, just like its α-word. For the example shown in Figure 5 b), the fifth and sixth columns of the code matrix are $[(c, 1), (a, 1), (b, 1), (f, 1), (d, 1), (b, 1)]^T$ and $[(e, 1), (e, 1), (a, 2), (c, 2), (a, 3), (c, 3)]^T$, respectively.

2.2. Opening the gate

All PHC^j are open paths except for $j = n$. But, if we "remove" the gate which belongs to the HC, we can achieve that all PHC^j are open paths. In order to impose the same treatment as for all the previous columns on the n-th column, we use a *modified code matrix* instead of the code matrix. They differ only in the alpha-word for the last column. Instead of "removing" the gate, we "open" the gate as follows:

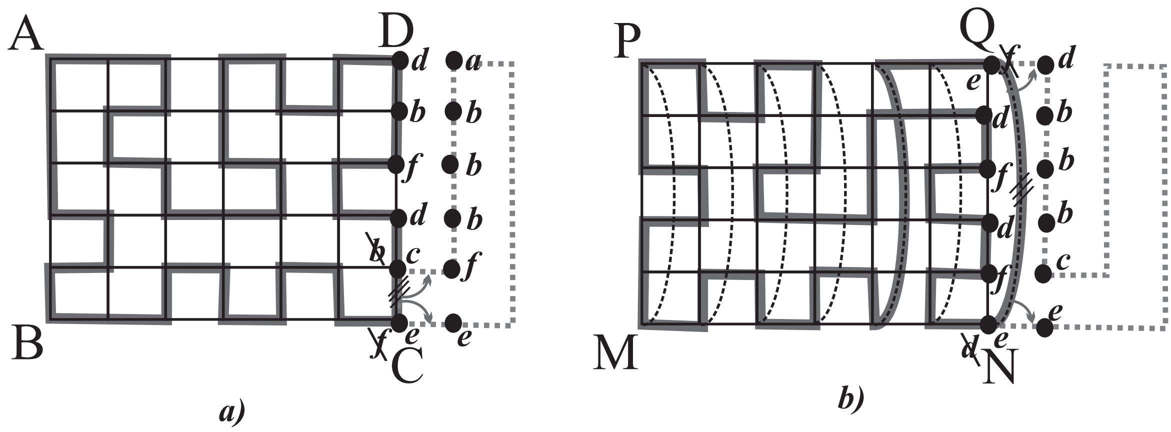

Figure 5: Opening gates

Case of graph $P_m \times P_n$:

- if $\alpha_{m-1,n}\alpha_{m,n} = df$, then we replace it with $\alpha_{m-1,n}\alpha_{m,n} = ee$;

- if $\alpha_{m-1,n}\alpha_{m,n} = bf$ (Figure 5a), then we replace it with $\alpha_{m-1,n}\alpha_{m,n} = ce$;

Case of graph $C_m \times P_n$:

- if $\alpha_{m-1,n}\alpha_{m,n} = df$, then we replace it with $\alpha_{m-1,n}\alpha_{m,n} = ee$;

- if $\alpha_{m-1,n}\alpha_{m,n} = bf$, then we replace it with $\alpha_{m-1,n}\alpha_{m,n} = ce$;
- if $\alpha_{m-1,n}\alpha_{m,n} = db$, then we replace it with $\alpha_{m-1,n}\alpha_{m,n} = ea$;
- if $\alpha_{m-1,n}\alpha_{m,n} = bb$, then we replace it with $\alpha_{m-1,n}\alpha_{m,n} = ca$;
- if $\alpha_{m-1,n}\alpha_{m,n} = fd$ (Figure 5 b), then we "open" the lower gate:
  - if $\alpha_{m,n}\alpha_{1,n} = db$, then we replace it with $\alpha_{m,n}\alpha_{1,n} = ea$;
  - if $\alpha_{m,n}\alpha_{1,n} = df$, then we replace it with $\alpha_{m,n}\alpha_{1,n} = ee$;

Other letters in the alpha-word for the n-th column remain unchanged. In this way we have joined to each HC a uniquely determined matrix $[(\alpha_{i,j}, n_{i,j})]_{m\times n}$ assigned only to the considered HC.

2.3. Properties of the modified code matrix

From the definition of the modified code matrix $[(\alpha_{i,j}, n_{i,j})]_{m\times n}$ for given HC of the graph $P_m \times P_n$ $(C_m \times P_n)$ we can easily obtain the following properties of that matrix (in the case of a thin cylinder grid graph, we adopt $\alpha_{m+1,j} \stackrel{\text{def}}{=} \alpha_{1,j}$ and $n_{m+1,j} \stackrel{\text{def}}{=} n_{1,j}$ for $1 \le j \le n$):

1. <u>*First column conditions:*</u>
   The alpha-word for the first column consists of letters from the set $\{a, b, c\}$. In the case of a rectangular grid graph, $\alpha_{1,1} = a$ and $\alpha_{m,1} = c$. In the case of a thin cylinder grid graph, this alpha-word is different from the word b^m (our assumption $n > 1$).
   The corresponding piece-word is uniquely determined, it is the word $1^{p_1}2^{p_2-p_1}3^{p_3-p_2}\ldots k^{p_k-p_{k-1}}1^{m-p_k}$, where k is the number of the letters c in the alpha-word, and p_1, $p_2, \ldots, p_k$ are positions of the first, second, $\ldots$, k-th letter c in the corresponding alpha-word.

2. <u>*Column conditions:*</u> *For every fixed j $(1 \le j \le n)$,*

   (a) *the ordered pairs $(\alpha_{i,j}, \alpha_{i+1,j})$, where $1 \le i \le m-1$ $(1 \le i \le m$, for thin cylinder), must be arcs in digraph $\mathcal{D}_{ud}$.*

   (b) *If $\alpha_{i,j} \in \{a, b, d\}$, where $1 \le i \le m-1$ $(1 \le i \le m$, for thin cylinder), then $n_{i,j} = n_{i+1,j}$.*

(c) In the case of rectangular grid graph, $\alpha_{1,j} \in \{a, d, e\}$ *and* $\alpha_{m,j} \in \{c, e, f\}$.

(d) Consider the letters in the piece-word for the j*-th column, where* $2 \le j \le n$*, with the first appearances of any number from above (from* $n_{1,j}$ *to* $n_{m,j}$*). Let them be* $n_{p_1,j}, n_{p_2,j}, \ldots, n_{p_l,j}$ $(l \in N)$. *Then,* $n_{p_i,j} = i$.

(e) $n_{i,j} \le \lfloor m/2 \rfloor$ *for* $1 \le i \le m$.

3. *Adjacency of column conditions:* *For every fixed* j*, where* $2 \le j \le n$,

(a) the ordered pairs $(\alpha_{i,j-1}, \alpha_{i,j})$*, where* $1 \le i \le m$*, must be arcs in digraph* $\mathcal{D}_{lr}$.

(b) If $n_{i,j-1} = n_{l,j-1}$ *and* $\alpha_{i,j-1}, \alpha_{l,j-1} \in \{a, c, e\}$*, where* $1 \le i < l \le m$*, then* $n_{i,j} = n_{l,j} \ \wedge \ \alpha_{i,j}\alpha_{i+1,j}\ldots\alpha_{l,j} \ne db^{l-i-1}f \ \wedge \ (\alpha_{1,j}\alpha_{2,j}\ldots\alpha_{i,j} \ne b^{i-1}f \vee \alpha_{l,j}\alpha_{l+1,j}\ldots\alpha_{m,j} \ne db^{m-l})$.

(c) If $1 \le i < l \le m$*,* $n_{i,j} = n_{l,j}$ *and vertices* (i, j) *and* (l, j) *belong to different fragments* v *and* u*, respectively, then there is exactly one sequence* $v = v_1, v_2, \ldots, v_p = u$ *of* p $(p > 1)$ *different fragments in the* j*-th column which satisfies: for every* t*, where* $1 \le t \le p-1$*, in the* $(j-1)$*-th column there exists exactly one vertex* $(x_t, j-1)$ *with* $\alpha_{x_t,j} \in \{d, e, f\}$ *for which* $(x_t, j) \in v_t$ *and there exists exactly one vertex* $(y_{t+1}, j-1)$ *with* $\alpha_{y_{t+1},j} \in \{d, e, f\}$ *for which* $(y_{t+1}, j) \in v_{t+1}$ *and* $n_{x_t,j-1} = n_{y_{t+1},j-1}$.

