

COFINAL TYPES ON ω_2

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ABSTRACT. In this paper we start the analysis of the class \mathcal{D}_{\aleph_2} , the class of cofinal types of directed sets of cofinality at most \aleph_2 . We compare elements of \mathcal{D}_{\aleph_2} using the notion of Tukey reducibility. We isolate some simple cofinal types in \mathcal{D}_{\aleph_2} , and then proceed to show which of these types have an immediate successor in the Tukey ordering of \mathcal{D}_{\aleph_2} .

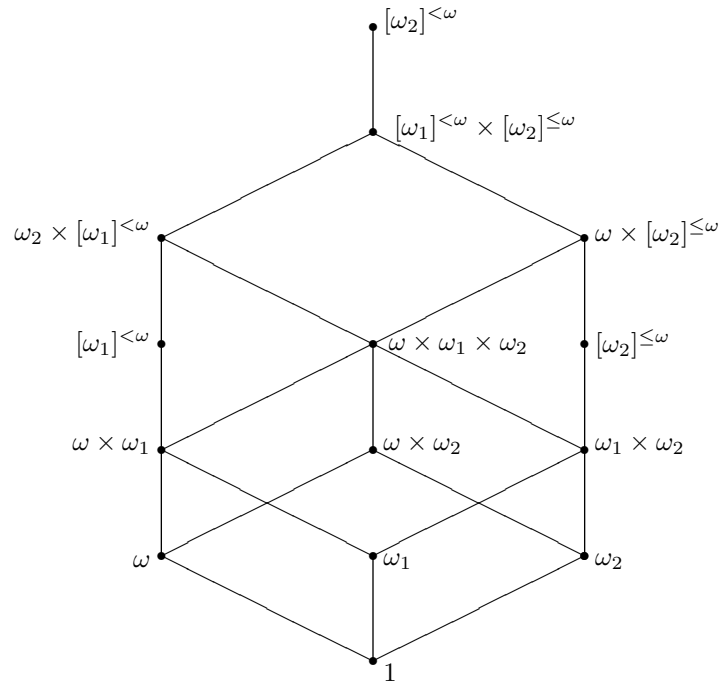


FIGURE 1. Tukey ordering of simple elements of the class \mathcal{D}_{\aleph_2}

1. INTRODUCTION

The purpose of this paper is to start the analysis of the class \mathcal{D}_{\aleph_2} , the class of directed sets of cofinality at most \aleph_2 . Up to now, there is a satisfactory understanding of cofinal types of directed sets of cardinality at most \aleph_1 . This line of research originates in the work of Tukey in [19], and in the work of Birkhoff in [1] and Day in [2]. Afterwards, Isbell in [7, 8] and Schmidt in [13], continued with finer

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analysis of the class of cofinal types of all directed sets. In the early 1980s, the second author in [18] completed Isbell's investigation of cofinal types of directed sets of cardinality at most \aleph_1 , by proving that only five cofinal types in \mathcal{D}_{\aleph_1} can be found in ZFC without additional set-theoretic assumptions. These are 1 , ω , ω_1 , $\omega \times \omega_1$, and $[\omega_1]^{<\omega}$. In the same paper, the second author proved that, assuming the Continuum Hypothesis, this class can be very rich. In particular, under CH, the class \mathcal{D}_{\aleph_1} contains 2^c pairwise cofinally non-equivalent directed sets. After that, he was also able to extend those results to all transitive relations on ω_1 in [17].

It is worth mentioning, although it is not directly connected to our work in this paper, that there has been a significant amount of work on cofinal types of definable directed sets, and cofinal types of ultrafilters when viewed as directed sets. The work on definable directed sets is due to the second author and Solecki in [14, 15]. The work on cofinal types of ultrafilters viewed as directed sets started with the work of Milovich in [10], and continued through papers of the second author with Dobrinen [5, 4, 3], and with Raghavan in [12]. Most recently, Raghavan and Shelah in [11], and the first author and Raghavan in [9] contributed to this topic.

The structure of the simplest directed sets in \mathcal{D}_{\aleph_2} is presented in Figure 1. Note that the order of a directed set in this picture is always assumed to be the standard one, $<$ for ordinals, \subseteq for families of sets, and the product ordering for products of directed sets. At the top is $[\omega_2]^{<\omega}$, the maximal cofinal type of directed sets of cardinality \aleph_2 . At the bottom is 1 , the minimal cofinal type of all directed sets. All the other simple types are obtained as products of ω , ω_1 , ω_2 , $[\omega_1]^{<\omega}$, and $[\omega_2]^{\leq\omega}$. Note that the set $[\omega_2]^{\leq\omega}$ is not of cardinality \aleph_2 , but it contains a cofinal subset \mathfrak{D} of cardinality \aleph_2 whose every uncountable subset is unbounded in $[\omega_2]^{\leq\omega}$, as Lemma 2.3 shows.

The paper is organized as follows. In the second section we introduce all the relevant notions. In the third section we prove some basic inequalities between cofinal types of directed set in \mathcal{D}_{\aleph_2} . In this section we prove that all these inequalities are strict, i.e. these cofinal types are all different with respect to cofinal equivalence. In the fourth section we prove that there are gaps in the Tukey ordering of \mathcal{D}_{\aleph_2} . In particular, we show that besides the already known gaps, there is neither a directed set D such that $[\omega_1]^{<\omega} <_T D <_T \omega_2 \times [\omega_1]^{\omega_1}$ nor D such that $[\omega_2]^{\leq\omega} <_T D <_T \omega \times [\omega_2]^{\leq\omega}$. In the fifth section we prove that, under certain set-theoretic assumptions, there is a directed set between $\omega_1 \times \omega_2$ and $[\omega_2]^{\leq\omega}$. Namely, we prove:

Theorem 1.1. *Assume GCH and that there is a non-reflecting stationary subset of S_0^2 . Then there is a directed set D such that $\omega_1 \times \omega_2 <_T D <_T [\omega_2]^{\leq\omega}$*

Finally, in the last section, we show that, under the same set-theoretic assumptions, there is a directed set between $[\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega}$ and $[\omega_2]^{<\omega}$.

Theorem 1.2. *Assume GCH and that there is a non-reflecting stationary subset of S_0^2 . Then there is a directed set D such that $[\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega} <_T D <_T [\omega_2]^{<\omega}$.*

2. PRELIMINARIES

We use standard set theoretic notation. In particular, if A is a set and κ is a cardinal, then $[A]^\kappa$ denotes the set of all subsets of A of cardinality κ . Thus, for example, $[A]^1 = \{\{x\} : x \in A\}$, whereas $[A]^\omega$ is the set of all countably infinite

subsets of the set A , and $[A]^{\leq\omega}$ is the set of all at most countable subsets of A . For a function $f : X \rightarrow Y$ and a set $A \subseteq X$, we denote $f''A = \{f(x) : x \in A\}$.

