# ANTICHAINS OF COPIES OF ULTRAHOMOGENEOUS STRUCTURES

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**Abstract**

We investigate possible cardinalities of maximal antichains in the poset of copies $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ of a countable ultrahomogeneous relational structure $\mathbb{X}$. It turns out that if the age of $\mathbb{X}$ has the strong amalgamation property, then, defining a copy of $\mathbb{X}$ to be large iff it has infinite intersection with each orbit of $\mathbb{X}$, the structure $\mathbb{X}$ can be partitioned into countably many large copies, there are almost disjoint families of large copies of size continuum and, hence, there are (maximal) antichains of size continuum in the poset $\mathbb{P}(\mathbb{X})$. Finally, we show that the posets of copies of all countable ultrahomogeneous partial orders contain maximal antichains of cardinality continuum and determine which of them contain countable maximal antichains. That holds, in particular, for the random (universal ultrahomogeneous) poset.

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## 1 Introduction

In this paper we investigate antichains in the posets of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where $\mathbb{P}(\mathbb{X}) := \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$ is the set of the substructures of a countable ultrahomogeneous relational structure $\mathbb{X}$ which are isomorphic to $\mathbb{X}$. Recall that a structure $\mathbb{X}$ is *ultrahomogeneous* iff for each isomorphism $\varphi : \mathbb{A} \to \mathbb{B}$ between finite substructures $\mathbb{A}$ and $\mathbb{B}$ of $\mathbb{X}$, there is an automorphism $f$ of $\mathbb{X}$ extending $\varphi$. These posets were analyzed from various viewpoints recently. Typically, the results obtained would be compared to the poset $\langle [\omega]^\omega, \subset \rangle$ of all infinite subsets of a countable set, ordered by inclusion. Set theorists thoroughly investigated this object and, most often, an antichain in this context is a set of pairwise incompatible elements, i.e. a collection of sets in $[\omega]^\omega$ with pairwise finite intersections (an *almost disjoint family*). Two basic facts are that there is no countable maximal antichain in $[\omega]^\omega$, whereas there is a maximal antichain of size continuum in that poset. We follow this approach. So, in this paper, an antichain is always a set of pairwise incompatible elements of the partial order in question.

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Section 2 contains definitions and facts which are used in the paper. Defining a copy of $\mathbb{X}$ to be *large* iff it has infinite intersection with each orbit of $\mathbb{X}$, in Sections 3 and 4 we prove the following general statement.

**Theorem 1.1** *If $\mathbb{X}$ is a countable ultrahomogeneous relational structure satisfying the strong amalgamation property, then*

- (a) *$\mathbb{X}$ can be partitioned into countably many large copies of $\mathbb{X}$;*
- (b) *There are almost disjoint families of large copies of $\mathbb{X}$ of size continuum;*
- (c) *There are (maximal) antichains of size continuum in the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$.*

In Section 5 we take a closer look on the case of countable ultrahomogeneous posets, using the following well-known classification due to Schmerl [19].

**Theorem 1.2 (Schmerl)** *Each countable ultrahomogeneous partial order is isomorphic to one of the following:*

- $\mathbb{A}_\omega$, *a countable antichain (that is, the empty relation on $\omega$);*
- $\mathbb{B}_n = n \times \mathbb{Q}$, $1 \leq n \leq \omega$, *where $\langle i_1, q_1 \rangle < \langle i_2, q_2 \rangle \Leftrightarrow i_1 = i_2 \ \wedge \ q_1 <_\mathbb{Q} q_2$;*
- $\mathbb{C}_n = n \times \mathbb{Q}$, $1 \leq n \leq \omega$, *where $\langle i_1, q_1 \rangle < \langle i_2, q_2 \rangle \Leftrightarrow q_1 <_\mathbb{Q} q_2$;*
- $\mathbb{D}$, *the random poset.*

So, in Section 5 we prove the following theorem.

**Theorem 1.3** *Let $\mathbb{X}$ be a countable ultrahomogeneous partial order. Then*

- (a) *There are maximal antichains of size continuum in the poset $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$;*
- (b) *There are countable maximal antichains in $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ if and only if $\mathbb{X}$ is neither isomorphic to $\mathbb{A}_\omega$ nor to $\mathbb{B}_\omega$.*

At this point we mention some related concepts. First, antichains in the poset of copies of the random (Rado) graph were analyzed in [12]. Second, forcing-related properties of the posets of copies of ultrahomogeneous structures were investigated in [13, 14, 15]. Third, in [7, 8, 9] a classification of relational structures with respect to the properties of posets $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ is given. Fourth, the order types of the maximal chains in the posets of copies of countable ultrahomogeneous graphs and countable ultrahomogeneous partial orders are described in [10, 11]. Finally, if $\mathbb{X}$ is a first order structure and $\preceq^R$ right Green's pre-order on its self-embedding monoid, $\mathrm{Emb}\,\mathbb{X}$, the corresponding antisymmetric quotient $\langle \mathrm{Emb}\,\mathbb{X}/ \approx^R, \preceq^R \rangle$ (right Green's order) is isomorphic to the partial order $\langle \mathbb{P}(\mathbb{X}), \supset \rangle$. Hence, our results provide some information about self-embedding monoids of structures.

## 2 Preliminaries

If $L = \langle R_i : i \in I \rangle$ is a relational language, where $\mathrm{ar}(R_i) = n_i \in \mathbb{N}$, for $i \in I$, and $\mathbb{X} = \langle X, \rho \rangle$ is an $L$-structure, where $\rho = \langle \rho_i : i \in I \rangle$ and $\rho_i \subset X^{n_i}$, for $i \in I$, then, for a subset $A$ of $X$, by $\rho \restriction A$ we will denote the sequence $\langle \rho_i \restriction A : i \in I \rangle$, where $\rho_i \restriction A := \rho_i \cap A^{n_i}$. Then the structure $\mathbb{A} = \langle A, \rho \restriction A \rangle$ is a substructure of $\mathbb{X}$.