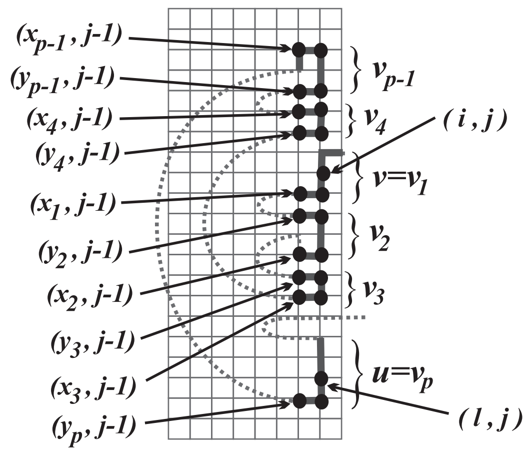

Figure 6: Property (3c) of modified code matrix

4. *Last column conditions:*

In the case of rectangular grid graph, the alpha-word for the last column can be any *alpha word preceding the word* $ab^{m-3}fe$ $(m \geq 3)$ *(in a sense that it obeys the outlined conditions (1)-(3) above (Figure 5a). In the case of thin cylinder grid graph, the alpha-word for the last column can be any alpha word preceding the words* $ab^{m-3}fe$ $(m \geq 3)$ *or any alpha word preceding the words* $db^{m-3}ce$ $(m \geq 3)$ *which satisfies* $\alpha_{m-1,n}\alpha_{m,n} = fe$ *(Figure 5b).*

In both cases, $n_{1,n}n_{2,n}n_{3,n}\dots n_{m,n} = 1^m$.

Proof. The properties above can be easily proved. Here are a few remarks.

(1) This is a description of the first column of the modified code matrix. For the graph $P_m \times P_n$, it can be obtained by concatenation of some words of the type: ab^tc, where $t \geq 0$. We have similar situation for the graph $C_m \times P_n$.

(2 a) Digraph $\mathcal{D}_{ud}$ partially determines alpha-words.

(b) All vertices of the same fragment belonging to the same PHC^j.

(c) Vertices $(1, j)$ and (m, j) in the graph $P_m \times P_n$ are of valency 2 or 3.

(d) It follows from the definition of $n_{i,j}$.

(e) Every PHC^j is associated with at least two different fragments for j-th column.

(3 a) Digraph $\mathcal{D}_{lr}$ partially determines which alpha-words can occur in two adjacent columns in the modified code matrix.

(b) In contrary, we get a cycle as a piece of HC. The expression in brackets refers to the graph $C_m \times P_n$ and is always correct for the graph $P_m \times P_n$ considering the condition (2c).

(c) There is unique path between two vertices from the same PHC^j which is an open path (Figure 6).

(4) Note that $\alpha_{i,n}$ must be in set $\{b, d, f\}$ for $2 \leq i \leq m-3$. In the case of the graph $P_m \times P_n$, $\alpha_{1,n} = d$ and $\alpha_{m,n} = e$, $\alpha_{m-1,n} \in \{c, e\}$. $\square$

Hence, every HC in the considered graph $P_m \times P_n$ $(C_m \times P_n)$ determines exactly one matrix $[(\alpha_{i,j}, n_{i,j})]_{m\times n}$ which fulfills properties described above and vice versa: every matrix $[(\alpha_{i,j}, n_{i,j})]_{m\times n}$ with entries in $\{a, b, c, d, e, f\} \times \{1, 2, \dots, \lfloor m/2 \rfloor\}$ which satisfies conditions (1) - (4) determines exactly one HC in the considered graph $P_m \times P_n$ $(C_m \times P_n)$, which is not so hard to prove. Namely, alpha-letters and the possibility of their contact, expressed by digraphs $\mathcal{D}_{ud}$ and $\mathcal{D}_{lr}$, provide that the subgraph of the graph $P_m \times P_n$ $(C_m \times P_n)$ determined by the matrix (after returning the gate) is spanning 2-regular graph, i.e. union of cycles. Properties related to the piece-letters provide that the number of the cycles is exactly one.

2.4. Technique for enumeration of Hamiltonian cycles

Now, we can create for each number m $(m \geq 3)$ a digraph $\mathcal{D}_m \stackrel{\text{def}}{=} (V(\mathcal{D}_m), E(\mathcal{D}_m))$ ($\mathcal{D}^R_m$ or $\mathcal{D}^C_m$, depending on a choice of grid graph) in the following way: the set of vertices $V(\mathcal{D}_m)$ consists of all possible columns in the modified code matrix $[(\alpha_{i,j}, n_{i,j})]_{m\times n}$ with entries in $\{a, b, c, d, e, f\} \times \{1, 2, \ldots, \lfloor m/2 \rfloor\}$ which satisfy the conditions above; an arrow joins the vertex v with the vertex u $((v, u) \in E(\mathcal{D}_m))$, i.e. $v \to u$ iff the vertex v might be the previous column for the vertex u, i.e. these columns satisfy conditions (3a)-(3c). We call the subset of $V(\mathcal{D}_m)$ which consists of all possible first columns in the matrix (condition (1)) the set of the *emphasized* vertices, and denote it by $\mathcal{F}_m$ ($\mathcal{F}^R_m$ or $\mathcal{F}^C_m$). The subset of $V(\mathcal{D}_m)$ which consist of all possible last columns in the matrix (condition (4)) is denoted by $\mathcal{L}_m$ ($\mathcal{L}^R_m$ or $\mathcal{L}^C_m$).

So, in this way, our problem of enumeration of all Hamiltonian cycles in $P_m \times P_n$ or $C_m \times P_n$ is reduced to enumeration of all oriented walks of the length $n - 1$ in the digraph $\mathcal{D}_m$ with the emphasized initial vertices and the final vertices in set $\mathcal{L}_m$. Therefore, applying a transfer matrix approach [20], we obtain that the both sequences r_m and c_m satisfy difference equations (linear, homogeneous, and with constant coefficients).

We can simplify digraphs $\mathcal{D}^R_m$ and $\mathcal{D}^C_m$, i.e. reduce their adjacency (transfer) matrices in the following way: consider the set $\mathcal{S}$ of all vertices from $\mathcal{D}_m$ ($\mathcal{D}_m$ is $\mathcal{D}^R_m$ or $\mathcal{D}^C_m$) having a given arrangement of symbols from $\{a, c, e\}$ (symbols with "output" to the right) in alpha-word with the same corresponding piece-numbers in positions of these symbols. For example, for $m = 9$, the vertex with alpha word *acdbcabce* and piece-word 112223332 and the vertex with alpha word *eabfeedce* and piece-word 111123332 are in the same set $\mathcal{S}$, because theirs "outputs" are in positions: $1, 2, 5, 6, 8$ and 9 in both alpha words and corresponding subsequence of their piece-words is the same: 112332. Actually, each vertex $(\alpha_1\alpha_2 \ldots \alpha_m, n_1n_2 \ldots n_m)$ corresponds to an *outlet word* $o_1o_2 \ldots o_m$ in the following way $o_i \stackrel{\text{def}}{=} \begin{cases} 0, & if\ \ \alpha_i \in \{b, d, f\} \\ n_i, & if\ \ \alpha_i \in \{a, c, e\}\,. \end{cases}$
In above example both vertices has the same outlet word: 110023032. Note that two vertices from $\mathcal{D}_m$ with the same outlet word have the same set of adjacent vertices.

The set $\mathcal{S}$ can be substituted by a unique vertex s (corresponding outlet word) in such a way that all edges starting from (terminating at) $\mathcal{S}$ now start from (terminate at) s.