A partially ordered set $\langle X, < \rangle$ is *directed* if for any x and y in X there is some z in X such that $x < z$ and $y < z$. A directed partially ordered set is called just a *directed set*. A subset Y of a directed set X is *bounded* if there is some x in X such that $y \leq x$ for each y in Y . Otherwise, Y is *unbounded* in X . If D and E are two directed sets, we say that $f : D \rightarrow E$ is a *Tukey function* if $f''D$ is unbounded in E whenever D is unbounded in D . When there is a Tukey function from a directed set D into a directed set E , we say that D is *Tukey reducible to E* , and write $D \leq_T E$. If there is a Tukey function $f : D \rightarrow E$, but there is no Tukey function from E to D , then we write $D <_T E$. Note that \leq_T is a transitive relation. For two directed sets D and E , by results in [13] and [19], $D \leq_T E$ if and only if there is a function $g : E \rightarrow D$ such that for every $d \in D$ there is an $e \in E$ such that $g(e) \geq d$ for each $e' \geq e$. A function g with this property is called a *convergent map* from E to D . A subset C of a directed set D is said to be *cofinal* in D if for every d in D there is some c in C such that $d \leq c$. We say that two directed sets D and E are *cofinally similar* if they are isomorphic to cofinal subsets of a single partially ordered set. Recall that by results of Tukey in [19], two directed sets D and E are cofinally similar ($D \equiv_T E$) if and only if $D \leq_T E$ and $E \leq_T D$. He also showed that \equiv_T is an equivalence relation. The equivalence classes of this relation are called *cofinal types*. For a directed set D , its *cofinality* is the minimal cardinality of a cofinal subset of D . Recall that $\text{cf}(D)$ denotes the cofinality of a directed set D , and that \mathcal{D}_κ denotes the class of cofinal types of directed sets of cofinality at most κ .

Suppose that κ is a cardinal. A set $C \subseteq \kappa$ is a club in κ if it is closed and unbounded in κ . A set $S \subseteq \kappa$ is stationary in κ if $S \cap C \neq \emptyset$ for every club C in κ . A set S stationary in κ is said to be *non-reflecting* if for every $\gamma < \kappa$ of uncountable cofinality, the set $S \cap \gamma$ is not stationary in γ . Regarding notation, we will also be using $S_0^2 = \{\alpha < \omega_2 : \text{cf}(\alpha) = \omega\}$ and $S_1^2 = \{\alpha < \omega_2 : \text{cf}(\alpha) = \omega_1\}$.

For directed sets D_i ($i \in I$), their product is the set $\prod_{i \in I} D_i$ equipped with a relation \leq defined as follows: $\langle d_i : i \in I \rangle \leq \langle e_i : i \in I \rangle$ iff $d_i \leq_{D_i} e_i$ for each $i \in I$. If $j \in I$, then $\pi_{D_j} : \prod_{i \in I} D_i \rightarrow D_j$ denotes the projection to D_j , i.e. $\pi_{D_j}(d_i : i \in I) = d_j$.

Lemma 2.1. *Suppose that D_1, \dots, D_n are directed sets. If $D_1, \dots, D_n \leq_T D$, then $D_1 \times \dots \times D_n \leq_T D$.*

Proof. By [16, Proposition 2], $D_1 \times \dots \times D_n$ is the least upper bound of D_1, \dots, D_n . Since $D_1, \dots, D_n \leq_T D$, it must be that $D_1 \times \dots \times D_n \leq_T D$. \square

Lemma 2.2. *Suppose that D and E are Tukey incomparable cofinal types. Then there is no Tukey map from $D \times E$ to either D or E .*

Proof. By Proposition 2 of [16], both D and E are Tukey below $D \times E$. If there were a Tukey map $f : D \times E \rightarrow E$, then we would have $D \leq_T D \times E \leq_T E$, and consequently $D \leq_T E$ which is in contradiction with the assumption of the lemma. \square

Lemma 2.3. *Directed set $[\omega_2]^{\leq\omega}$ contains a cofinal subset \mathfrak{D} of size \aleph_2 with the property that every uncountable subset of \mathfrak{D} is unbounded in $[\omega_2]^{\leq\omega}$. In particular, $[\omega_2]^{\leq\omega}$ belongs to \mathcal{D}_{\aleph_2} , i.e. $\text{cf}([\omega_2]^{\leq\omega}) \leq \aleph_2$.*

Proof. First, for each $\alpha < \omega_2$ fix an injection $e_\alpha : \alpha + 1 \rightarrow \omega_1$ such that $e_\alpha(\alpha) = 0$. Now for every $\gamma < \omega_1$ denote

$$F_\gamma(\alpha) = \{\xi \leq \alpha : e_\alpha(\xi) < \gamma\}.$$

Note that for every $\omega_1 \leq \alpha < \omega_2$ there is an unbounded set $C_\alpha \subseteq \omega_1$ such that $\gamma \subseteq F_\gamma(\alpha)$ whenever $\gamma \in C_\alpha$. To see this suppose it is not the case, i.e. that there is a $\delta < \omega_1$ such that for each $\gamma > \delta$ we have $\gamma \not\subseteq F_\gamma(\alpha)$. This means that for every $\gamma > \delta$ there is $\xi_\gamma < \gamma$ such that $e_\alpha(\xi_\gamma) \geq \gamma$. Let $f : \omega_1 \setminus (\delta + 1) \rightarrow \omega_1$ be given by $f(\gamma) = \xi_\gamma$. By the pressing down lemma, there is a stationary set $S \subseteq \omega_1 \setminus (\delta + 1)$ (stationary subset of ω_1), such that $\xi_\gamma = \xi$ for each $\gamma \in S$. But then $e_\alpha(\xi) > \gamma$ for every $\gamma \in S$ which is not possible because S is unbounded in ω_1 . Thus, we showed that there is an unbounded $C_\alpha \subseteq \omega_1$ such that $\gamma \subseteq F_\gamma(\alpha)$ for every $\gamma \in C_\alpha$. Define now

$$\mathfrak{D} = \{F_\gamma(\alpha) : \omega_1 \leq \alpha < \omega_2 \text{ \& } \gamma \in C_\alpha\}.$$

We will prove that the set \mathfrak{D} is as required. Since α ranges over a subset of ω_2 and γ ranges over a subset of ω_1 , it is clear that \mathfrak{D} is of cardinality \aleph_2 . Since every e_α is 1-1 function, and every $\gamma \in C_\alpha$ is a countable ordinal, it follows that each $F_\gamma(\alpha)$ is a countable set. Thus $\mathfrak{D} \subseteq [\omega_2]^{\leq \omega}$. Next, we prove that \mathfrak{D} is cofinal in $[\omega_2]^{\leq \omega}$. Take any countable $B \subseteq \omega_2$. Then there is some $\omega_1 \leq \alpha < \omega_2$ such that $B \subseteq \alpha$. Since B is countable, e_α is 1-1, and C_α is unbounded in ω_1 , there is some $\gamma \in C_\alpha$ such that $e_\alpha'' B \subseteq \gamma$, i.e. $B \subseteq F_\gamma(\alpha) \in \mathfrak{D}$. Thus, \mathfrak{D} is cofinal in $[\omega_2]^{\leq \omega}$.