If $\mathbb{Y} = \langle Y, \sigma \rangle$ is also an $L$-structure, an injection $f : X \to Y$ is called an *embedding* (we write $f : \mathbb{X} \hookrightarrow \mathbb{Y}$ or $f \in \mathrm{Emb}(\mathbb{X}, \mathbb{Y})$) iff for each $i \in I$ and $\bar{x} \in X^{n_i}$ we have: $\bar{x} \in \rho_i$ iff $f\bar{x} \in \sigma_i$. If, in addition, $f$ is a surjection, it is an *isomorphism*, the structures $\mathbb{X}$ and $\mathbb{Y}$ are *isomorphic*, and we write $\mathbb{X} \cong \mathbb{Y}$. If, in particular, $\mathbb{Y} = \mathbb{X}$, then $f$ is an *automorphism* of the structure $\mathbb{X}$. $\mathrm{Aut}(\mathbb{X})$ will denote the set of all automorphisms of $\mathbb{X}$ and $\mathrm{Emb}(\mathbb{X})$ is $\mathrm{Emb}(\mathbb{X}, \mathbb{X})$. By $\mathbb{P}(\mathbb{X})$ we denote the set of domains of substructures of $\mathbb{X}$ isomorphic to $\mathbb{X}$, that is $\mathbb{P}(\mathbb{X}) = \{A \subset X : \langle A, \rho \restriction A \rangle \cong \langle X, \rho \rangle\} = \{f[X] : f \in \mathrm{Emb}(\mathbb{X})\}$.

If $\langle P, \leq \rangle$ is a poset, the elements $x$ and $y$ of $P$ are *compatible* iff there is an element $z \in P$ such that $z \leq x$ and $z \leq y$. Otherwise, $x$ and $y$ are *incompatible* and we write $x \perp y$. A set $A \subset P$ is *an antichain* in $P$ if its elements are pairwise incompatible. An antichain $A$ is a *maximal antichain* in $P$ iff each $z \in P$ is compatible with some $x \in A$.

We recall some basic facts from Fraïssé theory. The *age*, $\mathrm{Age}\,\mathbb{X}$, of an ultrahomogeneous $L$-structure $\mathbb{X}$ (i.e., the class of all finite $L$-structures embeddable in $\mathbb{X}$) satisfies the *amalgamation property* (AP): if $\mathbb{A}, \mathbb{B}, \mathbb{C} \in \mathrm{Age}\,\mathbb{X}$ and $f_0 : \mathbb{A} \hookrightarrow \mathbb{B}$ and $g_0 : \mathbb{A} \hookrightarrow \mathbb{C}$ are embeddings, then there are $\mathbb{D} \in \mathrm{Age}\,\mathbb{X}$ and embeddings $f_1 : \mathbb{B} \hookrightarrow \mathbb{D}$ and $g_1 : \mathbb{C} \hookrightarrow \mathbb{D}$ such that $f_1 \circ f_0 = g_1 \circ g_0$. If, in addition, the amalgam $\mathbb{D}$ and the embeddings $f_1$ and $g_1$ can be chosen so that $f_1[B] \cap g_1[C] = f_1[f_0[A]] = g_1[g_0[A]]$, then (the age of) $\mathbb{X}$ satisfies the *strong amalgamation property* (SAP). We will use the following classical results of Fraïssé (see [4], p. 332–333).

**Theorem 2.1** *(a) A countable structure $\mathbb{X}$ is ultrahomogeneous iff for each finite substructure $\mathbb{A}$ of $\mathbb{X}$, each $f \in \mathrm{Emb}(\mathbb{A}, \mathbb{X})$ and each $x \in X \setminus A$ there is $y \in X$ such that $f \cup \{\langle x, y \rangle\} \in \mathrm{Emb}(\mathbb{A} \cup \{x\}, \mathbb{X})$.*

*(b) Countable ultrahomogeneous structures with the same age are isomorphic.*

If $\mathbb{X}$ is an $L$-structure, the *pointwise stabilizer* of a finite set $F \subset X$ is the subgroup $\mathrm{Aut}_F(\mathbb{X}) := \{g \in \mathrm{Aut}(\mathbb{X}) : \forall x \in F \; g(x) = x\}$ of the group $\mathrm{Aut}(\mathbb{X})$. The binary relation $\sim_F$ on the set $X \setminus F$ defined by $x \sim_F y$ iff there is $g \in \mathrm{Aut}_F(\mathbb{X})$ such that $g(x) = y$, is an equivalence relation and the equivalence class of an $x \in X \setminus F$ is denoted by $\mathrm{orb}_F(x)$ and called the *orbit of $x$ under* $\mathrm{Aut}_F(\mathbb{X})$. Thus

$$\mathrm{orb}_F(x) = \left\{ y \in X \setminus F : \exists g \in \mathrm{Aut}(\mathbb{X}) \;\; (g \restriction F = \mathrm{id}_F \wedge g(x) = y) \right\}.$$

The sets $\mathrm{orb}_F(x)$, where $F \in [X]^{<\omega}$ and $x \in X \setminus F$, are called *the orbits of* $\mathbb{X}$. Later in the paper, the strong amalgamation property will play a significant role and the next theorem provides convenient characterizations of this property.

**Theorem 2.2** *(see [4] p. 399 and [1] p. 37) For a countable ultrahomogeneous relational structure* $\mathbb{X}$ *the following conditions are equivalent:*
   *(a)* $\mathbb{X}$ *satisfies the strong amalgamation property,*
   *(b)* $\mathbb{X}$ *is strongly inexhaustible, that is,* $X \setminus F \in \mathbb{P}(\mathbb{X})$, *for each finite* $F \subset X$,
   *(c) The orbits of* $\mathbb{X}$ *are infinite.*

The following characterization of copies of ultrahomogeneous structures is, most likely, a known fact. We include its proof for completeness of the paper.

**Theorem 2.3** *If* $\mathbb{X}$ *is a countable ultrahomogeneous* $L$-*structure and* $A \subset X$, *then* $A \in \mathbb{P}(\mathbb{X})$ *iff*

$$\forall F \in [A]^{<\omega} \ \forall x \in X \setminus F \ \mathrm{orb}_F(x) \cap A \neq \emptyset. \tag{1}$$

*Consequently, if the set* $A$ *intersects all orbits of* $\mathbb{X}$, *then* $A \in \mathbb{P}(\mathbb{X})$.

**Proof.** Let $\mathbb{A} \cong \mathbb{X}$, $F \in [A]^{<\omega}$ and $x \in \mathbb{X} \setminus F$. Then there are $F' \subset A$, $a' \in A$ and an isomorphism $\psi : F' \cup \{a'\} \to F \cup \{x\}$ such that $\psi(a') = x$. Since the structure $\mathbb{A}$ is ultrahomogeneous, by Theorem 2.1(a) there is $a \in A$ such that $\varphi := (\psi \restriction F') \cup \{\langle a', a \rangle\} : F' \cup \{a'\} \to F \cup \{a\}$ is an isomorphism. Now $\eta := \varphi \circ \psi^{-1}$ is an isomorphism, $\eta \restriction F = \mathrm{id}_F$ and $\eta(x) = a$. Since $\mathbb{X}$ is ultrahomogeneous, there is $g \in \mathrm{Aut}(\mathbb{X})$ extending $\eta$; so $g \in \mathrm{Aut}_F(\mathbb{X})$, $g(x) = a$ and $a \in \mathrm{orb}_F(x) \cap A$.