After such simplifications, for each arrangement of symbols from $\{a, c, e\}$ in alpha-word with the same corresponding piece-numbers in positions of these symbols, i.e. for each outlet word, we get a vertex of the digraph $\mathcal{D}_m^{*R}$ (multidigraph $\mathcal{D}_m^{*C}$). It is possible that two different arcs starting from the same vertex of $\mathcal{D}_m$ terminate at two different vertices with the same outlet word. In such a case, the multiplicity of arches in $\mathcal{D}_m^*$ is calculated. For example, in $\mathcal{D}_6^{*C}$ two arcs starting from the vertex 101202 and terminating at the vertex 010010 appear.

It is not difficult to see that the numbers of walks of a given length with given initial and final vertices in digraph $\mathcal{D}_m$ (or corresponding vertices in $\mathcal{D}_m^*$) are equal in $\mathcal{D}_m$ and $\mathcal{D}_m^*$.

In [13]-[15] and [18] the transfer matrix T is constructed by combining all so-called *connectivity* or *column states* with all *bond distributions* and finding the resulting connectivity states formed by their combinations. These "connectivity states" would be corresponded to our outlet words. Here, the transfer matrix T is the adjacency matrix of multidigraph $\mathcal{D}_m^*$ obtained from digraph $\mathcal{D}_m$ by vertex contraction described above.

Further reduction of transfer-matrices (i.e. vertex-contractions) can be done using property of reflection symmetry for both graphs $P_m \times P_n$ and $C_m \times P_n$ (its representation developed in plane) and the property of rotational symmetry for $C_m \times P_n$. In this way we obtain the multidigraphs $\mathcal{D}_m^{**R}$ and $\mathcal{D}_m^{**C}$.

The computation of generating functions $\mathcal{R}_m(x) \stackrel{\text{def}}{=} \sum_{n\geq 1}^{\infty} r_m(n)x^n$ and $\mathcal{C}_m(x) \stackrel{\text{def}}{=} \sum_{n\geq 1}^{\infty} c_m(n)x^n$ is a matter of routine: Recall that the number $g_{ij}(n)$ of all oriented walks in any multidigraph $\mathcal{D}$ of length n from v_i to v_j $(v_i, v_j \in V(\mathcal{D}))$ is equal to the (i, j)-entry of nth degree of the adjacency matrix T of the multidigraph $\mathcal{D}$. The generating function $\mathcal{G}_{ij}(x) = \sum_{n\geq 0} g_{ij}(n)x^n$ is given by $\mathcal{G}(x) = \dfrac{(-1)^{i+j} det(I - xT : j, i)}{det(I - xT)}$ where $(B : j, i)$ denotes the matrix obtained by removing the jth row and ith column of B ([20], 4.7.2 Theorem).

2.5. Set of emphasized vertices

Theorem 1 . *The number of all vertices of the digraph $\mathcal{D}_m^R$ $(m \geq 2)$ that can occur as the first column in the modified code matrix of a Hamiltonian cycle of a rectangular grid graph $P_m \times P_n$ is equal to the $(m-2)$-nd Fibonacci number F_{m-2} $(F_0 = 1,\ F_1 = 1,\ F_k = F_{k-1} + F_{k-2}$ for $k \geq 2)$.*

Proof. The adjacency matrix of the subdigraph of $\mathcal{D}_{ud}$ induced by vertices from the set $\{a,b,c\}$ is $A \stackrel{\text{def}}{=} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Note that the number of all walks of length k $(k \geq 1)$, denoted by f_k, in this subgraph whose first and last vertices are a and c, respectively, is equal to the $(1,3)$-entry of the k-th degree of the matrix above. Since this matrix is annihilated by its characteristic polynomial $P(x) = x(x^2 - x - 1)$ (Cayley-Hamilton theorem), we obtain recurrence relation $f_{k+2} - f_{k+1} - f_k = 0$ for $k \geq 1$. This recurrence relation and initial conditions $f_1 = 1 = F_0$ (the unique alpha-word ac) and $f_2 = 1 = F_1$ (the unique alpha-word abc) impose that $f_k = F_{k-1}$ for $k \geq 1$. □

Theorem 2 . *The number of all vertices from the digraph $\mathcal{D}_m^C$ $(m \geq 2)$ that can occur as the first column in the modified code matrix of a Hamiltonian cycle of a thin cylinder grid graph $C_m \times P_n$ $(n \geq 2)$ is equal to the number $L_m - 1$, where L_m is the m-th Lucas number $(L_0 = 2,\ L_1 = 1,\ L_k = L_{k-1} + L_{k-2}$ for $k \geq 2)$.*

Proof. Let l_m denote the number of all closed walks of length m $(m \geq 2)$ in the subdigraph of $\mathcal{D}_{ud}$ induced by vertices from the set $\{a,b,c\}$ (every such walk is assigned to an alpha-word for the first column of the modified code matrix of a HC in a graph $C_m \times P_n$ for $n \geq 2$). This number is equal to the sum of $(1,3)$, $(2,1)$, $(2,2)$, $(3,1)$ and $(3,2)$ -entries of the m-th degree of the matrix A (defined in the previous theorem) minus one (the only excess word is b^m, which can not appear for $n > 1$). Thus, the sequence $s_m \stackrel{\text{def}}{=} l_m + 1$, where $m \geq 2$ satisfies the recurrence relation $s_{m+2} - s_{m+1} - s_m = 0$ for $m \geq 2$. Since initial conditions are $s_2 - 1 = l_2 = 2 = L_2 - 1$ (the alpha-words ac and ca) and $s_3 - 1 = l_3 = 3 = L_3 - 1$ (the alpha-words abc, bca and cab), we obtain that $s_m = L_m$, i.e. $l_m = L_m - 1$ for $m \geq 2$. □

3. Computational results

On the base of previous considerations we wrote computer programs, written in Pascal language for computation of the adjacency matrices of these (multi)digraphs. The numbers of vertices for $\mathcal{D}_m^R$, $\mathcal{D}_m^{*R}$, $\mathcal{D}_m^{**R}$, $\mathcal{D}_m^C$, $\mathcal{D}_m^{*C}$ and $\mathcal{D}_m^{**C}$, and orders of recurrence relations for some values of m are given in Tab.1 and Tab.2. Results obtained here confirm the data previously obtained in another way.

3.1. Rectangular grid graph $P_m \times P_n$

m	3	4	5	6	7	8	9	10	11	12	13	14
$\mid \mathcal{F}_m^R \mid = F_{m-2}$	1	2	3	5	8	13	21	34	55	89	144	233
$\mid V(\mathcal{D}_m^R) \mid$	5	16	61	165	761	1923	10044	24448	138848	$\ll$	$\ll$	$\ll$
$\mid V(\mathcal{D}_m^{*R}) \mid$	3	6	19	32	113	182	706	1117	4647	7280	31886	49625
$\mid V(\mathcal{D}_m^{**R}) \mid$	2	4	11	22	61	108	364	606	2354	3773	16019	25187
order	1	4	3	14	18	66	104	-	-	-	-	-

Tab. 1. The numbers of vertices for $\mathcal{D}_m^R$, $\mathcal{D}_m^{*R}$ and $\mathcal{D}_m^{**R}$, and orders od recurrence relations for rectangular grid graphs $P_m \times P_n$

$$\mathcal{R}_3(x) = \frac{x^2}{1-2x^2} = x^2+2x^4+4x^6+8x^8+16x^{10}+32x^{12}+64x^{14}+128x^{16}+256x^{18}+512x^{20}+\dots$$

$$\mathcal{R}_4(x) = \frac{-x^2}{-1+2x+2x^2-2x^3+x^4} = x^2+2x^3+6x^4+14x^5+37x^6+92x^7+236x^8+596x^9+ 1517x^{10}+3846x^{11}+9770x^{12}+24794x^{13}+62953x^{14}+159800x^{15}+405688x^{16}+1029864x^{17}+ 2614457x^{18}+6637066x^{19}+16849006x^{20}+\dots$$

$$\mathcal{R}_5(x) = \frac{x^2+3x^4}{1-11x^2-2x^6} = x^2+14x^4+154x^6+1696x^8+18684x^{10}+205832x^{12}+2267544x^{14}+ 24980352x^{16}+275195536x^{18}+3031685984x^{20}+\dots$$