We still have to prove that every uncountable subset of \mathfrak{D} is unbounded in $[\omega_2]^{\leq \omega}$. Take any uncountable $X \subseteq \mathfrak{D}$. Let us enumerate $X = \{F_{\gamma_\xi}(\alpha_\xi) : \xi < \omega_1\}$. We consider two cases: when the set $\Lambda = \{\alpha_\xi : \xi < \omega_1\}$ is uncountable or when Λ is countable and the set $\Gamma = \{\gamma_\xi : \xi < \omega_1\}$ is uncountable. There is no other case possible because if both Λ and Γ were countable, then the set X would also be countable. If Λ is uncountable, then from definition of e_α , in particular from $e_\alpha(\alpha) = 0$, it follows that $\Lambda \subseteq \bigcup X$. Thus $\bigcup X$ is uncountable, so X cannot be bounded in $[\omega_2]^{\leq \omega}$. Suppose now that Γ is uncountable with Λ countable. Since each $\gamma_\xi \in \Gamma$ belongs to C_{α_ξ} , we know that $\gamma_\xi \subseteq F_{\gamma_\xi}(\alpha_\xi)$ for $\xi < \omega_1$. Since Γ is an uncountable set of ordinals, $\bigcup \Gamma$ is also uncountable. Now we have $\bigcup \Gamma = \bigcup_{\xi < \omega_1} \gamma_\xi \subseteq \bigcup_{\xi < \omega_1} F_{\gamma_\xi}(\alpha_\xi) = \bigcup X$, so $\bigcup X$ is again uncountable. Consequently, X cannot be bounded in $[\omega_2]^{\leq \omega}$. We showed that in both cases X is unbounded, so we conclude that every uncountable subset of \mathfrak{D} is unbounded in $[\omega_2]^{\leq \omega}$ \square

For the remaining of this paper \mathfrak{D} will denote the directed set defined in the proof of Lemma 2.3, thus \mathfrak{D} is a cofinal subset of $[\omega_2]^{\leq \omega}$, the cardinality of \mathfrak{D} is \aleph_2 , and every uncountable subset of \mathfrak{D} is unbounded.

Remark 2.4. Note that since $[\omega_2]^\omega$ is cofinal in $[\omega_2]^{\leq \omega}$ we have

$$[\omega_2]^\omega \equiv_T [\omega_2]^{\leq \omega} \equiv_T \mathfrak{D},$$

and so, depending on the situation, we will be using these three forms of the same cofinal type of a directed set \mathfrak{D} .

3. BASIC INEQUALITIES IN \mathcal{D}_{\aleph_2}

In this section we list why all the inequalities claimed in Figure 1 hold. Some were already known (the ones about \mathcal{D}_{\aleph_1}), and some follow in a similar manner. We also explain why some orders from Figure 1 are Tukey incomparable.

Lemma 3.1 (see [16]). *There are Tukey maps:*

- $f_0 : 1 \rightarrow \omega_1$;
- $f_1 : 1 \rightarrow \omega$;
- $f_2 : \omega \rightarrow \omega \times \omega_1$;
- $f_3 : \omega_1 \rightarrow \omega \times \omega_1$;
- $f_4 : \omega \times \omega_1 \rightarrow [\omega_1]^{<\omega}$.

Lemma 3.2. *There are Tukey maps:*

- $f_5 : 1 \rightarrow \omega_2$;
- $f_6 : \omega_2 \rightarrow \omega \times \omega_2$;
- $f_7 : \omega_2 \rightarrow \omega_1 \times \omega_2$;
- $f_8 : \omega_1 \rightarrow \omega_1 \times \omega_2$;
- $f_9 : [\omega_1]^{<\omega} \rightarrow \omega_2 \times [\omega_1]^{<\omega}$;
- $f_{10} : \omega \rightarrow \omega \times \omega_2$;
- $f_{11} : \omega \times \omega_1 \times \omega_2 \rightarrow \omega_2 \times [\omega_1]^{<\omega}$;
- $f_{12} : \omega \times \omega_2 \rightarrow \omega \times \omega_1 \times \omega_2$;
- $f_{13} : \omega \times \omega_1 \rightarrow \omega \times \omega_1 \times \omega_2$;
- $f_{14} : \omega_1 \times \omega_2 \rightarrow \omega \times \omega_1 \times \omega_2$;
- $f_{15} : [\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega} \rightarrow [\omega_2]^{<\omega}$.

Proof. To see that inequality (11) is true, first note that by [16, Proposition 2], $\omega, \omega_1, \omega_2 \leq_T \omega_2 \times [\omega_1]^{\leq\omega}$. Now Lemma 2.1 implies that $\omega \times \omega_1 \times \omega_2 \leq_T \omega_2 \times [\omega_1]^{<\omega}$, i.e. there is a required Tukey map. Inequalities (5) and (15) are obvious, and inequalities (6)-(10) and (12-14) follow directly from [16, Proposition 2]. \square

Lemma 3.3. *There is a Tukey map $f_{16} : \omega_1 \times \omega_2 \rightarrow [\omega_2]^{\leq\omega}$.*

Proof. Fix a bijection $g : \omega_1 \times \omega_2 \rightarrow \omega_2$, and let $f(\alpha, \beta) = \{g(\xi, \beta) : \xi < \alpha\}$ for $\langle \alpha, \beta \rangle \in \omega_1 \times \omega_2$. Note that $f(\alpha, \beta)$ is a countable subset of ω_2 for each $\langle \alpha, \beta \rangle \in \omega_1 \times \omega_2$. To see that f is a Tukey map, let X be an unbounded subset of $\omega_1 \times \omega_2$. Then, either $\pi''_{\omega_1} X$ is unbounded in ω_1 , or $\pi''_{\omega_2} X$ is unbounded in ω_2 .

Suppose first that $\pi''_{\omega_1} X$ is unbounded in ω_1 , i.e. $\pi''_{\omega_1} X$ is of cardinality \aleph_1 . For each $\alpha \in \pi''_{\omega_1} X$ choose β_α such that $\langle \alpha, \beta_\alpha \rangle \in X$. Since

$$\bigcup_{\alpha \in \pi''_{\omega_1} X} \{g(\xi, \beta_\alpha) : \xi < \alpha\} \subseteq f'' X,$$

since g is a bijection, and since $\pi''_{\omega_1} X$ is of size \aleph_1 , it must be that $\bigcup f'' X$ is of cardinality at least \aleph_1 , thus unbounded in $[\omega_2]^{\leq\omega}$.

Suppose now that $\pi''_{\omega_2} X$ is unbounded in ω_2 , i.e. $\pi''_{\omega_2} X$ is of cardinality \aleph_2 . Then there is $\alpha < \omega_1$ such that $\Sigma = \{\beta < \omega_2 : \langle \alpha, \beta \rangle \in X\}$ is of cardinality \aleph_2 , so since g is a bijection and

$$\bigcup_{\beta \in \Sigma} \{g(\xi, \beta) : \xi < \alpha\} \subseteq f'' X,$$

it must be that $\bigcup f'' X$ is of cardinality \aleph_2 , thus unbounded in $[\omega_2]^{\leq\omega}$. \square

Lemma 3.4. *There are Tukey maps:*

- $f_{17} : [\omega_2]^{\leq\omega} \rightarrow \omega \times [\omega_2]^{\leq\omega}$;
- $f_{18} : \omega \times \omega_1 \times \omega_2 \rightarrow \omega \times [\omega_2]^{\leq\omega}$;
- $f_{19} : \omega \times [\omega_2]^{\leq\omega} \rightarrow [\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega}$;
- $f_{20} : \omega_2 \times [\omega_1]^{<\omega} \rightarrow [\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega}$.

Proof. Inequality (17) follows from [16, Proposition 2]. Inequality (18) follows Lemma 2.1 applied to inequality (16). Inequality (19) follows Lemma 2.1 applied to inequalities (2) and (4). Inequality (20) follows Lemma 2.1 applied to inequalities (7) and (16). \square

Lemma 3.5. *There is no Tukey map from $[\omega_2]^{\leq\omega}$ to $\omega_1 \times \omega_2$.*

Proof. Suppose that $f : [\omega_2]^{\leq\omega} \rightarrow \omega_1 \times \omega_2$ is a Tukey map.

