Counter-intuitive answers to some questions concerning minimal-palindromic extensions of binary words

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Abstract

In [Š. Holub & K. Saari, On highly palindromic words, *Discrete Appl. Math.* **157** (2009), 953–959] the authors proposed to measure the degree of "palindromicity" of a binary word w by ratio $\frac{|rws|}{|w|}$, where the word rws is minimal-palindromic—that is, does not contain palindromic subwords of length greater than $\lceil \frac{|w|}{2} \rceil$—and the length $|r|+|s|$ is as small as possible. It was asked whether the words of a given length n which reach the maximal possible ratio $\frac{|rws|}{|w|}$ among the words of length n are always palindromes. It was further asked whether it can be assumed, w.l.o.g., that r and s are of form 0^* or 1^*, or at least 0^*1^* or 1^*0^*. We negatively answer these questions, and also one further question of a similar kind.

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1 Introduction

Combinatorics on words is a branch of mathematics having a very wide scope of applications. A similar thing can be said about palindromic words, that is, words that can be read indistinctly from left to right or from right to left. Namely, they play a major role in the study of so-called Sturmian sequences [12, 7], which in turn have applications in number theory, routing optimization, computer graphics and image processing, pattern recognition and more [1, Chapter 9]. Palindromes further have applications in seemingly unrelated fields such as quantum physics [8, 2, 6], molecular biology [11, 10] [13, Chapter 4] and recently even music theory [14, 5, 3].

Thus, a more detailed knowledge about the behavior of palindromes is of a growing importance. One of the questions arising is determining which of two given words (not necessarily palindromes) is "more palindromic" than the other one, that is, defining a measure for the degree of "palindromicity" of a word. Clearly, different approaches can be imagined, depending on the interpretation of "more palindromic". Holub and Saari [9] chose the following one. Restricting themselves to the binary words, they observed that each word w contains a palindromic (scattered) subword of length at least $\lceil \frac{|w|}{2} \rceil$: a subword consisting of the dominant letter. On the basis of this observation, they called words w which do not contain palindromic subwords of length greater than $\lceil \frac{|w|}{2} \rceil$ *minimal-palindromic*: intuitively, these are the least palindromic words. The degree of "palindromicity" of a word w is then measured by the so-called *MP-ratio*, defined with the following conception in mind: the word is more palindromic the harder it is to extend it to a minimal-palindromic word (a strict definition is given in the next section). In the end of their paper, Holub and Saari posed a few plausible-looking questions, which would have, if answered positively, made computations of MP-ratios significantly simpler.

The aim of this paper is to answer these questions, and also one further question of a similar kind. Rather surprisingly, all the answers turn out to be negative.

2 Preliminaries

Let us present the notation and necessary definitions. In case of any unclear notions, we direct the reader to, for example, [4].

Given a set Σ called the *alphabet*, we call its elements *letters*, and finite sequences of letters are called *words*. For words $w = a_1a_2\dots a_n$ and $u = b_1b_2\dots b_m$ (where $a_1,\dots,a_n,b_1,\dots,b_m \in \Sigma$), with wu we denote the concatenation of words w and u, that is, $wu = a_1a_2\dots a_nb_1b_2\dots b_m$. Given a word w and an integer $k \geqslant 0$, we write w^k for $\underbrace{ww\dots w}_{k\text{ times}}$ (called the *k-th power* of a word w).

The set of all words over the alphabet Σ is denoted with Σ^*. If a is a letter, we write a^* for the set $\{a^k : k \geqslant 0\}$, and if b is an additional letter, we write a^*b^* for the set $\{a^k b^l : k, l \geqslant 0\}$.

The length of a word w is denoted with $|w|$. Notation $|w|_a$, where a is a letter, stands for the total number of occurrences of a in w. The unique word of length equal to 0, called the *empty word*, is denoted with ε.

A word $w = a_1 a_2 \dots a_n$ is a *subword* of a word v if there exist words $u_1, u_2, \dots, u_{n+1}$ such that $v = u_1 a_1 u_2 a_2 \dots u_n a_n u_{n+1}$. Remark: some authors require that the letters $a_1, a_2, \dots, a_n$ appear in the word v consecutively, and the herein presented notion of subword is then called *scattered* (or *sparse*) *subword*. We proceed with the former convention.

A *palindromic word* (or *palindrome*) is a word w that can be read indistinctly from left to right or from right to left, that is, $w = a_1 a_2 \dots a_n = a_n a_{n-1} \dots a_1$.