$\mathcal{R}_6(x) = \dfrac{U_6(x)}{V_6(x)}$, where

$U_6(x) = -x^2(1-x+3x^2-24x^3+24x^4-3x^5+3x^7-15x^8+9x^9+4x^{10}-2x^{11}+x^{12}$ and

$V_6(x) = -1+5x+14x^2-63x^3+12x^4+90x^5-35x^6-66x^7+118x^8-8x^9-82x^{10}+42x^{11}+ 28x^{12}-4x^{13}+2x^{14}$

$\mathcal{R}_6(x) = x^2+4x^3+37x^4+154x^5+1072x^6+5320x^7+32675x^8+175294x^9+1024028x^{10}+ 5668692x^{11}+32463802x^{12}+181971848x^{13}+1033917350x^{14}+5824476298x^{15}+32989068162x^{16}+ 186210666468x^{17}+1053349394128x^{18}+5950467515104x^{19}+33643541208290x^{20}+\dots$

$\mathcal{R}_7(x) = \dfrac{U_7(x)}{V_7(x)}$, where

$U_7(x) = x^2(-1-7x^2+568x^4-6525x^6+33250x^8-87046x^{10}+111603x^{12}+40229x^{14}-453054x^{16}+ 797154x^{18}-643288x^{20}+252197x^{22}-64012x^{24}+9162x^{26}+4592x^{28}+48x^{30}-96x^{32})$ and

$V_7(x) = -1+85x^2-1932x^4+20403x^6-116734x^8+386724x^{10}-815141x^{12}+1251439x^{14}- 1690670x^{16}+2681994x^{18}-4008954x^{20}+3390877x^{22}-1036420x^{24}-178842x^{26}+92790x^{28}+ 17732x^{30}-5972x^{32}+1728x^{34}+144x^{36}$

$\mathcal{R}_7(x) = x^2+92x^4+5320x^6+301384x^8+17066492x^{10}+966656134x^{12}+54756073582x^{14}+ 3101696069920x^{16}+175698206778318x^{18}+9952578156814524x^{20}+\dots$

$\mathcal{R}_8(x) = \dfrac{U_8(x)}{V_8(x)}$, where

$U_8(x) = x^2\ (-1+8x-49x^2+728x^3-2309x^4-22582x^5+136279x^6+66818x^7- 1949741x^8+3034428x^9+9047953x^{10}-30747404x^{11}+14203168x^{12}+53685844x^{13}- 159419927x^{14}+389023844x^{15}-182495382x^{16}-1834446618x^{17}+3377655817x^{18}+2848593746x^{19}+ 31571028176x^{25}+36720403183x^{26}-44848335674x^{27}-22305149141x^{28}+40947017226x^{29}-$

$1135195559x^{30} - 15888016092x^{31} + 12775242404x^{32} - 4486856598x^{33} - 9844952781x^{34} + 5171581220x^{35} + 5549491219x^{36} - 1095705448x^{37} - 3086017297x^{38} + 490344896x^{39} + 663060588x^{40} - 261275234x^{41} + 306110646x^{42} + 113438464x^{43} - 276490810x^{44} - 84101040x^{45} + 75291501x^{46} + 110093154x^{47} - 36881268x^{48} - 43622030x^{49} - 42380527x^{50} + 13162292x^{51} + 15938854x^{52} + 4508256x^{53} + 4393029x^{54} + 2200412x^{55} - 944944x^{56} - 1005136x^{57} - 274486x^{58} - 17328x^{59} + 3068x^{60} - 5554x^{61} - 1783x^{62} + 74x^{63} + 2x^{64})$ and

$V_8(x) = -1 + 16x + 59x^2 - 1824x^3 + 3898x^4 + 55218x^5 - 243282x^6 - 545916x^7 + 4861689x^8 - 2576498x^9 - 43488068x^{10} + 94333210x^{11} + 141446298x^{12} - 752431432x^{13} + 377840445x^{14} + 2789611474x^{15} - 4656548198x^{16} - 5258354388x^{17} + 18170944298x^{18} + 3512822542x^{19} - 45026326037x^{20} + 9980240588x^{21} + 84208620015x^{22} - 44876200668x^{23} - 121497215791x^{24} + 102246696772x^{25} + 117755621290x^{26} - 145213823124x^{27} - 60571088405x^{28} + 136877858022x^{29} + 3649170978x^{30} - 100110796416x^{31} + 42689760462x^{32} + 39482359310x^{33} - 72614614806x^{34} + 27495494908x^{35} + 40732692257x^{36} - 38863698070x^{37} + 9092063794x^{38} + 5076214026x^{39} - 9600155591x^{40} + 4294619636x^{41} - 1463899423x^{42} + 4331661320x^{43} - 2669382577x^{44} - 998576578x^{45} + 1722204514x^{46} - 1646502104x^{47} + 1188567443x^{48} - 143652474x^{49} - 380794039x^{50} - 27735814x^{51} + 132682964x^{52} + 79877148x^{53} + 41238077x^{54} - 16408310x^{55} - 42867025x^{56} - 18129698x^{57} + 4261277x^{58} + 4951334x^{59} + 985598x^{60} - 103168x^{61} - 13629x^{62} + 34282x^{63} + 6952x^{64} - 532x^{65} + 36x^{66}$

$\mathcal{R}_8(x) = x^2 + 8x^3 + 236x^4 + 1696x^5 + 32675x^6 + 301384x^7 + 4638576x^8 + 49483138x^9 + 681728204x^{10} + 7837276902x^{11} + 102283239429x^{12} + 1220732524976x^{13} + 15513067188008x^{14} + 188620289493918x^{15} + 2365714170297014x^{16} + 29030309635705054x^{17} + 361749878496079778x^{18} + 4459396682866920534x^{19} + 55391169255983979555x^{20} + \ldots$

$\mathcal{R}_9(x) = \dfrac{U_9(x)}{V_9(x)}$, where

$U_9(x) = x^2(-1 + 76x^2 + 46277x^4 - 11548367x^6 + 1325421944x^8 - 95715087677x^{10} + 4859171011903x^{12} - 183816217573920x^{14} + 5368018490246675x^{16} - 123755973124732086x^{18} + 2281522925708311107x^{20} - 33738923148287148608x^{22} + 395151015727738248919x^{24} - 3475736646566489911459x^{26} + 18147939897685309761008x^{28} + 60158226018645837707742x^{30} - 2986187787003911697781056x^{32} + 45999255179007152850533077x^{34} - 509040001441828544210867475x^{36} + 4581032248571763054306877265x^{38} - 35036034446802394740370464984x^{40} + 232555074120593721776644424627x^{42} - 1354690295434896902195285947371x^{44} + 6964468841076052151291920560645x^{46} - 31670314801614166030144261653349x^{48} + 127504014097238456857774048278234x^{50} - 455152234130485129170722531607931x^{52} + 1446174664330915685438473436719834x^{54} - 4119751217588378651650173404901121x^{56} + 10632170257631912956920469041089184x^{58} - 25131801031918228818438516207785957x^{60} + 54771603279521323947168784370056762x^{62} - 109815087647735609793577708219278154x^{64} + 200281958048192565870604542031881878x^{66} - 327306840608740592644893152674643301x^{68} + 474809914758340724032041981176303383x^{70} - 618096615550352667041516688099291682x^{72} + 762228433958987493905985694072126202x^{74} - 988687686930174677694990723899479879x^{76} + 1445930192226446138564317612381426778x^{78} - 2224381008961352413879498456974998798x^{80} + 3148548605883239476985582221722793518x^{82} - 3669050541021588697672094688801451735x^{84} + 3096365122040079373073434370012532718x^{86} - 1168472990653216827484575434764169041x^{88} - 1522006680906067597531720378673231092x^{90} + 3770957630177312257064278594496200583x^{92} - 4539996176798649845695700938005468737x^{94} + 3624246462552169661320262194651061187x^{96} - 1961868469465782537010622230443461642x^{98} + 1634009967979219333656821032693890278x^{100} - 5008174258554575238212385596078878018x^{102} + 12625219618435609464224474492594254070x^{104} - 21382940236847329558017150738411833168x^{106} + 25683807947399702019984595398064343876x^{108} - 21800210470318871372468512723037184929x^{110} + 11706744276040580240600634214808599766x^{112} - 1949208282214342780375050075955961458x^{114} -$