If $|\pi_{\omega_2}(f''[\omega_2]^1)| \leq \aleph_1$, then there is a set $Y \subseteq [\omega_2]^1$ of cardinality \aleph_2 , and an ordinal $\gamma < \omega_2$ such that $\pi_{\omega_2}(f(x)) = \gamma$ for each x in Y . Since Y is of cardinality \aleph_2 , there is a set $Z \subseteq Y$ of cardinality \aleph_2 , and an ordinal $\beta < \omega_1$ such that $f(x) = \langle \beta, \gamma \rangle$ for each x in Z . So $f^{-1}(\beta, \gamma)$ is unbounded in $[\omega_2]^{\leq\omega}$, contradicting the fact that f is a Tukey map.

If $|\pi_{\omega_2}(f''[\omega_2]^1)| = \aleph_2$, then there is a set $Y \subseteq \omega_2$ of cardinality \aleph_2 , and an ordinal $\beta < \omega_1$ such that $\langle \beta, \gamma \rangle \in f''[\omega_2]^1$ for each γ in Y . Let W be any subset of $\{\beta\} \times Y$ of cardinality \aleph_1 . The set W is bounded in $\omega_1 \times \omega_2$. Since W is uncountable, the set $f^{-1}W$ is unbounded in $[\omega_2]^{\leq\omega}$, again contradicting the assumption that f is a Tukey map. \square

Lemma 3.6. *$\omega \times \omega_1 \times \omega_2$ and $[\omega_2]^{\leq\omega}$ are incomparable cofinal types.*

Proof. Since $\omega \times \{\omega\} \times \{\omega\}$ is unbounded in $\omega \times \omega_1 \times \omega_2$, and every countable set in $[\omega_2]^{\leq\omega}$ is bounded, there is no Tukey function from $\omega \times \omega_1 \times \omega_2$ into $[\omega_2]^{\leq\omega}$. Thus $\omega \times \omega_1 \times \omega_2 \not\leq_T [\omega_2]^{\leq\omega}$. So suppose that there is a Tukey map $f : [\omega_2]^{\leq\omega} \rightarrow \omega \times \omega_1 \times \omega_2$.

If $|f''[\omega_2]^1| \leq \aleph_1$, then there is a set $Y \subseteq [\omega_2]^1$ of cardinality \aleph_2 , and a triple $\langle \beta, \gamma, \delta \rangle$ such that $f(x) = \langle \beta, \gamma, \delta \rangle$ for each x in Y . Thus $f^{-1}(\beta, \gamma, \delta)$ is unbounded in $[\omega_2]^{\leq\omega}$, contradicting the assumption that f is a Tukey function.

If $|f''[\omega_2]^1| = \aleph_2$, then there is a set $Y \subseteq \omega_2$ of cardinality \aleph_2 , an integer $n < \omega$, and an ordinal $\beta < \omega_1$ such that the set $Z = \{n\} \times \{\beta\} \times Y$ is contained as a subset in $f''[\omega_2]^1$. Let W be any subset of Z of cardinality \aleph_1 . Thus W is bounded in $\omega \times \omega_1 \times \omega_2$. Since W is uncountable $f^{-1}W$ is unbounded in $[\omega_2]^{\leq\omega}$, again contradicting the assumption that f is a Tukey function. \square

Lemma 3.7. *$\omega \times \omega_1 \times \omega_2$ and $[\omega_1]^{<\omega}$ are incomparable cofinal types.*

Proof. Suppose first that there is a Tukey function f from $[\omega_1]^{<\omega}$ into $\omega \times \omega_1 \times \omega_2$. For every unbounded set $X \subseteq [\omega_1]^{<\omega}$, $\pi''_{\omega_2}(f''X)$ is bounded in ω_2 , thus $\pi''_{\omega \times \omega_1}(f''X)$ is unbounded in $\omega \times \omega_1$. Hence $\pi_{\omega \times \omega_1} \circ f$ is a Tukey function from $[\omega_1]^{<\omega}$ into $\omega \times \omega_1$, which is impossible.

Suppose now that $f : \omega \times \omega_1 \times \omega_2 \rightarrow [\omega_1]^{<\omega}$ is a Tukey function. Since $[\omega_1]^{<\omega}$ is of size \aleph_1 , there is a set $Y \subseteq \omega_2$ of cardinality \aleph_2 , an integer $n < \omega$, an ordinal $\beta < \omega_1$, and a finite set $F \subseteq \omega_1$ such that $f(n, \beta, \gamma) = F$ for each γ in Y . Since F is bounded in $[\omega_1]^{<\omega}$ and $\{n\} \times \{\beta\} \times Y$ is unbounded in $\omega \times \omega_1 \times \omega_2$, this contradicts the assumption that f is a Tukey function. \square

Lemma 3.8. *There is no Tukey function from $[\omega_2]^{\leq\omega}$ into $\omega_2 \times [\omega_1]^{<\omega}$.*

Proof. Assume that $f : [\omega_2]^{\leq\omega} \rightarrow \omega_2 \times [\omega_1]^{<\omega}$ is a Tukey function.

If $|\pi''_{\omega_2}(f''[\omega_2]^1)| \leq \aleph_1$, then there is a set $Y \subseteq [\omega_2]^1$ of cardinality \aleph_2 , and an ordinal $\beta < \omega_2$ such that $\pi_{\omega_2}(f(x)) = \beta$ for each x in Y . Since $[\omega_1]^{<\omega}$ is of size \aleph_1 , there is a set $Z \subseteq Y$ of size \aleph_2 , and a finite set $F \subseteq \omega_1$ such that $f(x) = \langle \beta, F \rangle$ for

each x in Z . Thus $f^{-1}(\beta, F)$ is unbounded in $[\omega_2]^{\leq \omega}$, contradicting the assumption that f is a Tukey map.

If $|\pi''_{\omega_2}(f''[\omega_2]^1)| = \aleph_2$, then since $[\omega_1]^{< \omega}$ is of size \aleph_1 , there is a set $Y \subseteq \omega_2$ of size \aleph_2 , and a finite set $F \subseteq \omega_1$ such that the set $Z = Y \times \{F\}$ is a subset of $f''[\omega_2]^1$. Let W be any subset of Z of size \aleph_1 . Then W is bounded in $\omega_2 \times [\omega_1]^{< \omega}$. Since W is uncountable, $f^{-1}W$ is unbounded in $[\omega_2]^{\leq \omega}$, again contradicting the assumption that f is a Tukey map. \square

Lemma 3.9. *There is no Tukey function from $[\omega_1]^{< \omega}$ into $\omega \times [\omega_2]^{\leq \omega}$.*

Proof. Suppose that there is a Tukey map $f : [\omega_1]^{< \omega} \rightarrow \omega \times [\omega_2]^{\leq \omega}$. If the set $f''([\omega_1]^{< \omega})$ is countable, then $\pi''_{[\omega_2]^{\leq \omega}}(f''([\omega_1]^{< \omega}))$ is bounded in $[\omega_2]^{\leq \omega}$. Thus for every unbounded $X \subseteq [\omega_1]^{< \omega}$, the set $\pi''_{\omega}(f''X)$ is unbounded in ω . Then $\pi_{\omega} \circ f$ would be a Tukey function from $[\omega_1]^{< \omega}$ into ω , which is impossible. Hence, the set $f''([\omega_1]^{< \omega})$ is uncountable. Then there is $n < \omega$, and an infinite set $X \subseteq [\omega_2]^{\leq \omega}$ such that $W = \{n\} \times X \subseteq f''([\omega_1]^{< \omega})$. Clearly, W is bounded in $\omega \times [\omega_2]^{\leq \omega}$. Since W is infinite, the set $f^{-1}W$ is unbounded in $[\omega_1]^{< \omega}$. This contradicts the assumption that f is a Tukey function. \square