Since we are concerned only with binary words, from now on we fix the alphabet $\Sigma = \{0, 1\}$.

Clearly, each word $w \in \{0,1\}^*$ contains a palindromic subword of length at least $\lceil \frac{|w|}{2} \rceil$: a subword consisting of the dominant letter. We say that w is *minimal-palindromic* if it does not contain palindromic subwords of length greater than $\lceil \frac{|w|}{2} \rceil$. For a word $w \in \{0,1\}^*$, a pair (r, s), where $r, s \in \{0,1\}^*$, such that rws is minimal-palindromic, is called an *MP-extension* of w (an MP-extension always exists [9, Theorem 4]). If the length $|r| + |s|$ is minimal possible, then we call the pair (r, s) a *shortest MP-extension* or *SMP-extension*. We measure the degree of "palindromicity" of w by the ratio $\frac{|rws|}{|w|}$, called *MP-ratio*, where (r, s) is an SMP-extension of w.

Holub and Saari asked the following questions about MP-extensions:

Question 1. *Consider all the binary words of a given length n. Are those among them which reach the maximal possible MP-ratio necessarily palindromes?*

Question 2. *Does every binary word possess an SMP-extension (r, s) with $r, s \in 0^* \cup 1^*$?*

Question 3. *Does every binary word possess an SMP-extension (r, s) with $r, s \in 0^*1^* \cup 1^*0^*$?*

To these three questions we append another one of a similar kind.

Question 4. *Does every binary word possess an SMP-extension (r, s) such that r and s do not have a letter in common?*

Let us say a few words about the intuition behind these questions.

Clearly, the minimal possible MP-ratio equals 1 and is reached precisely for minimal-palindromic words, which are thought of as the least palindromic words. Question 1 deals with the words on the opposite end: since they are thought of as the most palindromic words, it is quite expected, as Question 1 predicts, that they must be palindromes. However, in the following section we show that this is not always the case.

Questions 2, 3 and 4 deal with the possible forms of SMP-extensions. Question 2 is based on the following intuition: since we are avoiding palindromic subwords longer than necessary, it seems reasonable to assume that r and s are as simple as possible, that is, powers of a single letter; indeed, other forms of r and s would give rise to more different subwords, thus increasing the chance of a palindrome being among them. Question 3 is just a weaker form of Question 2. Finally, Question 4, arguably the most plausible of all, predicts that it is safe to assume that r and s do not have a letter in common, based on the fact that a common letter to r and s actually increases the length of a longest palindromic subword of a starting word. Nevertheless, in the following section we disprove all these intuitions. Note that, although any counterexample to Question 3 also is a counterexample to Question 2, and furthermore, our counterexample to Question 4 also is another counterexample to Question 2—we still resolve Question 2 separately. The reason is that, while Questions 3 and 4 are resolved by a single counterexample each, we provide an infinite family of counterexamples to Question 2.

3 Main results

Let us first prove a useful lemma.

Lemma 1. *If (r, s) is an SMP-extension of w and $|r| + |s| > 0$, then $|rws|$ is odd.*

Proof. Suppose the opposite: (r, s) is an SMP-extension of w, $|r|+|s| > 0$ and $|rws|$ is even. Assume, w.l.o.g., $|r| > 0$ (the case $|s| > 0$ is analogous). Let r' be the word obtained by erasing any letter from r. Since rws is minimal-palindromic, it does not contain palindromic subwords of length greater than

$\lceil \frac{|rws|}{2} \rceil$. Since $|rws|$ is even and $|r'ws| = |rws| - 1$, we have $\lceil \frac{|r'ws|}{2} \rceil = \lceil \frac{|rws|}{2} \rceil$. Finally, since $r'ws$ clearly cannot contain palindromic subword longer than the palindromic subwords of rws, we have that (r', s) is an MP-extension of w shorter than (r, s), which is impossible. ∎

We are now ready for the main theorems.

Theorem 1. *The answer to Question 1 is negative.*

Proof. A counterexample will be given for $n = 6$. We claim that the maximal possible MP-ratio of words of length 6 equals $\frac{11}{6}$, and that one of the words achieving it is

$$v = 010110,$$

a non-palindrome.