$$
20571326482235941545142938978043484 59x^{116}+400431877139257073390172997816775377x^{118}+
$$

$$
279452193857989670362721870587765 1516x^{120}-409052180231879332666961700612870 3096x^{122}+
$$

$$
31291294063006495333088533118420906 12x^{124}-13756786731575933350029827517438 14396x^{126}+
$$

$$
7311195457547929606557569811517625 7x^{128}+4645385699539185703926456723042320 47x^{130}-
$$

$$
48963070117327629894945740746587723 5x^{132}+3316713561190178166759318645823914 21x^{134}-
$$

$$
17805350040290456524162445280994534 0x^{136}+8005552291049587001851336892119196 3x^{138}-
$$

$$
2926854364384207195974587763118218 8x^{140}+7281977194735054968661862622763643x^{142}+
$$

$$
1562749937489249908678343476601 4x^{144}-1315951676535726625186717873909901x^{146}+
$$

$$
941847278207730443397694992226521x^{148}-45445797221737126278543711500146 4x^{150}+
$$

$$
173989533668758681932554152791460x^{152}-55067916604094729458688541316117x^{154}+
$$

$$
14221207079451638523783325018195x^{156}-2664603148205212070863265871155x^{158}+
$$

$$
1422927167160149957767513806 20x^{160}+16114544627803464336581734347 8x^{162}-
$$

$$
9882297864272153232727781610 0x^{164}+388545036744975678198984233 40x^{166}-
$$

$$
12286068548487742072524098046x^{168}+330999301461670771224879877 5x^{170}-
$$

$$
769883893496642235472079656x^{172}+152997215197939598556986147x^{174}-
$$

$$
25467277614674240074182557x^{176}+3466457754647493514043729x^{178}-
$$

$$
370835556753680898551237x^{180}+28492099430428788345252x^{182}-
$$

$$
11222318044348211681 08x^{184}-57974993679037202755x^{186}+1447644974787326706 5x^{188}-
$$

$$
1411891237821847568x^{190}+92983644642547230x^{192}-4333116252101212x^{194}+
$$

$$
127878213541748x^{196}-926491809488x^{198}-150497066680x^{200}+7798177376x^{202}-78759936x^{204}-
$$

$$
368640x^{206})
$$

and

$$
V_9(x)=-1+672x^2-178941x^4+26786039x^6-2607448600x^8+
$$

$$
179022506347x^{10}-9138846694357x^{12}+360041299997972x^{14}-
$$

$$
11254854430370909x^{16}+285239012592685968x^{18}-5964627217090541641x^{20}+
$$

$$
104500678360781697484x^{22}-1556583951761808187351x^{24}+20014735589628148063803x^{26}-
$$

$$
225840870982639685350870x^{28}+2275592733721786744418588x^{30}-
$$

$$
20826364708844211419088048x^{32}+175698356667789807902833571x^{34}-
$$

$$
1381174156518847754742200917x^{36}+10170019003804901336735147471x^{38}-
$$

$$
70003420053325632588023367766x^{40}+446182037050452191079109199615x^{42}-
$$

$$
2595362044476627757245437008109x^{44}+13570008625005415621556838250183x^{46}-
$$

$$
63003395189524492106909601816507x^{48}+257826103840415278692445505871098x^{50}-
$$

$$
927795089970952084248323277475301x^{52}+2943063243792739889950387942270474x^{54}-
$$

$$
8284388338421319713668314321950849x^{56}+20893786955948014423103382099606436x^{58}-
$$

$$
47682931456935989016644226476248441x^{60}+99034722216970869411718009120972998x^{62}-
$$

$$
186613940860788357047700590145469850x^{64}+314393511785306230125922905225687470x^{66}-
$$

$$
461228773076139092991049045910233189x^{68}+568163799314454613889626216489802291x^{70}-
$$

$$
569970237446092330623145821872270554x^{72}+516255441745874003918772527423187876x^{74}-
$$

$$
750331973988610457686979424425455695x^{76}+1948116315614897591684683097566788710x^{78}-
$$

$$
4767578165656000132898694536173303552x^{80}+9223068331940449503246199380170797588x^{82}-
$$

$$
14439385882606881084375341082872500069x^{84}+19203524833778237619399199496120112344x^{86}-
$$

$$
22654155027324560919450394582691204737x^{88}+24342554197365645052552314094292020138x^{90}-
$$

$$
24340773477750862776080869834954798051x^{92}+24250658103545708573796143054316829733x^{94}-
$$

$$
27745190966510447840996071368294727573x^{96}+38425792204525402615949097274689190884x^{98}-
$$

$$
55422759326895948871535222743427159802x^{100}+70729055476730900234366793432472266368x^{102}-
$$

$$
73819925880373004637572018001559769310x^{104}+63388514129546493372164181497486524518x^{106}-
$$

$$
52759270432980368768927960250795764010x^{108}+55764118845777226484391752561108715665x^{110}-
$$

$$
66464113509700746109349441075277770500x^{112}+62296605320562742399955687633954554900x^{114}-
$$

$$
31148391366039709828008192258625920077x^{116}-12485250186916140101609953912898081887x^{118}+
$$

$$
42654862914755984553959255801657245314x^{120}-47023712901001741125118508732822852170x^{122}+
$$

$$
33080927717174510775217853281082076598x^{124}-15494466120988713368893421376058986544x^{126}+
$$

$$3429254057650617087578787175065609089x^{128}+1834366466922000360932519537787508153x^{130}-2847750979275136270288226785862119971x^{132}+2216810876719448894152498968621570249x^{134}-1347444141266719076559545050826163790x^{136}+701841127814802063228662479499782493x^{138}-318066936221517953502258428878290012x^{140}+121105551713136925328282829822866983x^{142}-34745081077056040606914781189637450x^{144}+4499432686690403495320601923345141x^{146}+2575385020956666440077901987225623x^{148}-2619480426445702741842509277432650x^{150}+1531700770701230953980399995413110x^{152}-725941992725792269897852489297623x^{154}+293308884467487194944446092523363x^{156}-99272941541573765316896500953947x^{158}+26610547639802501699214550716520x^{160}-4823713154410742640789125247946x^{162}+749307900979298593081424016 62x^{164}+39552920254619157085413885137 6x^{166}-2140117095133203932001458962 20x^{168}+78239618982805866166560174399x^{170}-22992955661092007469888280252x^{172}+5643220564094431894769771279x^{174}-1159808414772210919562895201x^{176}+197576217930011633432855397x^{178}-27350727342373286714221107x^{180}+2950281377202644726344372x^{182}-220666390717767574487088x^{184}+5787537137476979667629x^{186}+1229475105352798691453x^{188}-232763105542097450138x^{190}+23427163147889339094x^{192}-1633355302567880268x^{194}+82645890727987184x^{196}-2982658741842664x^{198}+72036310273096x^{200}-1019997566464x^{202}+5772791568x^{204}-24126720x^{206}+628224x^{208}$$

$$\mathcal{R}_9(x) = x^2+596x^4+175294x^6+49483138x^8+13916993782x^{10}+3913787773536x^{12}+1100831164969864x^{14}+309656520296472068x^{16}+87106950271042689032x^{18}+24503579727182933530758x^{20}+\ldots$$

$$\mathcal{R}_{10}(x) = x^2+16x^3+1517x^4+18684x^5+1024028x^6+17066492x^7+681728204x^8+13916993782x^9+467260456608x^{10}+10754797724124x^{11}+328076475659033x^{12}+8091313110371792x^{13}+233977398720987284x^{14}+6002042996016384360x^{15}+168435972906750526954x^{16}+4418118886987754341770x^{17}+121913396076344218930045x^{18}+3238352620436399748512108x^{19}+88514516642574170326003422x^{20}+2367968394724809617269954460x^{21}+64378679819118973995552999424x^{22}+1729205339062710545306506216594x^{23}+46870790094757966307797465706733x^{24}+1261799178695476718104974498991926x^{25}+34143413041293215097879140823296997x^{26}+920342971626015231111461725973168608x^{27}+24879955233880910452208887587176427276x^{28}+671127944788297237626112190665810784834x^{29}+18133019824359066053505296359700754385407x^{30}+\ldots$$