Lemma 3.10. *$\omega_2 \times [\omega_1]^{< \omega}$ and $\omega \times [\omega_2]^{\leq \omega}$ are Tukey incomparable cofinal types.*

Proof. First notice that by Proposition 2 of [16], if $\omega_2 \times [\omega_1]^{< \omega} \leq_T \omega \times [\omega_2]^{\leq \omega}$, then $[\omega_1]^{< \omega} \leq_T \omega \times [\omega_2]^{\leq \omega}$ which is impossible by Lemma 3.9. On the other hand, if $\omega \times [\omega_2]^{\leq \omega} \leq_T \omega_2 \times [\omega_1]^{< \omega}$, then $[\omega_2]^{\leq \omega} \leq_T \omega_2 \times [\omega_1]^{< \omega}$ contradicting Lemma 3.8. \square

Lemma 3.11. *All the Tukey inequalities (0)-(20) are strict.*

Proof. First note that $\omega, \omega_1, \omega_2$ are incomparable cofinal types, and that they are strictly above 1, i.e. inequalities (0), (1), and (5) are strict. Lemma 2.2 implies that inequalities (2),(3),(6),(7),(8), and (10) are strict. For the same reason, and the fact that $\omega \times \omega_1, \omega \times \omega_2, \omega_1 \times \omega_2$ are obviously incomparable, inequalities (12),(13), and (14) are strict. Lemma 3.7 implies that inequalities (9) and (11) are strict, and strictness of the inequality (4) is known since the introduction of the notion of cofinal equivalence (see [16]). By Lemma 3.5 the inequality (16) is strict. Next, Lemma 3.6 implies that the inequalities (17) and (18) are strict. Similarly, that the inequalities (19) and (20) are strict follows from Lemma 3.10.

We still have to prove that the inequality (15) is strict. Suppose that f is a Tukey function from $[\omega_2]^{< \omega}$ into $[\omega_2]^{\leq \omega} \times [\omega_1]^{< \omega}$.

If $f''[\omega_2]^1$ is of size less than \aleph_2 , then there is a set $Y \subseteq [\omega_2]^1$ of cardinality \aleph_2 , a countable set $S \subseteq \omega_2$, and a finite set $F \subseteq \omega_1$ such that $f(x) = \langle S, F \rangle$ for each x in $[\omega_2]^1$. Thus $f^{-1}(S, F)$ is unbounded in $[\omega_2]^{< \omega}$ contradicting the assumption that f is a Tukey function.

If $f''[\omega_2]^1$ is of cardinality \aleph_2 , then there is a finite $F \subseteq \omega_1$ and an infinite set $Y \subseteq [\omega_2]^{\leq \omega}$ such that $Y \times \{F\} \subseteq f''[\omega_2]^1$. Let W be any countably infinite subset of $Y \times \{F\}$. Then W is bounded in $[\omega_2]^{\leq \omega} \times [\omega_1]^{< \omega}$ while $f^{-1}W$ is unbounded in $[\omega_2]^{< \omega}$, contradicting the assumption that f is a Tukey map. \square

4. GAPS IN \mathcal{D}_{\aleph_2}

Theorem 4.1. *There is no directed set D such that $[\omega_1]^{< \omega} <_T D <_T \omega_2 \times [\omega_1]^{< \omega}$.*

Proof. Note that it is enough to prove that if D is a directed set such that $[\omega_1]^{<\omega} <_T D \leq_T \omega_2 \times [\omega_1]^{<\omega}$, then $D \equiv_T \omega_2 \times [\omega_1]^{<\omega}$. So suppose that D is a directed set such that there are Tukey maps $f : [\omega_1]^{<\omega} \rightarrow D$ and $g : D \rightarrow \omega_2 \times [\omega_1]^{<\omega}$. Since $[\omega_1]^{<\omega} <_T D$, it must be that D is of cofinality \aleph_2 . For $s \in D$ let us denote $s^\uparrow = \{x \in D : s \leq x\}$. Denote also $E = f''([\omega_1]^{<\omega})$. Since f is a Tukey map, every infinite subset of E is unbounded in D . Enumerate $[\omega_1]^{<\omega} = \{s_\xi : \xi < \omega_1\}$. For $\alpha < \omega_1$ consider sets $D_\alpha = g^{-1}[\omega_2 \times \{s_\alpha\}]$. Clearly $D = \bigcup_{\alpha < \omega_1} D_\alpha$. Since g is a Tukey map, for each $\alpha < \omega_1$ if $X \subseteq D_\alpha$ is of cardinality \aleph_1 , then X is bounded in D . For each $\xi < \omega_1$ pick $\alpha(\xi)$ such that $\alpha(\xi) > \alpha(\nu)$ for all $\nu < \xi$, and that $f(s_\xi)^\uparrow \cap D_{\alpha(\xi)}$ is of cardinality \aleph_2 . Let us enumerate $f(s_\xi)^\uparrow \cap D_{\alpha(\xi)} = \{d_{\xi\gamma} : \gamma < \omega_2\}$ for each $\xi < \omega_1$. Now define $h : \omega_2 \times [\omega_1]^{<\omega} \rightarrow D$ so that $h(\gamma, s_\xi) = d_{\xi\gamma}$. Clearly h is well-defined, and we proceed to show that h is a Tukey map.

Suppose that $A \subset \omega_2 \times [\omega_1]^{<\omega}$ is unbounded. Then either there is an unbounded set $\Gamma \subseteq \omega_2$ and $\xi < \omega_1$ such that $\langle \gamma, \xi \rangle \in A$ whenever $\gamma \in \Gamma$, or there is an infinite set $\Sigma \subset \omega_1$ such that for every $\xi \in \Sigma$ there is $\gamma \in \omega_2$ such that $\langle \gamma, s_\xi \rangle \in A$. Assume first that there is such unbounded $\Gamma \subseteq \omega_2$, i.e. Γ is of cardinality \aleph_2 and $\Gamma \times \{s_\xi\} \subseteq A$ for a fixed $\xi < \omega_1$. Then $h''A \cap D_{\alpha(\xi)}$ is of size \aleph_2 , thus it is unbounded in D . Assume now that there is such infinite Σ , i.e. that $\Sigma \subseteq \pi''_{[\omega_1]^{<\omega}} A$. Then the set $\{\xi < \omega_1 : h''A \cap D_{\alpha(\xi)} \cap f(s_\xi)^\uparrow \neq \emptyset\}$ is infinite, so if it were bounded by some $d \in D$, there would be an infinite subset of E bounded by d which is impossible. Thus $h''A$ is unbounded in this case also. \square

Theorem 4.2. *There is no directed set D such that $[\omega_2]^{\leq\omega} <_T D <_T \omega \times [\omega_2]^{\leq\omega}$.*

Proof. Let, by Lemma 2.3, $\mathfrak{D} = \{s_\xi : \xi < \omega_2\}$ be a cofinal subset of $[\omega_2]^{\leq\omega}$ of size \aleph_2 such that every uncountable subset of \mathfrak{D} is unbounded. We may suppose that D is a directed set such that $\mathfrak{D} <_T D$, and that $D \leq_T \omega \times \mathfrak{D}$, and in that case it is enough to prove that $\omega \times \mathfrak{D} \leq_T D$. Let f be a Tukey function from \mathfrak{D} to D . Note that then $E = f''\mathfrak{D}$ is such that every uncountable subset of E is unbounded in D . On the other hand, let g be a Tukey function from D to $\omega \times \mathfrak{D}$. Denote $D_n = g^{-1}(\{n\} \times \mathfrak{D})$ for each $n < \omega$. Then $D = \bigcup_{n < \omega} D_n$. Now, for $k < \omega$ define

$$D'_k = D_k \setminus \{d \in D : (\exists n < k)(\exists x \in D_n) d \leq x\}.$$