Assuming (1) we prove that the set $\mathrm{Pi}(\mathbb{A}, \mathbb{X})$ of all finite partial isomorphisms from $\mathbb{A}$ into $\mathbb{X}$ has the back-and-forth property. So, let $\varphi \in \mathrm{Pi}(\mathbb{A}, \mathbb{X})$. First, if $a \in A \setminus \mathrm{dom}\,\varphi$, then by Theorem 2.1(a) there is $x \in X$ such that $\psi := \varphi \cup \{\langle a, x \rangle\}$ is an isomorphism and, clearly, $\psi \in \mathrm{Pi}(\mathbb{A}, \mathbb{X})$. Second, if $x \in X \setminus \mathrm{ran}\,\varphi$, then by Theorem 2.1(a) there is $x' \in X$ such that $\psi := \varphi^{-1} \cup \{\langle x, x' \rangle\}$ is an isomorphism. Since $x' \notin \mathrm{dom}\,\varphi$, by (1) there exists $a \in \mathrm{orb}_{\mathrm{dom}\,\varphi}(x') \cap A$ and, hence, there is $g \in \mathrm{Aut}_{\mathrm{dom}\,\varphi}(\mathbb{X})$ such that $g(x') = a$. Now, $\eta := g \circ \psi$ is a finite isomorphism with domain $\mathrm{ran}\,\varphi \cup \{x\}$ and $\eta[\mathrm{ran}\,\varphi \cup \{x\}] = g[\psi[\mathrm{ran}\,\varphi]] \cup g[\psi[\{x\}]] = \mathrm{dom}\,\varphi \cup \{a\}$. In addition, for $y \in \mathrm{ran}\,\varphi$ we have $\eta(y) = g(\psi(y)) = g(\varphi^{-1}(y)) = \varphi^{-1}(y)$, which gives $\eta = \varphi^{-1} \cup \{\langle x, a \rangle\}$. So $\eta^{-1} = \varphi \cup \{\langle a, x \rangle\} \in \mathrm{Pi}(\mathbb{A}, \mathbb{X})$. $\square$

## 3   Partitions into large copies

Here we make some observations about copies of ultrahomogeneous structures incompatible in a very strong way. By $DC$ we denote the class of countable structures *having disjoint copies* (there are copies $A, B \in \mathbb{P}(\mathbb{X})$ such that $A \cap B = \emptyset$) and by $SAP$ the class of countable ultrahomogeneous structures satisfying SAP.

**Example 3.1** *A countable ultrahomogeneous structure without disjoint copies.* Let $\mathbb{X} = \mathbb{Q} \cup \mathbb{Y}$, where $\mathbb{Q}$ is the rational line and $\mathbb{Y} := \langle \{y\}, \{\langle y, y \rangle\} \rangle$, where $y \notin \mathbb{Q}$. Then $y \in A$, for each $A \in \mathbb{P}(\mathbb{X})$.

A structure $\mathbb{X}$ is called *indivisible* (resp. *strongly indivisible*) iff for each partition $X = A \cup B$ there is $C \in \mathbb{P}(\mathbb{X})$ such that $C \subset A$ or $C \subset B$ (resp. $A \in \mathbb{P}(\mathbb{X})$ or $B \in \mathbb{P}(\mathbb{X})$). Let $UH$, $I$, and $SI$, denote the classes of ultrahomogeneous, indivisible and strongly indivisible countable relational structures respectively.

Confirming a conjecture of Fraïssé, Pouzet proved that each countable indivisible structure $\mathbb{X}$ has disjoint copies [18]; thus, $I \subset DC$. Here we prove that more holds for countable ultrahomogeneous structures satisfying SAP; thus $SAP \subset DC$. We note that $SAP \not\subset I$, for example, $\mathbb{B}_n \in SAP \setminus I$, for $1 < n < \omega$.

**Theorem 3.2** *Each countable ultrahomogeneous structure $\mathbb{X}$ satisfying SAP can be partitioned into countably many large copies of $\mathbb{X}$.*

**Proof.** W.l.o.g. we assume that $X = \omega$. By Theorem 2.2, the set of orbits, $\Omega := \{\mathrm{orb}_F(x) : F \in [\omega]^{<\omega} \wedge x \in \omega \setminus F\}$, is a countable subfamily of $[\omega]^\omega$. Let $\Omega = \{O_n : n \in \omega\}$ be an enumeration of $\Omega$ and let the sequence $\langle m_{n,k} : n \le k < \omega \rangle$ in $\omega$ be constructed by recursion as follows. First, let $m_{0,0} = \min O_0$.

Second, if $0 < k < \omega$ and $m_{n',k'}$ are defined for $n' \le k' < k$, then we define $m_{n,k}$ for $n \le k$ by: $m_{0,k} = \min[O_0 \setminus \{m_{n',k'} : n' \le k' < k\}]$ and, for $0 < n \le k$,

$$m_{n,k} = \min \left[ O_n \setminus \left( \{m_{n',k'} : n' \le k' < k\} \cup \{m_{n',k} : n' < n\} \right) \right]. \quad (2)$$

The recursion works, since $|O_n| = \omega$, for all $n \in \omega$. By the construction, all the $m_{n,k}$'s are different. So, defining $A_i := \{m_{n,n+i} : n < \omega\}$, for all $i \in \omega$, we have $A_{i_1} \cap A_{i_2} = \emptyset$, for $i_1 \neq i_2$. By (2), for each $n \in \omega$ we have $m_{n,n+i} \in A_i \cap O_n$, thus the set $A_i$ intersects all the orbits of $\mathbb{X}$ and, by Theorem 2.3, $A_i \in \mathbb{P}(\mathbb{X})$.

Clearly we have $A := \bigcup_{i \in \omega} A_i = \{m_{n,k} : n \le k < \omega\} \subset \bigcup_{n \in \omega} O_n = X$. Suppose that $O_n \not\subset A$, for some $n \in \omega$ and let $m = \min(O_n \setminus A)$. By (2) we have $|A \cap O_n| = |\{m_{n,k} : k < \omega\}| = \omega$, so there is $k = \min\{k' < \omega : m < m_{n,k'}\}$ and we have $m < m_{n,k}$. Since $\{m_{n',k'} : n' \le k' < k\} \cup \{m_{n',k} : n' < n\} \subset A$ we have $S := O_n \setminus \{m_{n',k'} : n' \le k' < k\} \cup \{m_{n',k} : n' < n\} \supset O_n \setminus A$ and, by (2), $m_{n,k} = \min S \le \min(O_n \setminus A) = m$, which gives a contradiction. Thus $A = X$ and $\{A_i : i < \omega\}$ is a partition of $X (= \omega)$.