In the first place, let us prove that the MP-ratio of v is indeed $\frac{11}{6}$. Let (r, s) be an MP-extension of v. Since v contains palindromic subwords of length 5, 01010 and 01110, we have $|rvs| \geqslant 9$. Let us suppose that $|rvs| = 9$. In that case, rvs must not have palindromic subwords of length greater than 5. Notice that, if s contains the letter 0, then 001100 is a subword of rvs, a contradiction; if s contains the letter 1, then 101101 is a subword of rvs, and a contradiction again. Therefore, s is the empty word, and $|r| = 3$. Now, if 11 is a subword of r, then 1101011 is a subword of rvs, a contradiction. If $r = 000$, then 000000 is a subword of rvs, a contradiction. Therefore, r contains one letter 1 and two letters 0. If 01 is a subword of r, then 0101010 is a subword of rvs, a contradiction. That leaves only the possibility $r = 100$, but then 100001 is a subword of rvs, a contradiction. Altogether, it must hold $|rvs| > 9$, and therefore, by Lemma 1, $|rvs| \geqslant 11$. Since $\underbrace{000}_{r}\underbrace{010110}_{v}\underbrace{11}_{s}$ is minimal-palindromic, we have that $(000, 11)$ is an SMP-extension of v, and thus the MP-ratio of v is indeed $\frac{11}{6}$.

We now have to prove that all the other words of length 6 have MP-ratio at most $\frac{11}{6}$, that is, that for each word w there exists an MP-extension (r, s) such that $|rws| \leqslant 11$. Such extensions are shown in Table 1. (Only the words starting with the letter 0 are considered, since the other half are analogous. Proposed extensions are in fact SMP-extensions, though there is no need to prove that, we only need $|rws| \leqslant 11$.) ∎

Theorem 2. *The answer to Question 2 is negative.*

w	rws	w	rws	w	rws
000000	00000011111	001011	001011	010110	00001011011
000001	000001111	001100	00011001111	010111	0010111
000010	000010111	001101	0001101	011000	1011000
000011	0000111	001110	000111011	011001	00011001111
000100	000100111	001111	0001111	011010	11011010000
000101	0001011	010000	111010000	011011	000110111
000110	0001101	010001	010001111	011100	101110000
000111	000111	010010	00100101111	011101	000111011
001000	001000111	010011	0100111	011110	10111100000
001001	001001111	010100	110101000	011111	000011111
001010	001010111	010101	001010111		

Table 1: MP-extensions of words of length 6.

Proof. We claim that, for every $k \geqslant 4$, the only SMP-extension of the word

$$v = 010^k1010$$

is the pair $(\varepsilon, u) = (\varepsilon, 01^{k+2})$, thus providing an infinite family of counterexamples to Question 2.

In the first place, let us prove that (ε, u) is an MP-extension of v, that is, that $vu = 010^k101001^{k+2}$ does not contain palindromic subwords of length greater than $\lceil \frac{|vu|}{2} \rceil = \lceil \frac{2k+9}{2} \rceil = k+5$. Let p be a palindromic subword of vu. We shall distinguish a few cases:

Case 1: p begins with three or more letters 1. In this case, p is clearly a subword of 111001^{k+2}. It is now obvious that p cannot be longer than 1^{k+5}, that is, $|p| \leqslant k+5$.

Case 2: p begins with exactly two letters 1. In this case, p is clearly a subword of 11010011, and thus p cannot be longer than 1100011, that is, $|p| \leqslant 7 < k+5$.

Case 3: p begins with exactly one letter 1. In this case, p is clearly a subword of $10^k101001$, what has length $k+7$. Since the considered word is not a palindrome, and since it can be easily checked that erasing any one of its letters does not leave a palindrome (because $k \geqslant 4$), it follows that $|p| \leqslant k+5$.

Case 4: p begins with the letter 0. In this case, p is clearly a subword of 010^k10100, which has length $k+7$. Since the considered word is not a palindrome, and since it can be easily checked that erasing any one of its letters does not leave a palindrome, it follows that $|p| \leqslant k+5$.

Therefore, we have proved that (ε, u) is an MP-extension of v. Notice that $010^{k+1}10$ is a palindromic subword of v, of length $k+5$. It now follows that for any MP-extension (r,s) of v we have $|rvs| \geqslant 2k+9$, and thus (ε, u) is in fact an SMP-extension of v. We are left to prove that it is unique.