$$\mathcal{R}_{11}(x) = x^2+3846x^4+5668692x^6+7837276902x^8+10754797724124x^{10}+14746957510647992x^{12}+20223692320200140940x^{14}+27738606105535271640888x^{16}+38049128385426605236700966x^{18}+52194036750499722755908743018x^{20}+71598455565101470929617326988084x^{22}+98217523834843365306426848969040826x^{24}+134733398926676359394934062807293332148x^{26}+184825504599290362353590818010156561025202x^{28}+253541269372796154224081182226905935514531848x^{30}+\ldots$$

$$\mathcal{R}_{12}(x) = 1x^2+32x^3+9770x^4+205832x^5+32463802x^6+966656134x^7+102283239429x^8+3913787773536x^9+328076475659033x^{10}+14746957510647992x^{11}+1076226888605605706x^{12}+53540340738182687296x^{13}+3593066312119675283778x^{14}+190433654636776931262392x^{15}+12142048779807437697982030x^{16}+669350612142203868221516908x^{17}+41358164110391551396686680798x^{18}+2336166137451307902906141687868x^{19}+141586604507971463549601451270512x^{20}+8119084139970932725890345059891572x^{21}+486249226070068320381106413320344172x^{22}+28143966058629148023757475362429759374x^{23}+1673219200189416324422979402201514800461x^{24}+$$

$974034759833613292381283679934228511235540x^{25}+$
$576472410785213181323020535964728543174904x^{26}+$
$3367753073313424337447351796730102704613120 86x^{27}+$
$19876199193805759350802031055764292024927441076x^{28}+$
$116371184821587887687712098956612098925793678002 2x^{29}+$
$6856327963857041335656562473644407142349738446 6264x^{30}+\dots$

$\mathcal{R}_{13}(x) = 1x^2 + 24794x^4 + 181971848x^6 + 1220732524976x^8 + 8091313110371792x^{10} + 53540340738182687296x^{12} + 354282765498796010420944x^{14} + 2344813362310160031818110686x^{16} + 15521405796403235465091907705964x^{18} + 102751870895009099449398653093247696x^{20} + 680246633198251681729146832058319250524x^{22} + 4503514353827253863124963840235250766532140x^{24} + 29815394539834813572600735261571894552950941626x^{26} + 197392786967827213807155950129123558668321939744712x^{28} + 1306840963238784384619349355608060741019262932272741748x^{30} + \dots$

$\mathcal{R}_{14}(x) = 1x^2 + 64x^3 + 62953x^4 + 2267544x^5 + 1033917350x^6 + 54756073582x^7 + 15513067188008x^8 + 1100831164969864x^9 + 233977398720987284x^{10} + 20223692320200140940x^{11} + 3593066312119675283778x^{12} + 354282765498796010420944x^{13} + 56126499620491437281263608x^{14} + 6040964455632840415885507728x^{15} + 888511465584607682074513271223x^{16} + 101354577357676648372267541703156x^{17} + 14201999537428457960516291307794439x^{18} + 1683580734401860187030975255800837396x^{19} + 228534623217424294538253859744615069605x^{20} + 27788916535169530514682811124988076298342x^{21} + 3694358804813564823037251519890067057666526x^{22} + 456814938062062563268383716325925309787045462x^{23} + 59904497908037189781466337690475379740764762385x^{24} + 7489655073805004409777835709085005402872971727898x^{25} + 973349118813297870712015952553880377096571439409309x^{26} + 122584382136686634601254210293745864201939036422472456x^{27} + 15836807024750749574106724392556189684881848226515147589x^{28} + 2004096986760406486037181812041725816893182639157547319900x^{29} + 257903251378950763272345729816758156570336914027734193809464x^{30} + \dots$

3.2. Thin cylinder grid graph $C_m \times P_n$

m	3	4	5	6	7	8	9	10	11	12	13
$\mid \mathcal{F}_m^C \mid = L_m - 1$	3	6	10	17	28	46	75	122	198	321	520
$\mid V(\mathcal{D}_m^C) \mid$	9	28	110	292	1393	3368	18933	42449	$\ll$	$\ll$	$\ll$
$\mid V(\mathcal{D}_m^{*C}) \mid$	3	6	20	32	126	182	834	1117	5797	7280	41834
$\mid V(\mathcal{D}_m^{**C}) \mid$	1	2	4	8	15	22	64	90	311	383	1742
order	1	2	3	7	12	20	51	74	246	303	-

Tab. 2. The numbers of vertices for $\mathcal{D}_m^C$, $\mathcal{D}_m^{*C}$ and $\mathcal{D}_m^{**C}$, and orders od recurrence relations for thin cylinder grid graphs $C_m \times P_n$

$\mathcal{C}_3(x) = \frac{x(1+x)}{1-2x} = x + 3x^2 + 6x^3 + 12x^4 + 24x^5 + 48x^6 + 96x^7 + 192x^8 + 384x^9 + 768x^{10} + 1536x^{11} + 3072x^{12} + 6144x^{13} + 12288x^{14} + 24576x^{15} + 49152x^{16} + 98304x^{17} + 196608x^{18} + 393216x^{19} +$

$786432x^{20} + \dots$

$\mathcal{C}_4(x) = \dfrac{-x(-1-2x+x^2)}{x^2-4x+1} = x + 6x^2 + 22x^3 + 82x^4 + 306x^5 + 1142x^6 + 4262x^7 + 15906x^8 + 59362x^9 + 221542x^{10} + 826806x^{11} + 3085682x^{12} + 11515922x^{13} + 42978006x^{14} + 160396102x^{15} + 598606402x^{16} + 2234029506x^{17} + 8337511622x^{18} + 31116016982x^{19} + 116126556306x^{20} + \dots$

$\mathcal{C}_5(x) = \dfrac{-x(-1+x-4x^2+2x^3)}{2x^3-4x^2+6x-1} = x + 5x^2 + 30x^3 + 160x^4 + 850x^5 + 4520x^6 + 24040x^7 + 127860x^8 + 680040x^9 + 3616880x^{10} + 19236840x^{11} + 102313600x^{12} + 544168000x^{13} + 2894227280x^{14} + 15393318880x^{15} + 81871340160x^{16} + 435443220000x^{17} + 2315960597120x^{18} + 12317733383040x^{19} + 65513444349760x^{20} + \dots$

$\mathcal{C}_6(x) = \dfrac{x(1+x)(1-2x+16x^2-24x^3-2x^4-14x^5+6x^6)}{1-9x-10x^3+28x^4+36x^5+32x^6+12x^7} = x + 8x^2 + 86x^3 + 776x^4 + 7010x^5 + 63674x^6 + 578090x^7 + 5247824x^8 + 47640092x^9 + 432480632x^{10} + 3926091512x^{11} + 35641352528x^{12} + 323554871864x^{13} + 2937255393440x^{14} + 26664624744320x^{15} + 242063463190976x^{16} + 2197470272854016x^{17} + 19948799940346880x^{18} + 181096701955896896x^{19} + 1644009442040416928x^{20} + \dots$

$\mathcal{C}_7(x) = \dfrac{U_7(x)}{V_7(x)}$ where

$U_7(x) = x(1 - 5x + 24x^2 - 42x^3 - 400x^4 - 474x^5 + 700x^6 + 1996x^7 + 588x^8 - 192x^9 + 8x^{10} - 16x^{11} + 16x^{12})$ and

$V_7(x) = 1 - 12x - 18x^2 + 112x^3 + 440x^4 + 772x^5 + 196x^6 - 2064x^7 - 3724x^8 - 2040x^9 - 496x^{10} - 128x^{11} + 16x^{12}$