Now, let $\{S_j : j \in \omega\} \subset [\omega]^\omega$ be a partition of $\omega$ and $B_j := \bigcup_{i \in S_j} A_i$, for $j \in \omega$. Then $\{B_j : j \in \omega\} \subset \mathbb{P}(\mathbb{X})$ is a partition of $X$ and for $n \in \omega$ we have $\{m_{n,n+i} : i \in S_j\} \subset B_j \cap O_n$; thus $B_j$, $j \in \omega$, are large copies of $\mathbb{X}$.  □

**Example 3.3** *A countable ultrahomogeneous divisible structure which does not have the SAP, but has disjoint copies.* Let $\mathbb{X}$ be the wreath product $I_\omega[T_3]$ (see [2]),

that is the disjoint union $\bigcup_{n \in \omega} T_3^n$ of $\omega$-many copies of the oriented triangle. By Theorem 2.2 the structure $\mathbb{X}$ does not satisfy SAP, it is clear that $\mathbb{X}$ is not indivisible, but for each $S \in [\omega]^\omega$ we have $A_S := \bigcup_{n \in S} T_3^n \in \mathbb{P}(\mathbb{X})$. Concerning Theorems 2.3 and 3.2 we note that each one-element subset of $X$ is an orbit of $\mathbb{X}$. Hence $X$ is the only subset of $X$ intersecting all the orbits of $\mathbb{X}$.

Figure 1 shows the relationship between the mentioned five classes. For $\mathbb{X}_{Lach}$ see [4], p. 402. $\mathbb{G}_\omega$ is the linear graph on $\omega$, i.e. $\langle \omega, \sim \rangle$, where $m \sim n \Leftrightarrow |m - n| = 1$. $\mathbb{Q} \cup 1_{re}$ is the structure from Example 3.1.
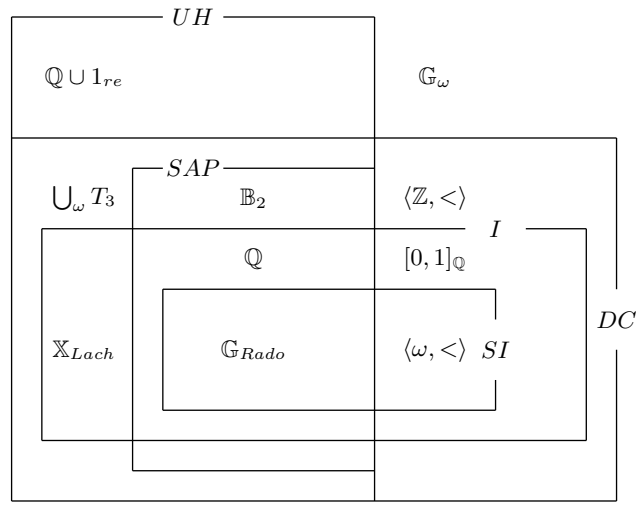


Figure 1: Countable relational structures

## 4   Large almost disjoint families of large copies

By Theorem 5.3 of [7], if $\mathbb{X}$ is a countable indivisible $L$-structure, then the set $\mathbb{P}(\mathbb{X})$ contains an almost disjoint family of size $\mathfrak{c}$ and, hence, the poset $\mathbb{P}(\mathbb{X})$ contains maximal antichains of size $\mathfrak{c}$. Here, proving Theorem 1.1(b) and (c), we show that more is true for countable ultrahomogeneous structures satisfying SAP.

**Theorem 4.1** *If $\mathbb{X}$ is a countable ultrahomogeneous $L$-structure satisfying SAP, then there is an almost disjoint family of large copies of size $\mathfrak{c}$.*

**Proof.** W.l.o.g. we assume that $X = \omega$. By Theorem 2.2, the set of orbits, $\Omega := \{\mathrm{orb}_F(x) : F \in [\omega]^{<\omega} \wedge x \in \omega \setminus F\}$, is a countable subfamily of $[\omega]^\omega$. Let

$\Omega = \{O_n : n \in \omega\}$ be an enumeration of $\Omega$ and let $\langle m_{n,k} : n \le k < \omega \rangle$ be the sequence in $\omega$ constructed in Theorem 3.2. Namely, $m_{0,0} = \min O_0$ and if $m_{n',k'}$ are defined for $n' \le k' < k$, then

$$m_{n,k} = \min \left[ O_n \setminus \left( \{m_{n',k'} : n' \le k' < k\} \cup \{m_{n',k} : n' < n\} \right) \right]. \quad (3)$$

Since all the $m_{n,k}$'s are different, defining $D_n := \{m_{n,k} : k \in [n, \omega)\}$, for $n \in \omega$, we have $D_n \in [O_n]^\omega$ and $D_m \cap D_n = \emptyset$, for $m \ne n$. Clearly we have $D := \bigcup_{n \in \omega} D_n = \{m_{n,k} : n \le k < \omega\}$ and (see the proof of Theorem 3.2) $D = \omega$. So $\{D_n : n < \omega\}$ is a partition of $\omega$ refining $\Omega$.

*Claim.* If $\{D_n : n < \omega\} \subset [\omega]^\omega$ is a partition of $\omega$, then there exists an almost disjoint family $\{A_\alpha : \alpha < \mathfrak{c}\} \subset [\omega]^\omega$, such that $|A_\alpha \cap D_n| = \omega$, for each $\alpha < \mathfrak{c}$ and each $n \in \omega$.

*Proof of Claim.* W.l.o.g. instead of $\omega$ we can take the set of rationals, $\mathbb{Q}$, and suppose that $D_n$, $n < \omega$, are dense suborders of $\mathbb{Q}$. Let $f : \omega \to \omega$ be a surjection such that $|f^{-1}[\{n\}]| = \omega$, for each $n \in \omega$. By recursion, for each real $x \in \mathbb{R}$ we construct an increasing sequence $\langle q_k^x : k \in \omega \rangle$ in $\mathbb{Q}$ converging to $x$ in the following way. First we take $q_0^x \in D_{f(0)} \cap (-\infty, x)$; if $q_0^x, \dots, q_k^x$ are defined, then we take $q_{k+1}^x \in D_{f(k+1)} \cap (\max\{q_k^x, x - \frac{1}{k+1}\}, x)$. By the density of the sets $D_n$ the recursion works. Now, defining the sets $A_x := \{q_k^x : k \in \omega\}$, for $x \in \mathbb{R}$, and $\mathcal{A} := \{A_x : x \in \mathbb{R}\} \in [\mathbb{Q}]^\omega$ we obtain an almost disjoint family of size $\mathfrak{c}$. In addition, for $x \in \mathbb{R}$, $n \in \omega$ and $k \in f^{-1}[\{n\}]$ we have $q_k^x \in A_x \cap D_{f(k)} = A_x \cap D_n$ and, since $|f^{-1}[\{n\}]| = \omega$ and $q_k^x$'s are different, we have $|A_x \cap D_n| = \omega$. $\quad\square$