Clearly, $\bigcup_{n < \omega} D'_n$ is a partition of D , and $D'_n \subseteq D_n$ for each $n < \omega$. Since g is a Tukey function, for each $n < \omega$ every countable subset of D_n is bounded in D . Thus, for each $n < \omega$ every countable subset of D'_n is bounded in D . Since E is of cardinality \aleph_2 , there is n_0 such that $|E \cap D'_{n_0}| = \aleph_2$. Denote $X_0 = E \cap D'_{n_0} = \{d_{\xi 0} : \xi < \omega_2\}$. For each $\xi < \omega_2$ pick $d'_{\xi 0} \in D \setminus D'_{n_0}$ such that $d_{\xi 0} < d'_{\xi 0}$. Denote $X'_0 = \{d'_{\xi 0} : \xi < \omega_2\}$. Now let $n_1 > n_0$ be such that $|X'_0 \cap D'_{n_1}| = \aleph_2$. Such an n_1 exists because for no $k < \omega$ is $\bigcup_{n < k} D'_n$ cofinal in D . Denote $X_1 = X'_0 \cap D'_{n_1}$ and enumerate $X_1 = \{d_{\xi 1} : \xi < \omega_2\}$. Again, for each $\xi < \omega_2$ pick $d'_{\xi 1} \in D \setminus \bigcup_{n \leq n_1} D'_n$ such that $d_{\xi 1} \leq d'_{\xi 1}$. Now suppose that $n_0 < \dots < n_m$, and sets $X_0, \dots, X_m, X'_0, \dots, X'_m$, and $D'_{n_0}, \dots, D'_{n_m}$ have been defined. Let us enumerate $X_m = \{d_{\xi m} : \xi < \omega_2\}$, and assume that $X'_m = \{d'_{\xi m} : \xi < \omega_2\}$ is such that for each $\xi < \omega_2$, $d'_{\xi m} \in D \setminus \bigcup_{n \leq m} D'_n$ is such that $d_{\xi m} < d'_{\xi m}$. As before, there is $n_{m+1} > n_m$ such that $|X'_m \cap D'_{n_{m+1}}| = \aleph_2$. Denote $X_{m+1} = X'_m \cap D'_{n_{m+1}} = \{d_{\xi m+1} : \xi < \omega_2\}$, and for each $\xi < \omega_2$ let $d'_{\xi m+1} \in D \setminus \bigcup_{n \leq n_{m+1}} D'_n$ be such that $d_{\xi m+1} \leq d'_{\xi m+1}$.

Then $X'_{m+1} = \{d'_{\xi_{m+1}} : \xi < \omega_2\}$. Finally, define $h : \omega \times \mathfrak{D} \rightarrow D$ as follows: for $m < \omega$ and $\xi < \omega_2$ let $h(m, s_\xi) = d_{\xi_{n_m}}$.

We will prove that h is a Tukey function which will conclude the proof. Suppose that $Y \subseteq \omega \times \mathfrak{D}$ is unbounded. Then either $\pi''_\omega Y$ is unbounded in ω or $\pi''_{\mathfrak{D}} Y$ is unbounded in \mathfrak{D} . If $\pi''_\omega Y$ is unbounded in ω , then the set $K = \{n < \omega : h''Y \cap D'_n \neq \emptyset\}$ is infinite. If there were $d \in D$ such that $x \leq d$ for all $x \in h''Y$, then there would be $m < \omega$ such that $d \in D'_m$. But this is impossible because by the definition of D'_m there is no $n > m$ and $x \in D'_n$ such that $x \leq d$, and consequently the set K cannot be infinite. Thus in this case $h''Y$ is unbounded. If $\pi''_{\mathfrak{D}} Y$ is unbounded in \mathfrak{D} , then for some $m < \omega$ the set $h''Y \cap D'_m$ is uncountable. Suppose that $d \in D$ is such that $x \leq d$ for each $x \in h''Y \cap D'_m$. But this is impossible because then there would be an uncountable subset of E bounded by d . Thus $h''Y$ is unbounded in this case also, and so h is a Tukey function from $\omega \times \mathfrak{D}$ to D . \square

5. DIRECTED SET BETWEEN $\omega_1 \times \omega_2$ AND $[\omega_2]^{\leq \omega}$

In this section we prove that, under certain set theoretic assumptions, there is a directed set D such that $\omega_1 \times \omega_2 <_T D <_T [\omega_2]^{\leq \omega}$. We will construct such a D using an \aleph_2 -Souslin tree. Assume, in this entire section, that GCH holds and that there is a non-reflecting stationary subset of S_0^2 . By the work of Gregory in [6], this implies that there is an \aleph_2 -Souslin tree. So suppose that $T \subseteq \omega_1^{<\omega_2}$ is an \aleph_2 -Souslin tree with the property that for each $t \in T$ and $\alpha < \omega_1$, the node $t^\frown \langle \alpha \rangle$ belongs to T . Let \leq denote the order in the tree T , and let T_α denote the α th level of the tree T . For $X \subseteq T$, let \widehat{X} be the downward closure in T of the set X . Finally, let D_T be the set of all subsets X of T such that for each $t \in T$, the set $\{\alpha < \omega_1 : t^\frown \langle \alpha \rangle \in \widehat{X}\}$ is non-stationary in ω_1 . We consider D_T ordered by inclusion. Since non-stationary sets in ω_1 form an ideal, D_T is a directed set.

Lemma 5.1. *Suppose that $X \subseteq T$ is of cardinality \aleph_2 . Then there is an element t of T such that for each $t' \geq t$ there is $x \in X$ such that $x \geq t'$.*

Proof. Suppose the contrary: that for each $t \in T$, there is $t' \geq t$ such that every $x \geq t'$ does not belong to X . By induction we construct an antichain $\langle t_\alpha : \alpha < \omega_2 \rangle$ in T . By applying the assumption on the root \emptyset we obtain t_0 such that every $x \geq t_0$ does not belong to X . Suppose now that we are in stage $\beta < \omega_2$, and that t_α has been chosen for each $\alpha < \beta$ in such a way that $\langle t_\alpha : \alpha < \beta \rangle$ is an antichain, and that for each $\alpha < \beta$, every $x \geq t_\alpha$ does not belong to X . Let δ be the supremum of levels of the nodes t_α ($\alpha < \beta$). Then $\widehat{T} \upharpoonright \delta$ is of cardinality \aleph_1 . Thus, there is a $t \in X$ on a level above δ , and such that $\{\widehat{t}\} \cap \{t_\alpha : \alpha < \beta\} = \emptyset$. By the assumption, there is $t_\beta \geq t$ such that every $x \geq t_\beta$ does not belong to X . Clearly, $\langle t_\alpha : \alpha \leq \beta \rangle$ is an antichain in T . This completes the definition of an antichain of size \aleph_2 in T , which is in contradiction with the assumption that T is an \aleph_2 -Souslin tree. \square

Corollary 5.2. *Every element X of D_T is of cardinality at most \aleph_1 . In particular, D_T is of cardinality \aleph_2 .*