$\mathcal{C}_7(x) = x + 7x^2 + 126x^3 + 1484x^4 + 18452x^5 + 229698x^6 + 2861964x^7 + 35663964x^8 + 444486280x^9 + 5539931796x^{10} + 69048910000x^{11} + 860620499760x^{12} + 10726732430288x^{13} + 133697577587000x^{14} + 1666401898058352x^{15} + 20769976722986288x^{16} + 258876295158900832x^{17} + 3226625529605854320x^{18} + 40216553455854426560x^{19} + 501257787787122948736x^{20} + \dots$

$\mathcal{C}_8(x) = \dfrac{U_8(x)}{V_8(x)}$ where

$U_8(x) = x - 13x^2 + 122x^3 - 515x^4 + 284x^5 + 164x^6 + 23x^7 + 4151x^8 + 4808x^9 + 4528x^{10} - 371x^{11} - 3646x^{12} - 5785x^{13} + 1951x^{14} - 340x^{15} - 3396x^{16} + 1918x^{17} + 232x^{18} - 180x^{19} + 136x^{20} + 40x^{21}$ and

$V_8(x) = 1 - 23x + 34x^2 + 345x^3 + 218x^4 - 22x^5 - 2919x^6 - 5041x^7 - 8806x^8 - 11998x^9 - 5873x^{10} + 1318x^{11} + 4467x^{12} + 11373x^{13} + 3848x^{14} - 584x^{15} + 1018x^{16} - 928x^{17} + 84x^{18} + 72x^{19} - 40x^{20}$

$\mathcal{C}_8(x) = x + 10x^2 + 318x^3 + 6114x^4 + 126426x^5 + 2588218x^6 + 53055038x^7 + 1087362018x^8 + 22286085818x^9 + 456763781330x^{10} + 9361593883038x^{11} + 191870363459178x^{12} + 3932475321605194x^{13} + 80597971743535618x^{14} + 1651894168575456078x^{15} + 33856364932336405826x^{16} + 693902471632291156946x^{17} + 14221864665640856614738x^{18} + 291483951760814319838934x^{19} + 5974103686936428822276538x^{20} + \dots$

$\mathcal{C}_9(x) = \dfrac{U_9(x)}{V_9(x)}$ where

$U_9(x) = x - 14x^2 + 23x^3 + 336x^4 - 16957x^5 - 89969x^6 + 403751x^7 + 4343059x^8 + 9811182x^9 - 22061744x^{10} - 136134923x^{11} - 101701342x^{12} + 484070914x^{13} + 739972904x^{14} - 1357384073x^{15} - 4045130085x^{16} + 15158032x^{17} + 11991138067x^{18} + 14172465702x^{19} -$

$12040169743x^{20} - 45693932330x^{21} - 41411061809x^{22} + 5291646774x^{23} + 58445286384x^{24} + 83288226276x^{25} + 46719537632x^{26} - 34353758002x^{27} - 81720695042x^{28} - 43813570064x^{29} + 38771285108x^{30} + 63529244300x^{31} + 2247036936x^{32} - 36434368112x^{33} - 23152276736x^{34} - 44952924832x^{35} + 38885961376x^{36} + 72912142896x^{37} - 42736188704x^{38} - 54626558656x^{39} + 35330998176x^{40} + 11973338368x^{41} - 9519445504x^{42} - 4745700864x^{43} + 3893091200x^{44} - 341817472x^{45} + 260547072x^{46} - 157725184x^{47} - 62935040x^{48} + 9509376x^{49} + 990208x^{50} - 743424x^{51} - 94208x^{52}$ and

$V_9(x) = 1 - 23x - 280x^2 + 2238x^3 + 38447x^4 + 152710x^5 - 334567x^6 - 6586040x^7 - 32475252x^8 - 63704924x^9 + 60031321x^{10} + 576571223x^{11} + 1239197497x^{12} + 1418765189x^{13} + 2026783228x^{14} + 4972136691x^{15} + 5975151103x^{16} - 6586175756x^{17} - 30318783786x^{18} - 30892900555x^{19} + 20062310737x^{20} + 90546530362x^{21} + 117036064104x^{22} + 70575876534x^{23} - 38397703554x^{24} - 152573320432x^{25} - 153311805598x^{26} - 20762733116x^{27} + 107865096688x^{28} + 88313767808x^{29} - 12462888472x^{30} - 33949381128x^{31} - 48309187400x^{32} - 43670159120x^{33} + 33605932304x^{34} + 68887152928x^{35} - 59186108112x^{36} - 74330191136x^{37} + 37908518240x^{38} + 52440361440x^{39} - 27959018048x^{40} - 11060535424x^{41} + 5036989056x^{42} + 2618432768x^{43} - 2732681344x^{44} + 1276157184x^{45} - 334600192x^{46} + 33238528x^{47} + 41793024x^{48} - 7955456x^{49} - 835584x^{50} + 188416x^{51}$

$\mathcal{C}_9(x) = x + 9x^2 + 510x^3 + 12348x^4 + 351258x^5 + 9806292x^6 + 276018090x^7 + 7769376972x^8 + 218915964618x^9 + 6169925169414x^{10} + 173923080282474x^{11} + 4903042542453720x^{12} + 138226113213225360x^{13} + 3896923927019062734x^{14} + 109864493967924549384x^{15} + 3097380080814655131414x^{16} + 87323767337933601800838x^{17} + 2461902328199084994926838x^{18} + 69407973132514050824027916x^{19} + 1956807009306757665486727506x^{20} + \dots$

$$\mathcal{C}_{10}(x) = \frac{U_{10}(x)}{V_{10}(x)}, \text{ where}$$

$U_{10}(x) = x - 39x^2 + 575x^3 - 3101x^4 - 38926x^5 + 593761x^6 + 399003x^7 - 17947512x^8 - 288879x^9 + 323157201x^{10} + 310973518x^{11} - 4083658154x^{12} - 6720263322x^{13} + 10576897592x^{14} - 16126940286x^{15} + 74634096666x^{16} + 570245737540x^{17} + 233741256554x^{18} - 21552795160x^{19} + 865114092100x^{20} - 9291121597729x^{21} - 26389448121020x^{22} - 5683264452455x^{23} + 6583791932659x^{24} - 49898750767273x^{25} + 315979941343118x^{26} + 504920031758716x^{27} - 747757751823236x^{28} + 365223269515742x^{29} + 468698992474310x^{30} - 4215576639581164x^{31} - 1637436089988972x^{32} + 1765376594205117x^{33} + 1233556858393443x^{34} + 12966503202249x^{35} + 8878712439532771x^{36} + 4251646252675155x^{37} - 13529844752214987x^{38} - 3165450940361060x^{39} + 6689496214624 6469x^{40} - 16614241293844 6496x^{41} + 247573811350407087x^{42} - 264672988176095508x^{43} + 259266400965909996x^{44} - 240928741545060400x^{45} + 280989690728315552x^{46} - 281908555670004818x^{47} + 193630123883711450x^{48} - 29759954615540638x^{49} - 90725428711029292x^{50} + 89898939081190546x^{51} - 36754884472967192x^{52} - 9484855356469664x^{53} + 14295310287528848x^{54} - 8207318634694492x^{55} + 2352200005546480x^{56} - 622789969896116x^{57} + 80772542517472x^{58} + 295323803822064x^{59} - 232644038667768x^{60} + 1352849538304x^{61} + 58124702155456x^{62} - 16775487444320x^{63} - 5041315838688x^{64} + 2109831946176x^{65} - 1290057205824x^{66} + 112646928896x^{67} + 147211061568x^{68} - 107184403456x^{69} - 20300055424x^{70} + 2712887040x^{71} + 84950528x^{72} + 176240640x^{73} + 26257408x^{74} - 73728x^{75}$ and