By Claim, there is an almost disjoint family $\{A_\alpha : \alpha < \mathfrak{c}\} \subset [\omega]^\omega$ such that for each $\alpha < \mathfrak{c}$ and each $n \in \omega$ we have $|A_\alpha \cap D_n| = \omega$ and, since $D_n \subset O_n$, $|A_\alpha \cap O_n| = \omega$. By Theorem 2.3 we have $A_\alpha \in \mathbb{P}(\mathbb{X})$. $\quad\square$

**Example 4.2** *Applications of Theorem 1.1.* The countable ultrahomogeneous digraphs (structures with one irreflexive and asymmetric binary relation) have been classified by Cherlin [2, 3]. Following the organization of the *Cherlin's list* given in [17], we mention some structures satisfying SAP. By Theorem 1.1 their posets of copies contain almost disjoint families and maximal antichains of size continuum.

- The posets $\mathbb{A}_\omega$, $\mathbb{B}_n$, for $n \le \omega$, and $\mathbb{D}$ from Schmerl's list (see Theorem 1.2);
- All countable ultrahomogeneous tournaments (*Lachlan's list* [16]): $\mathbb{Q}$; the random tournament, $\mathbb{T}^\infty$; the circular tournament, $S(2)$; (see [3], p. 18);
- All *Henson's digraphs* with forbidden sets of tournaments [5]; ([17], p. 11);
- Digraphs $\Gamma_n$, $n > 1$, where $\Gamma_n$ is the Fraïssé limit of the amalgamation class of all finite digraphs not embedding the empty digraph of size $n$;

- Two "sporadic" primitive digraphs $S(3)$ and $\mathcal{P}(3)$;
- The digraphs $n * I_\infty$, for $2 \leq n \leq \omega$, which are universal subject to the constraint that non-relatedness is an equivalence relation with $n$ classes.

We remark that some of these structures are not indivisible (so Theorem 5.3 of [7] can not be applied) for example: $S(2)$, $S(3)$, $\mathbb{B}_n$ and $\mathbb{C}_n$, for $1 < n < \omega$.

# 5   Ultrahomogeneous partial orders

Here we prove Theorem 1.3 showing that there are maximal antichains of copies of size $\mathfrak{c}$ for all ultrahomogeneous partial orders and that $\mathbb{A}_\omega$ and $\mathbb{B}_\omega$ are the only structures on Schmerl's list, for which there are no countable maximal antichains of copies.

First, since the poset $\mathbb{P}(\mathbb{A}_\omega)$ is isomorphic to the poset $\langle [\omega]^\omega, \subset \rangle$, it contains maximal antichains of size $\mathfrak{c}$, but does not contain countable maximal antichains.

## 5.1   The posets $\mathbb{B}_n$ (disjoint copies of the rational line)

It is evident that for each $n \leq \omega$ the poset $\mathbb{B}_n$ is strongly inexhaustible. So, by Theorem 2.2, the structure $\mathbb{B}_n$ satisfies the SAP and, by Theorem 1.1, its poset of copies, $\mathbb{P}(\mathbb{B}_n)$, contains maximal antichains of size $\mathfrak{c}$. Here we show that, concerning the existence of countable maximal antichains of copies, the finite unions $\mathbb{B}_n$, $n \in \mathbb{N}$, and the infinite union $\mathbb{B}_\omega$ are different. First, the basic case is $\mathbb{B}_1 \cong \mathbb{Q}$.

**Lemma 5.1** *The poset $\mathbb{P}(\mathbb{Q})$ contains both $\omega$-sized and $\mathfrak{c}$-sized maximal antichains.*

**Proof.** Clearly, the family $\mathcal{A} = \{I_n : n \in \mathbb{Z}\}$ of the open intervals in $\mathbb{Q}$ given by $I_n = ((2n-1)\sqrt{2}, (2n+1)\sqrt{2}) \cap \mathbb{Q}$, for $n \in \mathbb{Z}$, is an antichain in $\mathbb{P}(\mathbb{Q})$. If $C \in \mathbb{P}(\mathbb{Q})$, then $|C \cap I_n| > 1$, for some $n \in \mathbb{Z}$, (otherwise we would have $C \hookrightarrow \mathbb{Z}$). Thus, if $x, y \in C \cap I_n$ and $x < y$, then (since $C \cong \mathbb{Q}$) we have $(x, y)_C \in P(\mathbb{Q})$ and $(x, y)_C \subset I_n \cap C$. So, $\mathcal{A}$ is a countable maximal antichain in $\mathbb{P}(\mathbb{Q})$. $\square$

The following, more general consideration will be used in our analysis of the poset $\mathbb{P}(\mathbb{B}_\omega)$. For each $i \in \omega$, let $\mathbb{P}_i = \langle P_i, \leq_i \rangle$ be a partial order with a minimum $0_i$ and let $|P_i| \geq 2$. By $\prod_{i \in \omega} \mathbb{P}_i$ we denote the direct product of $\mathbb{P}_i$'s, the poset $\langle P, \leq \rangle$, where $P := \prod_{i \in \omega} P_i$ and $\langle x_i \rangle \leq \langle y_i \rangle$ iff $x_i \leq_i y_i$, for all $i \in \omega$. Defining the support of an element $x = \langle x_i \rangle \in P$ by $\mathrm{supp}(x) := \{i \in \omega : x_i \neq 0_i\}$ we consider the suborder $P^{\mathrm{cs}} := \{x \in P : |\mathrm{supp}(x)| = \omega\}$ of the product $\prod_{i \in \omega} \mathbb{P}_i$, call it the *countable support product* of $\mathbb{P}_i$'s and denote it by $\prod_{i \in \omega}^{\mathrm{cs}} \mathbb{P}_i$.

**Lemma 5.2** $\prod_{i \in \omega}^{\mathrm{cs}} \mathbb{P}_i$ *does not contain countable maximal antichains.*

**Proof.** Let $A = \{a^n : n \in \omega\}$ be an antichain in $P^{\text{cs}}$, where $a^n = \langle a_i^n \rangle$. First we show that for different $m, n \in \omega$ the set

$$K_{m,n} = \{i \in \operatorname{supp}(a^m) \cap \operatorname{supp}(a^n) : \exists b_i \in P_i \;\; 0_i <_i b_i \leq_i a_i^m, a_i^n\}$$

is finite. Otherwise, defining $c_i = b_i$, for $i \in K_{m,n}$ and $c_i = 0_i$, for $i \in \omega \setminus K_{m,n}$, we would have $c = \langle c_i \rangle \in P^{\text{cs}}$ and $c \leq a^m, a^n$, which is false.