Let (r,s) be an SMP-extension of v. We already know that $|rvs| = 2k+9$, that is, $|r|+|s| = k+3$. Notice that, if r contains the letter 1, then 1010^k101 is a palindromic subword of rvs of length $k+6$, a contradiction; if r contains the letter 0, then 0010^k100 is a palindromic subword of rvs of length $k+6$, and a contradiction again. Therefore, $r = \varepsilon$ and $|s| = k+3$. Since $|v|_1 = 3$ and $|v|_0 = k+3$, we have either $|s|_1 = k+1$ and $|s|_0 = 2$, or $|s|_1 = k+2$ and $|s|_0 = 1$ (because otherwise we would have $|vs|_0 > k+5$ or $|vs|_1 > k+5$, which would contradict the fact that vs is minimal-palindromic). If 10 is a subword of s, then $010^{k+2}10$ is a palindromic subword of vs of length $k+6$, a contradiction. Therefore, $s = 001^{k+1}$ or $s = 01^{k+2}$. Finally, in the former case $10^{k+4}1$ is a palindromic subword of vs of length $k+6$, which is a contradiction, and thus only the latter case remains, which was to be proved. ■

Theorem 3. *The answer to Question 3 is negative.*

Proof. We claim that the only SMP-extension of the word

$$v = 0010000010100111$$

is the pair $(\varepsilon, u) = (\varepsilon, 1011111)$, thus providing a counterexample to Question 3.

In the first place, let us prove that (ε, u) is an MP-extension of v, that is, that $vu = 00100000101001111011111$ does not contain palindromic subwords of length greater than $\lceil \frac{|vu|}{2} \rceil = \lceil \frac{23}{2} \rceil = 12$. Let p be a palindromic subword of vu. We shall distinguish a few cases:

<u>Case 1:</u> p begins with four or more letters 1. In this case, p is clearly a subword of 1111111011111, and since the considered word is not a palindrome, it follows that $|p| \leqslant 12$.

<u>Case 2:</u> p begins with exactly three letters 1. In this case, p is clearly a subword of 1110011110111, and since the considered word is not a palindrome, it follows that $|p| \leqslant 12$.

<u>Case 3:</u> p begins with exactly two letters 1. In this case, p is clearly a subword of 1101001111011, and since the considered word is not a palindrome, it follows that $|p| \leqslant 12$.

Case 4: p begins with exactly one letter 1. In this case, p is clearly a subword of 10000010100111101, and we may write $p = 10p'01$, where p' is a palindromic subword of 0000101001111. Obviously, p' is a palindromic subword of either 000010100 or 101001111, and since these two words are not palindromes, it follows that $|p'| \leqslant 8$ and therefore $|p| = |p'| + 4 \leqslant 12$.

Case 5: p begins with the letter 0 followed by two or more letters 1. In this case, p is clearly a subword of 011010011110, and it follows $|p| \leqslant 12$.

Case 6: p begins with the letter 0 followed by exactly one letter 1. In this case, p is clearly a subword of 01000001010010. Since the considered word is not a palindrome, and since it can be easily checked that erasing any one of its letters does not leave a palindrome, it follows that $|p| \leqslant 12$.

Case 7: p begins with two or more letters 0. In this case, p is clearly a subword of 00100000101000. Since the considered word is not a palindrome, and since it can be easily checked that erasing any one of its letters does not leave a palindrome, it follows that $|p| \leqslant 12$.

Therefore, we have proved that (ε, u) is an MP-extension of v. Notice that 001000000100 is a palindromic subword of v, of length 12. It now follows that for any MP-extension (r, s) of v we have $|rvs| \geqslant 23$, and thus (ε, u) is in fact an SMP-extension of v. We are left to prove that it is unique.

Let (r, s) be an SMP-extension of v. We already know that $|rvs| = 23$, that is, $|r| + |s| = 7$. Notice that, if r contains the letter 1, then 10010000001001 is a palindromic subword of rvs of length 14, a contradiction; if r contains the letter 0, then 0001000001000 is a palindromic subword of rvs of length 13, and a contradiction again. Therefore, $r = \varepsilon$ and $|s| = 7$. Since $|v|_1 = 6$ and $|v|_0 = 10$, we have either $|s|_1 = 5$ and $|s|_0 = 2$, or $|s|_1 = 6$ and $|s|_0 = 1$ (because otherwise we would have $|vs|_0 > 12$ or $|vs|_1 > 12$, which would contradict the fact that vs is minimal-palindromic). If 00 is a subword of s, then 00100000000100 is a palindromic subword of vs of length 14, a contradiction. Therefore, $|s|_1 = 6$ and $|s|_0 = 1$. If 110111 is a subword of s, then 1110111110111 is a palindromic subword of vs of length 13, a contradiction. If 111110 is a subword of s, then 0111111111110 is a palindromic subword of vs of length 13, a contradiction. That leaves only the possibilities: $s = 0111111$ or $s = 1011111$ or $s = 1111011$. In the first case 1111110111111 is a palindromic subword of vs of length 13, a contradiction. In the third case 11011111111011 is a palindromic subword of vs of length 14, a contradiction. Thus only the second case remains, which was to be proved. ■

Theorem 4. *The answer to Question 4 is negative.*

Proof. We claim that the only SMP-extension of the word

$$v = 01111101001$$

is the pair $(y, u) = (1, 1000000)$, thus providing a counterexample to Question 4.