$V_{10}(x) = 1 - 51x + 5x^2 + 11619x^3 + 9006x^4 - 699989x^5 - 3186749x^6 - 943346x^7 + 56265059x^8 + 96329991x^9 - 686508724x^{10} + 420419692x^{11} + 11362521740x^{12} + 11504194424x^{13} + 28390254218x^{14} + 200716249494x^{15} - 70175720860x^{16} - 1106584035436x^{17} - 629797297008x^{18} - 7045339131322x^{19} - 22125767687817x^{20} + 2788794635616x^{21} - 10810143567801x^{22} - 33374737493705x^{23} + 285004282069983x^{24} + 167552784317868x^{25} - 447677428069020x^{26} + 75031504308498x^{27} + 368824321835860x^{28} - 3720274929679452x^{29} + 530811108040698x^{30} +$

$$8337296380288724x^{31} - 8968627049790535x^{32} - 1380663470501809x^{33} + 23668225925233795x^{34} + 13552725950578467x^{35} - 52541117152534163x^{36} + 48939979544979641x^{37} - 57706258043492644x^{38} + 79025183999679033x^{39} - 236099893457915748x^{40} + 429446543965575441x^{41} - 546277493791345056x^{42} + 524592822363305950x^{43} - 486150511597333906x^{44} + 504867358626279588x^{45} - 551751927519078938x^{46} + 483726944250029916x^{47} - 279644671565378364x^{48} + 260835448180868 20x^{49} + 118526405306060888x^{50} - 124365466657513136x^{51} + 60423256264608584x^{52} - 3095157360308972x^{53} - 15425475066306008x^{54} + 14262276038298660x^{55} - 7190933116063176x^{56} + 2206515160624464x^{57} + 66552339356568x^{58} - 585866003830480x^{59} + 363197247113232x^{60} - 56483974881040x^{61} - 35541467146368x^{62} + 8571044515232x^{63} + 593468499872x^{64} - 1545390018752x^{65} + 872903387072x^{66} + 129864382720x^{67} - 139735383424x^{68} + 16263668096x^{69} + 15338542848x^{70} + 110205440x^{71} - 395481088x^{72} - 86820864x^{73} + 73728x^{74}$$

$$\mathcal{C}_{10}(x) = x + 12x^2 + 1182x^3 + 45502x^4 + 2127332x^5 + 95718442x^6 + 4343656672x^7 + 196769260362x^8 + 8917775068522x^9 + 404126474166012x^{10} + 18314237688963002x^{11} + 829962636335203152x^{12} + 37612209746663052792x^{13} + 1704508129504662739932x^{14} + 77244815889633863270612x^{15} + 3500576762912651494559832x^{16} + 158638966340047716575123082x^{17} + 7189192910739445419701828602x^{18} + 325799492558337775893472693122x^{19} + 14764565463315482740994905989762x^{20} + \dots$$

$$\mathcal{C}_{11}(x) = x + 11x^2 + 2046x^3 + 97328x^4 + 6355404x^5 + 387822094x^6 + 24320491316x^7 + 1519170232976x^8 + 95249624584400x^9 + 5973677282007402x^{10} + 374905251599545986x^{11} + 23534073657511178476x^{12} + 1477568095192517655932x^{13} + 92775355905853945839438x^{14} + 5825578147023937709240306x^{15} + 365810849961625116513720948x^{16} + 22971031488025813312501357724x^{17} + 1442473603938447988832876703906x^{18} + 90581015967424468590909550845132x^{19} + 5688104490132302736598641869726970x^{20} + 357189378064985771483960127872667064x^{21} + 22430028852033269340342876703334550776x^{22} + 1408514460343878873636186663058250171468x^{23} + 88448992006860478922476849198446411578408x^{24} + 5554238459362461178048398749863050396158798x^{25} + 348783703963454353503322592717244366836443212x^{26} + 21902206360114861718151027135402949262041067596x^{27} + 1375370031278160094338755405058136066909165196906x^{28} + 86367679790381191741973350325020872235324187994496x^{29} + 5423541295961746810679269624547348011502830631723548x^{30} + \dots$$

$$\mathcal{C}_{12}(x) = x + 14x^2 + 4478x^3 + 330838x^4 + 35085590x^5 + 3411202430x^6 + 340632046678x^7 + 33794298241774x^8 + 3360563350227504x^9 + 334009240038242920x^{10} + 33204360051870939552x^{11} + 3300767481388100805696x^{12} + 328127904170727818697864x^{13} + 32618970712571492117370784x^{14} + 3242635777680184511300857916x^{15} + 322348781990170719747864116760x^{16} + 32044535804530080338868609353380x^{17} + 3185531751226308600151118696849016x^{18} + 316672172685996222940257879842533984x^{19} + 31480227815320192058392635825802576736x^{20} + 3129434259568237682143299999217638922 96x^{21} + {} + 311095550153998314511310112883320031519752x^{22} + 30925858585921651586578080911160258417792756x^{23} + 3074324685501882957408624786639868964253211292x^{24} + 305617134159632679759194969301005821460164489592x^{25} + 30381251900391584641954801726055775209858628558968x^{26} + 3020185597844903427398369950732366430339199664170488x^{27} + 300235193578143900089692439077439155496268541547305216x^{28} + 29846235783442489972702555271808071110568676715259683148x^{29} + 2966999903725422576083624911414787814400116815530601861216x^{30} + \dots$$

4. Asymptotic relations

Let θ_m (φ_m) be the maximum eigenvalue of the adjacency matrix M_m^R (M_m^C). For large n we then have the following asymptotic relations for the numbers of Hamiltonian cycles in $P_m \times P_n$ and $C_m \times P_n$:

$$r_m(n) \sim a_m \theta_m^n \quad (n \to +\infty) \qquad \text{and} \qquad c_m(n) \sim b_m \varphi_m^n \quad (n \to +\infty)$$

where a_m and b_m are positive numbers. Some of the values θ_m and φ_m are given in Tab.3. The entropy per site for the given graphs are

$$\lim_{n\to\infty} \frac{\ln \ r_m(n)}{mn} = \ln \sqrt[m]{\theta_m} \qquad\qquad \lim_{n\to\infty} \frac{\ln \ c_m(n)}{mn} = \ln \sqrt[m]{\varphi_m}.$$

The values $\sqrt[m]{\theta_m}$ and $\sqrt[m]{\varphi_m}$ coincide with the data given in [18].

m	3	4	5	6	7	8	9
θ_m	1.4142135624	2.5386157635	3.3191082404	5.6520586485	7.5263454629	12.3823516416	16.7721681936
$\sqrt[m]{\theta_m}$	1.1224620483	1.2622612298	1.2711718975	1.3346511857	1.3342164993	1.3696209538	1.3679350192
φ_m	2	3.7320508076	5.3186282178	9.0780749969	12.4639668315	20.4954806289	28.1928327985
$\sqrt[m]{\varphi_m}$	1.2599210499	1.3899106635	1.3968826610	1.4443273251	1.4339114292	1.4586707132	1.4491939889

Tab. 3. The maximum eigenvalues θ_m and φ_m

5. Concluding remarks

The algorithm presented here for determination of the number of Hamiltonian cycles in a labeled graph $P_m \times P_n$ has simpler formulation than those presented in earlier articles [1, 21]. We demonstrate that this algorithm can be adapted for the enumeration of Hamiltonian cycles in a thin cylinder grid graph with minor modifications. Among other results, the obtained algorithm for grid cylinder $C_m \times P_n$ in a special case $m = 4$ confirms results obtained in [14] for the cubic lattice $2 \times 2 \times n$. The obtained asymptotic relations confirm data in [18] for both $P_m \times P_n$ and $C_m \times P_n$.

In addition, we offer the simple explanation for the fact that the numbers of Hamiltonian cycles in these grid graphs for any fixed m, as sequences with counter n, are determined by linear recurrences.

We believe that this approach ("step by step") to enumeration of Hamiltonian cycles can be extended for similar grids, and to the three-dimensional grids. For a cube grid, for example, in the first step, it would be necessary to determine $\binom{6}{2} = 15$ alpha-letters instead of $\binom{4}{2} = 6$ alpha-letters here. For determining the code matrix $[(\alpha_{i,j,k}, n_{i,j,k})]_{m\times n\times p}$ and its properties, it would be necessary to defined "back-forth" adjacency digraph $\mathcal{D}_{bf}$ beside the existing left-right and up-down adjacency digraphs $\mathcal{D}_{lr}$ i $\mathcal{D}_{ud}$.

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