Let $\langle i_n : n \in \omega \rangle$ be the sequence in $\omega$ defined by $i_0 = \min(\operatorname{supp}(a^0))$ and

$$i_n = \min \Big( \operatorname{supp}(a^n) \setminus (\{i_0, \ldots, i_{n-1}\} \cup K_{0,n} \cup \cdots \cup K_{n-1,n}) \Big). \qquad (4)$$

Let $c = \langle c_i : i \in \omega \rangle$, where $c_{i_n} = a_{i_n}^n$, for $n \in \omega$, and $c_i = 0_i$, if $i \notin \{i_n : n \in \omega\}$. For $n \in \omega$ we have $i_n \in \operatorname{supp}(a^n)$; thus, $c_{i_n} = a_{i_n}^n > 0_{i_n}$ and, hence, $c \in P^{\text{cs}}$.

Assuming that $d \leq a^m, c$, for some $d = \langle d_i \rangle \in P^{\text{cs}}$ and $m \in \omega$, we would have $\operatorname{supp}(d) \subset \operatorname{supp}(c)$ and, hence, $\operatorname{supp}(d) = \{i_n : n \in M\}$, where $M \in [\omega]^\omega$. For each $n \in M$ we would have $0_{i_n} < d_{i_n} \leq a_{i_n}^m, a_{i_n}^n$ and, hence, $i_n \in K_{m,n}$, which is, by (4), impossible for $n > m$. Thus the element $c$ of $P^{\text{cs}}$ is incompatible with all the elements of $A$ and, hence, $A$ is not a maximal antichain in $\prod_{i \in \omega}^{\text{cs}} \mathbb{P}_i$. $\square$

**Theorem 5.3** *(a) For each $n \in \mathbb{N}$ there is a countable maximal antichain in $\mathbb{P}(\mathbb{B}_n)$.*
*(b) Each infinite maximal antichain in $\mathbb{P}(\mathbb{B}_\omega)$ is uncountable.*

**Proof.** (a) It is evident that $\mathbb{P}(\mathbb{B}_n) = \{\bigcup_{i<n} \{i\} \times C_i : \forall i < n \;\; C_i \in \mathbb{P}(\mathbb{Q})\}$ (see the proof of Theorem 5.1 of [10]), which implies that $\mathbb{P}(\mathbb{B}_n) \cong \mathbb{P}(\mathbb{Q})^n$. By Lemma 5.1 there is a countable maximal antichain $A = \{A_j : j \in \omega\}$ in $\mathbb{P}(\mathbb{Q})$ and, defining $\bar{A}_j := \langle A_j, \mathbb{Q}, \ldots, \mathbb{Q} \rangle \in \mathbb{P}(\mathbb{Q})^n$, for $j \in \omega$, we obtain a countable antichain $\mathcal{A} := \{\bar{A}_j : j \in \omega\}$ in the product $\mathbb{P}(\mathbb{Q})^n$. Now, if $\bar{C} := \langle C_0, \ldots, C_{n-1} \rangle \in \mathbb{P}(\mathbb{Q})^n$, then, by the maximality of $A$, there are $j \in \omega$ and $C \in \mathbb{P}(\mathbb{Q})$ such that $C \subset C_0 \cap A_j$ and for $\bar{D} := \langle C, C_1, \ldots, C_{n-1} \rangle$ in the poset $\mathbb{P}(\mathbb{Q})^n$ we have $\bar{D} \leq \bar{C}$ and $\bar{D} \leq \bar{A}_j$. Thus $\mathcal{A}$ is a maximal antichain in the product $\mathbb{P}(\mathbb{Q})^n$.

(b) It is easy to see that the copies of $\mathbb{B}_\omega$ are of the form $\bigcup_{i \in S} \{i\} \times C_i$, where $S \in [\omega]^\omega$ and $C_i \in \mathbb{P}(\mathbb{Q})$, for all $i \in S$ (see the proof of Theorem 5.2 of [10]). Thus, the poset $\mathbb{P}(\mathbb{B}_\omega)$ is isomorphic to the countable support product $\prod_{i \in \omega}^{\text{cs}} \mathbb{P}_i$, where $\mathbb{P}_i = \langle \mathbb{P}(\mathbb{Q}) \cup \{\emptyset\}, \subset \rangle$, for all $i \in \omega$, and we apply Lemma 5.2. $\square$

## 5.2   The posets $\mathbb{C}_n$ (dense antichains)

It is easy to check (see, for example, [10], p. 96) that

$$\mathbb{P}(\mathbb{C}_n) := \{n \times A : A \in \mathbb{P}(\mathbb{Q})\}, \text{ for } n < \omega, \qquad (5)$$

$$\mathbb{P}(\mathbb{C}_\omega) := \{\textstyle\bigcup_{q \in A} C_q \times \{q\} : A \in \mathbb{P}(\mathbb{Q}) \wedge \forall q \in A \;\; C_q \in [\omega]^\omega\}. \qquad (6)$$

For $n \leq \omega$ and $Z \subset n \times \mathbb{Q}$, let $\operatorname{supp}(Z) = \{q \in \mathbb{Q} : Z \cap (n \times \{q\}) \neq \emptyset\}$. Notice that $Z \in \mathbb{P}(\mathbb{C}_n)$ implies $\operatorname{supp}(Z) \cong \mathbb{Q}$.

**Theorem 5.4** *For each $n$ satisfying $1 \leq n \leq \omega$ we have*

*(a) If $\mathcal{A}$ is a maximal antichain in $\mathbb{P}(\mathbb{Q})$, then $\mathcal{B} = \{n \times A : A \in \mathcal{A}\}$ is a maximal antichain in $\mathbb{P}(\mathbb{C}_n)$ and $|\mathcal{A}| = |\mathcal{B}|$;*

*(b) The poset $\mathbb{P}(\mathbb{C}_n)$ contains both $\omega$-sized and $\mathfrak{c}$-sized maximal antichains.*

**Proof.** (a) Clearly we have $\mathcal{B} \subset \mathbb{P}(\mathbb{C}_n)$. First we prove that $\mathcal{B}$ is an antichain. Assuming that for different $A, A' \in \mathcal{A}$ there is $Z \in \mathbb{P}(\mathbb{C}_n)$ such that $Z \subset n \times A, n \times A'$, we would have $Z \subset n \times (A \cap A')$ and $\mathbb{Q} \cong \mathrm{supp}(Z) \subset A \cap A'$, which is impossible since $\mathcal{A}$ is an antichain in $\mathbb{P}(\mathbb{Q})$.