In the first place, let us prove that (y, u) is an MP-extension of v, that is, that $yvu = 101111101001 1000000$ does not contain palindromic subwords of length greater than $\lceil \frac{|yvu|}{2} \rceil = \lceil \frac{19}{2} \rceil = 10$. Let p be a palindromic subword of yvu. We shall distinguish a few cases:

Case 1: p begins with two or more letters 1. In this case, p is clearly a subword of 111111010011. Since the considered word is not a palindrome, and since it can be easily checked that erasing any one of its letters does not leave a palindrome, it follows that $|p| \leqslant 10$.

Case 2: p begins with exactly one letter 1. In this case, p is clearly a subword of 101111101001. Since the considered word is not a palindrome, and since it can be easily checked that erasing any one of its letters does not leave a palindrome, it follows that $|p| \leqslant 10$.

Case 3: p begins with the letter 0. In this case, p is clearly a subword of 011111010011000000, and we may write $p = 0p'0$, where p' is a palindromic subword of 1111101001100000. Obviously, p' is a palindromic subword of either 11111010011 or 01001100000. It is not hard to see that the longest palindromic subwords of these two words are 11111111, and 00011000 or 00000000, respectively. It follows that $|p'| \leqslant 8$, and therefore $|p| = |p'| + 2 \leqslant 10$.

Therefore, we have proved that (y, u) is an MP-extension of v. Notice that 01111110 is a palindromic subword of v, of length 8. It now follows that for any MP-extension (r, s) of v we have $|rvs| \geqslant 15$, and thus, by Lemma 1, in order to prove that (y, u) is an SMP-extension of v, we have to show that there are no MP-extensions (r, s) of v such that $|rvs| = 15$ or $|rvs| = 17$.

Suppose that there exists an MP-extension (r, s) of v such that $|rvs| = 15$. Since $|v|_1 = 7$ and $|v|_0 = 4$, we have either $|r|_1 + |s|_1 = 1$ and $|r|_0 + |s|_0 = 3$, or $|r|_1 + |s|_1 = 0$ and $|r|_0 + |s|_0 = 4$. In both cases, at least one of r, s contains the letter 0. If r contains the letter 0, then 0011111100 is a palindromic subword of rvs of length 10, a contradiction; if s contains the letter 0, then 011111110 is a palindromic subword of rvs of length 9, and a contradiction again. Therefore, there are no MP-extensions (r, s) such that $|rvs| = 15$.

Suppose that there exists an MP-extension (r, s) of v such that $|rvs| = 17$. Notice that, if r contains the letter 0, then 0011111100 is a palindromic subword of rvs of length 10, a contradiction; if r contains the letter 1, then 1011111101 is a palindromic subword of rvs of length 10, and a contradiction again. Therefore, $r = \varepsilon$ and $|s| = 6$. Since $|v|_1 = 7$ and $|v|_0 = 4$, we have either $|s|_1 = 2$ and $|s|_0 = 4$, or $|s|_1 = 1$ and $|s|_0 = 5$. Notice that, if 10 is a subword of s, then 0111111110 is a palindromic subword of vs of length 10, a contradiction. Therefore, $s = 000011$ or $s = 000001$. In the former case, 11000000011 is a palindromic subword of vs of length 11, a contradiction; in the latter case, 1000000001 is a palindromic subword of vs of length 10, and a contradiction again. Therefore, there are no MP-extensions (r, s) such that $|rvs| = 17$, and thus (y, u) is in fact an SMP-extension of v. We are left to prove that it is unique.

Let (r, s) be an SMP-extension of v. We already know that $|rvs| = 19$, that is, $|r| + |s| = 8$. Notice that, if 00 is a subword of r, then 00011111000 is a palindromic subword of rvs of length 11, a contradiction; if 11 is a subword of r, then 11011111011 is a palindromic subword of rvs of length 11, a contradiction; if 01 is a subword of r, then 01011111010 is a palindromic subword of rvs of length 11, a contradiction; if 10 is a subword of r, then 100111111001 is a palindromic subword of rvs of length 12, and a contradiction again. Therefore, $|r| \leqslant 1$. Since $|v|_1 = 7$ and $|v|_0 = 4$, we have either $|r|_1 + |s|_1 = 3$ and $|r|_0 + |s|_0 = 5$, or $|r|_1 + |s|_1 = 2$ and $|r|_0 + |s|_0 = 6$. We further note that 110 cannot be a subword of s (this shall be needed later), since otherwise 01111111110 would be a palindromic subword of vs of length 11, a contradiction.