Proof. Let $X \in D_T$, and suppose that X is of cardinality \aleph_2 . By Lemma 5.1 there is a t in T such that for each $t' \geq t$ there is $x \in X$ such that $x \geq t'$. This means that $\{\alpha < \omega_1 : t^\frown \langle \alpha \rangle \in \widehat{X}\} = \omega_1$, in particular this set is non-stationary, contradicting

the fact that X is in D_T . Thus X must be of cardinality less than \aleph_2 . Now, the standing assumption of GCH in this section implies that $|D_T| = \aleph_2$. \square

Corollary 5.3. *For every partition $D_T = \bigcup_{\gamma < \omega_1} D_\gamma$, there is an ordinal $\gamma < \omega_1$, and an unbounded $E \subseteq D_\gamma$ of size \aleph_1 .*

Proof. Recall that GCH is assumed in this section. Thus $|D_T| = \aleph_2$. So there is a $\gamma < \omega_1$ such that $|D_\gamma| = \aleph_2$. Then $|\bigcup D_\gamma| = \aleph_2$. By Lemma 5.1, there is a t in T such that for each $t' \geq t$ there is x in $\bigcup D_\gamma$ such that $x \geq t'$. Again, this implies that $\{\alpha < \omega_1 : t^\frown \langle \alpha \rangle \in \widehat{\bigcup D_\gamma}\} = \omega_1$. Now for every $\alpha < \omega_1$ pick X_α in D_γ such that $t^\frown \langle \alpha \rangle \in \widehat{X_\alpha}$. Let $E = \{X_\alpha : \alpha < \omega_1\}$. Since any upper bound Y for E would have to contain the union of E as a subset, we would have $\{\alpha < \omega_1 : t^\frown \langle \alpha \rangle \in \widehat{Y}\} = \omega_1$, in particular this set would be non-stationary. Thus E is unbounded in D_T . Clearly E is of cardinality \aleph_1 and a subset of D_γ . \square

Lemma 5.4. *If D' is a cofinal subset of D_T , then there is an uncountable subset of D' bounded in D_T .*

Proof. Let D' be a cofinal subset of D_T , and let $\{t_\gamma : \gamma < \omega_2\}$ be a level set in T . Note that $L \subseteq T$ is a level set if $|L \cap T_\gamma| = 1$ for each $\gamma < \omega_2$. Clearly, for every $\gamma < \omega_2$, $\{t_\gamma\} \in D_T$. Thus, for each $\gamma < \omega_2$ we can choose $S_\gamma \in D'$ such that $t_\gamma \in S_\gamma$. Let, for each $X \in D_T$ and $t \in T$, N_t^X denote the non-stationary set $\{\alpha < \omega_1 : t^\frown \langle \alpha \rangle \in \widehat{X}\}$, and let C_t^X denote the club in ω_1 disjoint with N_t^X .

Now fix a sufficiently large regular cardinal θ , and an elementary submodel $M \prec H(\theta)$ of cardinality \aleph_1 containing all the relevant objects and such that $M^{\aleph_0} \subseteq M$. Denote $\delta = M \cap \omega_2$. Then δ is a limit ordinal which belongs to ω_2 , so let $\langle \gamma_\xi : \xi < \omega_1 \rangle$ be an increasing sequence in δ such that $\sup_{\xi < \omega_1} \gamma_\xi = \delta$. Enumerate $T \upharpoonright \delta = \{s_\xi : \xi < \omega_1\}$ in such a way that if $s_{\xi_1} \leq s_{\xi_2}$, then $\xi_1 \leq \xi_2$. In order to simplify notation, let C_ξ^γ denote the set $C_{s_\xi}^{S_\gamma}$ for each $\gamma < \omega_2$ and $\xi < \omega_1$.

Now, by induction, we construct three sequences $\langle \delta_\xi : \xi < \omega_1 \rangle$, $\langle \Gamma_\xi : \xi < \omega_1 \rangle$, and $\langle \eta_\xi : \xi < \omega_1 \rangle$ with the following properties:

- (1) $C_{\xi_1}^\delta \cap \eta_{\xi_2} = C_{\xi_1}^{\delta_{\xi_2}} \cap \eta_{\xi_2}$ for $\xi_1 \leq \xi_2 < \omega_1$,
- (2) $\delta_{\xi_1} < \delta_{\xi_2} < \delta$ for $\xi_1 < \xi_2 < \omega_1$,
- (3) $\sup \{\delta_\xi : \xi < \omega_1\} = \delta$.

First consider s_0 . Let η_0 be the first limit point of C_0^δ . Let

$$\Gamma_0 = \{\gamma < \omega_2 : C_0^\delta \cap \eta_0 = C_0^\gamma \cap \eta_0\}.$$

Since $\delta \in \Gamma_0$, the set Γ_0 is stationary in ω_2 . Let $\delta_0 = \min \Gamma_0$.

Suppose now that $\xi_0 < \omega_1$, and that δ_ξ, Γ_ξ , and η_ξ have been constructed for each $\xi < \xi_0$. Let η_{ξ_0} be the first limit point of $C_{\xi_0}^\delta \setminus \sup \{\eta_\xi : \xi < \xi_0\}$. Consider the set

$$\Gamma_{\xi_0} = \left\{ \gamma \in \bigcap_{\xi < \xi_0} \Gamma_\xi : (\forall \xi \leq \xi_0) C_\xi^\delta \cap \eta_{\xi_0} = C_\xi^\gamma \cap \eta_{\xi_0} \right\}.$$

Since Γ_{ξ_0} belongs to M , and since $\delta \in \Gamma_{\xi_0}$, it must be that Γ_{ξ_0} is stationary in ω_2 . Since Γ_{ξ_0} is cofinal in ω_2 and belongs to M , the set $\delta \cap \Gamma_{\xi_0}$ is cofinal in δ . Define δ_{ξ_0} to be the minimal ordinal in $\delta \cap \Gamma_{\xi_0}$ greater than both $\sup_{\xi < \xi_0} \delta_\xi$ and γ_{ξ_0} (recall

that $\langle \gamma_\xi : \xi < \omega_1 \rangle$ is chosen to be cofinal in δ). It is clear from the construction that conditions (1-3) are satisfied.

Now we prove that $\{S_{\delta_\xi} : \xi < \omega_1\}$ is as required. It is sufficient to prove that $S = \bigcup_{\xi < \omega_1} S_{\delta_\xi} \in D_T$, as it will witness that $\{S_{\delta_\xi} : \xi < \omega_1\}$ is an uncountable subset of D' bounded in D_T . Since, for each $\xi < \omega_1$, both δ_ξ and $\langle S_\gamma : \gamma < \omega_2 \rangle$ belong to M , it must be that $S_{\delta_\xi} \in M$. Since $|S_{\delta_\xi}| \leq \aleph_1$ ($\xi < \omega_1$), we have $S \subseteq M$. Thus $S \subseteq T \upharpoonright \delta$. This means that, in order to prove $S \in D_T$, it is enough to prove that for each $t \in T \upharpoonright \delta$, the set $\{\alpha < \omega_1 : t \frown \langle \alpha \rangle \in \widehat{S}\}$ is non-stationary in ω_1 . So take any $t \in T \upharpoonright \delta$. Let $\zeta < \omega_1$ be such that $s_\zeta = t$. Define

$$C = C_\zeta^\delta \cap \left(\bigcap_{\xi \leq \zeta} C_\xi^{\delta_\xi} \right) \cap \left(\Delta_{\xi < \omega_1} C_\xi^{\delta_\xi} \right).$$