Second we prove that $\mathcal{B}$ is a maximal antichain in $\mathbb{P}(\mathbb{C}_n)$. If $Z \in \mathbb{P}(\mathbb{C}_n)$, then $\mathrm{supp}(Z) \cong \mathbb{Q}$ and, by the maximality of $\mathcal{A}$, there are $A \in \mathcal{A}$ and $B \in \mathbb{P}(\mathbb{Q})$ such that $B \subset A \cap \mathrm{supp}(Z)$. Now, for $Y = \bigcup_{q \in B} Z \cap (n \times \{q\})$ we have $Y \subset Z \cap (n \times A)$ and $Y \in \mathbb{P}(\mathbb{C}_n)$ because for each $q \in \mathrm{supp}(Z)$ there is a bijection between $n$ and $Z \cap (n \times \{q\})$. So, $Y$ witnesses the compatibility of $Z$ and $n \times A \in \mathcal{B}$.

(b) follows from (a) and Lemma 5.1. □

## 5.3  The poset $\mathbb{D}$ (the random poset)

Recall that $\mathbb{D} = \langle D, < \rangle$ is the unique, up to isomorphism, countable ultrahomogeneous partial order which embeds all countable partial orders. Since the structure $\mathbb{D}$ satisfies the SAP, by Theorem 1.1 the poset $\mathbb{P}(\mathbb{D})$ contains maximal antichains of size $\mathfrak{c}$. So, Theorem 5.9 given below completes the proof of Theorem 1.3.

First we recall some definitions and facts from [10] which will be used in the proof of Theorem 5.9 (see Fact 3.1, Fact 3.2 and Lemma 3.3 of [10]) and note that $\|$ will denote the *incomparability relation*: $p \parallel q \Leftrightarrow p \neq q \wedge \neg p < q \wedge \neg q < p$.

**Definition 5.5** Let $\mathbb{P} = \langle P, < \rangle$ be a partial order. By $C(\mathbb{P})$ we denote the set of all triples $\langle L, G, U \rangle$ of pairwise disjoint finite subsets of $P$ such that:

(C1) $\forall l \in L \ \ \forall g \in G \ \ l < g$,

(C2) $\forall u \in U \ \forall l \in L \ \neg u < l$,

(C3) $\forall u \in U \ \forall g \in G \ \neg g < u$.

For $\langle L, G, U \rangle \in C(\mathbb{P})$, let $P_{\langle L, G, U \rangle}$ be the set of all $p \in P \setminus (L \cup G \cup U)$ satisfying:

(S1) $\forall l \in L \ \ p > l$,

(S2) $\forall g \in G \ p < g$,

(S3) $\forall u \in U \ p \parallel u$.

**Fact 5.6** *A countable partial order $\mathbb{P} = \langle P, < \rangle$ is (isomorphic to) a countable random poset iff $P_{\langle L, G, U \rangle} \neq \emptyset$, for each $\langle L, G, U \rangle \in C(\mathbb{P})$.*

**Fact 5.7** *Let $\mathbb{P} = \langle P, < \rangle$ be a partial order and $\emptyset \neq A \subset P$. Then*

*(a) $C(A, <) = \{\langle L, G, U \rangle \in C(\mathbb{P}) : L, G, U \subset A\}$;*

*(b) $A_{\langle L, G, U \rangle} = P_{\langle L, G, U \rangle} \cap A$, for each $\langle L, G, U \rangle \in C(A, <)$.*

**Fact 5.8** *Let $\mathbb{D} = \langle D, < \rangle$ be a countable random poset. Then*
- *(a) $D_{\langle L,G,U \rangle} \in \mathbb{P}(\mathbb{D})$, for each $\langle L, G, U \rangle \in C(\mathbb{D})$;*
- *(b) If $C \subset D$ and $A \not\subset C$, for each $A \in \mathbb{P}(\mathbb{D})$, then $D \setminus C \in \mathbb{P}(\mathbb{D})$;*
- *(c) If $\mathcal{L} \subset \mathbb{P}(\mathbb{D})$ is a chain, then $\bigcup \mathcal{L} \in \mathbb{P}(\mathbb{D})$.*

**Theorem 5.9** *There is a countable maximal antichain in $\mathbb{P}(\mathbb{D})$.*

**Proof.** Let $\mathcal{C}$ be a maximal chain in $\mathbb{D}$. Assuming that $x = \max \mathcal{C}$ (resp. $x = \min \mathcal{C}$) we would have $\mathbb{D}_{\langle \{x\}, \emptyset, \emptyset \rangle} = \emptyset$ (resp. $\mathbb{D}_{\langle \emptyset, \{x\}, \emptyset \rangle} = \emptyset$). Thus $\mathcal{C}$ is an unbounded chain in $\mathbb{D}$ and, since $|\mathcal{C}| = \omega$, it has a cofinal subset isomorphic to $\omega$ and a coinitial subset isomorphic to $\omega^*$. This implies that $\mathbb{D}$ contains an unbounded copy of the integers. W.l.o.g. we suppose that $\mathbb{Z}$ itself is that copy. For $m \in \mathbb{Z}$, let $A_m = \{x \in D : x < m\}$ and $X_m = A_m \setminus (A_{m-1} \cup \{m-1\})$; that is,

$$X_m = \{x \in D : x < m \wedge x \not\leq m - 1\}. \tag{7}$$

*Claim.* $X_m \in \mathbb{P}(\mathbb{D})$, for every $m \in \mathbb{Z}$.

*Proof of Claim.* We show that $X_m \cong \mathbb{D}$ using Fact 5.6. If $\langle L, G, U \rangle \in C(X_m)$, then by Fact 5.7(a) we have $\langle L, G, U \rangle \in C(\mathbb{D})$ and $L, G, U \subset X_m$. By Fact 5.7(b), we have to show that $D_{\langle L,G,U \rangle} \cap X_m \neq \emptyset$. There are four cases.

*Case I*: $L \neq \emptyset$ and $G \neq \emptyset$. Since $\langle L, G, U \rangle \in C(\mathbb{D})$, there is $d \in \mathbb{D}_{\langle L,G,U \rangle}$ and we show that $d \in X_m$. Since $\emptyset \neq G \subset X_m$, for $g \in G$ we have $d < g < m$. Assuming that $d \leq m - 1$, for $l \in L$ we would have $l < d \leq m - 1$, which is false because $L \subset X_m$ and, by (7), $l \not\leq m - 1$. So, $d \not\leq m - 1$ and, by (7), $d \in X_m$.