Suppose that $r = 0$. Since $|r|_0 + |s|_0 \geqslant 5$, it follows that 00 is a subword of s. Therefore, 00111111100 is a palindromic subword of rvs of length 11, a contradiction.

Suppose that $r = \varepsilon$. Therefore, we have either $|s|_1 = 3$ and $|s|_0 = 5$, or $|s|_1 = 2$ and $|s|_0 = 6$. Notice that, if 111 is a subword of s, then 11110001111 is a palindromic subword of vs of length 11, a contradiction. That leaves $|s|_1 = 2$ and $|s|_0 = 6$. We know that 110 is not a subword of s, that is, s ends with the letter 1. Therefore, 0000001 is a subword of s. However, 10000000001 is then a palindromic subword of vs of length 11, a contradiction.

Therefore, $r = 1$, and either $|s|_1 = 2$ and $|s|_0 = 5$, or $|s|_1 = 1$ and $|s|_0 = 6$. Notice that, if 01 is a subword of s, then 10111111101 is a palindromic subword of rvs of length 11, a contradiction. It now follows that $s = 1100000$ or $s = 1000000$. Finally, the former case contradicts the earlier observation that 110 is not a subword of s, and thus only the latter case remains, which was to be proved. ■

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References

[1] J.-P. Allouche & J. Shallit, *Automatic Sequences. Theory, Applications, Generalizations,* Cambridge University Press, Cambridge, 2003.

[2] M. Baake, A note on palindromicity, *Lett. Math. Phys.* **49** (1999), 217–227.

[3] N. Carey, On a class of locally symmetric sequences: The right infinite word Λ_θ, in C. Agon & M. Andreatta & G. Assayag & E. Amiot & J. Bresson & J. Mandereau (eds.), *Mathematics and Computation in Music. Proceedings of the Third International Conference (MCM 2011)*, Springer, Berlin, 2011, pp. 42–55.

[4] C. Choffrut & J. Karhumäki, Combinatorics of words, in G. Rozenberg & A. Salomaa (eds.), *Handbook of Formal Languages, Vol. 1. Word, Language, Grammar,* Springer, Berlin, 1997, pp. 329–438.

[5] D. Clampitt & T. Noll, Regions and standard modes, in E. Chew & A. Childs & C.-H. Chuan (eds.), *Mathematics and Computation in Music. Proceedings of the Second International Conference (MCM 2009)*, Springer, Berlin, 2009, pp. 81–92.

[6] D. Damanik & J.-M. Ghez & L. Raymond, A palindromic half-line criterion for absence of eigenvalues and applications to substitution Hamiltonians, *Ann. Henri Poincaré* **2** (2001), 927–939.

[7] X. Droubay & G. Pirillo, Palindromes and Sturmian words, *Theoret. Comput. Sci.* **223** (1999), 73–85.

[8] A. Hof & O. Knill & B. Simon, Singular continuous spectrum for palindromic Schrödinger operators, *Comm. Math. Phys.* **174** (1995), 149–159.

[9] Š. Holub & K. Saari, On highly palindromic words, *Discrete Appl. Math.* **157** (2009), 953–959.

[10] L. Kari & K. Mahalingam, Watson-Crick palindromes in DNA computing, *Nat. Comput.* **9** (2010), 297–316.

[11] M.-Y. Leung & K. P. Choi & A. Xia & L. H. Y. Chen, Nonrandom clusters of palindromes in herpesvirus genomes, *J. Computational Biology* **12** (2005), 331–354.

[12] A. de Luca, Sturmian words: structure, combinatorics, and their arithmetics, *Theoret. Comput. Sci.* **183** (1997), 45–82.

[13] D. Nolan & T. Speed, *Stat Labs. Mathematical Statistics Through Applications,* Springer, Berlin, 2000.

[14] T. Noll, Sturmian sequences and morphisms. A music-theoretical application, in A. Yves (ed.), *Mathématique et musique, Journée annuelle de la Société Mathématique de France*, 2008, pp. 79–102.