*Case II*: $L = \emptyset$ and $G \neq \emptyset$. Suppose that $\langle \emptyset, G, U \cup \{m-1\} \rangle \notin C(\mathbb{D})$. Then (C3) fails and, since $\langle \emptyset, G, U \rangle \in C(\mathbb{D})$, there is $g \in G$ such that $g < m - 1$. But, since $G \subset X_m$, this is impossible by (7). Thus $\langle \emptyset, G, U \cup \{m-1\} \rangle \in C(\mathbb{D})$ and, hence, there is $d \in \mathbb{D}_{\langle \emptyset, G, U \cup \{m-1\} \rangle}$, which implies $d \in \mathbb{D}_{\langle \emptyset, G, U \rangle}$. We prove that $d \in X_m$. First, for $g \in G$ we have $d < g < m$. Second, since $d \parallel m - 1$, we have $d \not\leq m - 1$ and, by (7), $d \in X_m$ indeed.

*Case III*: $L \neq \emptyset$ and $G = \emptyset$. Consider the condition $\langle L, \{m\}, U \rangle$. Since $L \subset X_m$, by (7) we have $l < m$, for all $l \in L$, and (C1) is true. (C2) is true because $\langle L, \emptyset, U \rangle \in C(\mathbb{D})$. Since $U \subset X_m$, by (7) for each $u \in U$ we have $\neg m < u$ and (C3) is true. So, $\langle L, \{m\}, U \rangle \in C(\mathbb{D})$, there is $d \in \mathbb{D}_{\langle L, \{m\}, U \rangle} \subset \mathbb{D}_{\langle L, \emptyset, U \rangle}$ and we show that $d \in X_m$. Clearly $d < m$ and $d \leq m - 1$ would imply that $l < d \leq m - 1$, for some $l \in L$, which is false because $\emptyset \neq L \subset X_m$. So, $d \not\leq m - 1$ and $d \in X_m$.

*Case IV*: $L = \emptyset$ and $G = \emptyset$. Suppose that $\langle \emptyset, \{m\}, U \cup \{m-1\} \rangle \notin C(\mathbb{D})$. Then (C3) fails so there is $u \in U \cup \{m-1\}$ such that $m < u$. Since $m \not< m - 1$ we have $u \in U$, which is false because $u \in X_m$. Thus $\langle \emptyset, \{m\}, U \cup \{m-1\} \rangle \in$

$C(\mathbb{D})$, there is $d \in \mathbb{D}_{\langle \emptyset, \{m\}, U \cup \{m-1\}\rangle} \subset \mathbb{D}_{\langle \emptyset, \emptyset, U\rangle}$ and we show that $d \in X_m$. Clearly $d < m$ and $d \parallel m - 1$ gives $d \not\leq m - 1$; so, $d \in X_m$ indeed. $\qquad \square$

*Claim.* $X = \bigcup_{m \in \mathbb{Z}} X_m$ is isomorphic to the random poset.

*Proof of Claim.* By Fact 5.8(a) we have $A_m = \mathbb{D}_{\langle \emptyset, \{m\}, \emptyset \rangle} \in \mathbb{P}(\mathbb{D})$, for all $m \in \mathbb{Z}$. Since $\{A_m : m \in \mathbb{Z}\}$ is a chain in $\mathbb{P}(\mathbb{D})$, Fact 5.8(c) gives $A = \bigcup_{m \in \mathbb{Z}} A_m \in \mathbb{P}(\mathbb{D})$ Since for each $m \in \mathbb{Z}$ we have $X_m \subset A_m$ and $X_m \cap \mathbb{Z} = \emptyset$ it follows that $X \subset A \setminus \mathbb{Z}$. Conversely, since $\bigcap_{m \in \mathbb{Z}} A_m = \emptyset$ (by the unboundedness of $\mathbb{Z}$), for $x \in A \setminus \mathbb{Z}$ there is $m \in \mathbb{Z}$ such that $x \in A_m \setminus A_{m-1}$ and, since $x \neq m - 1$, we have $x \in X_m \subset X$; so $X = A \setminus \mathbb{Z}$. Since $A \cong \mathbb{D}$ and $\mathbb{Z}$ does not contain copies of $\mathbb{D}$, by Fact 5.8(b) we have $A \setminus \mathbb{Z} \in \mathbb{P}(\mathbb{D})$ that is, $X \in \mathbb{P}(\mathbb{D})$. $\qquad \square$

Finally, we prove that $\mathcal{A} = \{X_m : m \in \mathbb{Z}\}$ is a maximal antichain in $\mathbb{P}(X)$. Since $X_m \cap X_n = \emptyset$, for different $m, n \in \mathbb{Z}$, $\mathcal{A}$ is an antichain in $\mathbb{P}(X)$. For a proof of its maximality we take $C \in \mathbb{P}(X)$ and first, towards a contradiction, suppose that

$$\forall m \in \mathbb{Z} \ \ \forall x, y \in C \cap X_m \ \ (x \neq y \Rightarrow x \parallel y). \tag{8}$$

Let us fix $x \in C$ and $m_0 \in \mathbb{Z}$, where $x \in C \cap X_{m_0}$. Since $C_{\langle \{x\}, \emptyset, \emptyset \rangle} \neq \emptyset$ there are $m \geq m_0$ and $z \in C \cap X_m$ such that $x < z$, which by (8) implies that $m > m_0$. Thus the set $M := \{m > m_0 : \exists z \in C \cap X_m \ \ x < z\}$ is non-empty. Let $m_1 = \min M$ and let us pick $z \in C \cap X_{m_1}$ such that $x < z$. Then, since $C_{\langle \{x\}, \{z\}, \emptyset \rangle} \neq \emptyset$ there are $m \in \mathbb{Z}$ and $y \in C \cap X_m$ such that $x < y < z$. Now, $m \in M$, we have $m_0 \leq m \leq m_1$, and, by (8), $m_0 < m < m_1$, which is impossible because $m_1 = \min M$.

Thus (8) is false and, hence, there are $m \in \mathbb{Z}$ and $x, y \in X_m \cap C$ such that $x < y$. Since $C \in \mathbb{P}(X)$, by Fact 5.8(a) we have $C_{\langle \{x\}, \{y\}, \emptyset \rangle} \in \mathbb{P}(X)$. If $t \in C_{\langle \{x\}, \{y\}, \emptyset \rangle}$, then $x < t < y < m$ and $t \leq m - 1$ would imply $x < m - 1$, which is false because $x \in X_m$; so, $t \in X_m$. Thus, $C_{\langle \{x\}, \{y\}, \emptyset \rangle} \subset C \cap X_m$, that is, $C$ and $X_m$ are compatible elements of $\mathbb{P}(X)$ and $\mathcal{A}$ is a maximal antichain in $\mathbb{P}(X)$. $\quad \square$

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