Since countable intersection of clubs is a club, and since diagonal intersection of ω_1 many clubs is a club, we know that C is a club in ω_1 . We will prove that $C \cap \{\alpha < \omega_1 : s_\zeta \frown \langle \alpha \rangle \in \widehat{S}\} = \emptyset$. So suppose that an ordinal $\alpha < \omega_1$ is such that $\alpha \in C \cap \{\alpha < \omega_1 : s_\zeta \frown \langle \alpha \rangle \in \widehat{S}\}$. This means that $\alpha \in C$ and that for some $\mu < \omega_1$ and some $x \in S_{\delta_\mu}$, $s_\zeta \frown \langle \alpha \rangle < x$. Note that this implies that

$$\alpha \notin C_\zeta^{\delta_\mu}. \quad (5.1)$$

If $\mu \leq \zeta$, then since $\alpha \in \bigcap_{\xi \leq \zeta} C_\xi^{\delta_\xi}$, we have $\alpha \in C_\zeta^{\delta_\mu}$ which is clearly contradicting (5.1). Thus, it must be that $\zeta < \mu$. We consider two subcases, either $\alpha < \eta_\mu$ or $\alpha \geq \eta_\mu$. If $\alpha \geq \eta_\mu > \mu$, then $\alpha \in C$ implies that $\alpha \in \Delta_{\xi < \omega_1} C_\xi^{\delta_\xi}$, which together with $\alpha > \mu$ implies that $\alpha \in C_\zeta^{\delta_\mu}$. But this is in contradiction with (5.1). If $\alpha < \eta_\mu$, then by the property of δ_μ we have $C_\zeta^\delta \cap \eta_\mu = C_\zeta^{\delta_\mu} \cap \eta_\mu$. By (5.1), $\alpha \notin C_\zeta^{\delta_\mu}$, so since $\alpha < \eta_\mu$, it must be that $\alpha \notin C_\zeta^\delta$. But, by the definition of the set C , this means that $\alpha \notin C$, which is in contradiction with the initial assumption on α . \square

Theorem 5.5. *If $T \subseteq \omega_1^{<\omega_2}$ is an \aleph_1 -branching \aleph_2 -Souslin tree, then*

$$\omega_1 \times \omega_2 <_T D_T <_T [\omega_2]^{\leq \omega}.$$

Proof. Let us enumerate $T = \{t_\alpha : \alpha < \omega_2\}$.

First we prove $\omega_1 \times \omega_2 \leq_T D_T$. We will find a Tukey function $f : \omega_1 \times \omega_2 \rightarrow D_T$. So, for $\langle \beta, \gamma \rangle \in \omega_1 \times \omega_2$, define

$$f(\beta, \gamma) = \{t_\gamma\} \cup \{t_\alpha \frown \langle \alpha \rangle \in T : \alpha < \beta\}.$$

The function f is properly defined because the image of every element of $\omega_1 \times \omega_2$ is countable, thus belongs to D_T . Now we prove that f is Tukey, i.e. the image of every unbounded set is unbounded. Let X be unbounded in $\omega_1 \times \omega_2$. The either $|\pi''_{\omega_1} X| = \aleph_1$ or $|\pi''_{\omega_2} X| = \aleph_2$. Suppose first that $|\pi''_{\omega_2} X| = \aleph_2$, and let $Y = \{\gamma < \omega_2 : t_\gamma \in \pi''_{\omega_2} X\}$. Then $Y \subseteq \bigcup f'' X$, in particular $\bigcup f'' X$ is of cardinality \aleph_2 , so $f'' X$ cannot be bounded in D_T . Suppose now that $|\pi''_{\omega_1} X| = \aleph_1$. Then $\{t_\alpha \frown \langle \alpha \rangle : \alpha < \omega_1\} \subseteq \bigcup f'' X$ thus showing that $f'' X$ is not bounded in D_T .

Next we prove $D_T \leq_T [\omega_2]^{\leq \omega}$. For $X \in D_T$ define $g(X) = \{\alpha < \omega_2 : t_\alpha \in X\}$. Suppose now that \mathcal{A} is unbounded in D_T . Then $\bigcup \mathcal{A}$ is uncountable because otherwise \mathcal{A} would be bounded. This means that

$$\left| \bigcup g'' \mathcal{A} \right| = \left| \bigcup \{g(X) : X \in \mathcal{A}\} \right| = \left| \left\{ \alpha < \omega_2 : t_\alpha \in \bigcup \mathcal{A} \right\} \right| \geq \aleph_1.$$

Thus, $g''\mathcal{A}$ is unbounded in $[\omega_2]^{\leq\omega}$.

Now we prove that $D_T \not\leq_T \omega_1 \times \omega_2$. Suppose the contrary, that there is a Tukey function $h : D_T \rightarrow \omega_1 \times \omega_2$. For $\alpha < \omega_1$, let

$$D_\alpha = \{X \in D_T : (\exists \beta < \omega_2) h(X) = \langle \alpha, \beta \rangle\}.$$

Since h is a function, $D_T = \bigcup_{\alpha < \omega_1} D_\alpha$ is a partition. By Corollary 5.3, there is an $\alpha < \omega_1$ and an unbounded set $E \subseteq D_\alpha$ of cardinality \aleph_1 . Enumerate $E = \{X_\xi : \xi < \omega_1\}$, and let $\beta_\xi < \omega_2$ be such that $h(X_\xi) = \langle \alpha, \beta_\xi \rangle$, and let $\beta < \omega_2$ be such that $\beta_\xi < \beta$ for every $\xi < \omega_1$. Then $h''E = \{\langle \alpha, \beta_\xi \rangle : \xi < \omega_1\}$. Since E is unbounded and h is Tukey, $h''E$ is unbounded. This is a contradiction because $h''E \leq \langle \alpha, \beta \rangle$ in $\omega_1 \times \omega_2$.

Finally, we prove $[\omega_2]^{\leq\omega} \not\leq_T D_T$. So suppose the contrary that $[\omega_2]^{\leq\omega} \leq_T D_T$. We already know that $D_T \leq_T [\omega_2]^{\leq\omega}$, so the assumption gives us $D_T \equiv_T [\omega_2]^{\leq\omega}$. This means that there is a directed set E such that both D_T and $[\omega_2]^{\leq\omega}$ are cofinal subsets of E . By Lemma 2.3 there is a set $A \subseteq [\omega_2]^{\leq\omega}$ cofinal in $[\omega_2]^{\leq\omega}$, and such that every uncountable $B \subseteq A$ is unbounded in $[\omega_2]^{\leq\omega}$. Now, for each $x \in A$ take $d_x \in D_T$ such that $x \leq d_x$. Since A is cofinal in E , the set $D' = \{d_x : x \in A\}$ is also cofinal in E , and consequently cofinal in D_T . By Lemma 5.4, there is an uncountable subset $D^* \subseteq D'$ bounded in D_T . Let $d^* \in D_T$ be such that $d \leq d^*$ for each $d \in D'$. Consider the set $B = \{x \in A : d_x \in D^*\}$. Since D' is uncountable, the set B is also uncountable. Thus, by the assumption on A , the set B is unbounded in $[\omega_2]^{\leq\omega}$, but also in E because $[\omega_2]^{\leq\omega}$ is a cofinal subset of E . Then, for each $x \in B$ we have $x \leq d_x \leq d^*$, contradicting the unboundedness of B in E . This shows that $D_T \not\equiv_T [\omega_2]^{\leq\omega}$, and consequently $D_T \not\leq_T \omega_1 \times \omega_2$. \square

This, together with the following theorem, concludes the proof of Theorem 1.1

Theorem 5.6 (Gregory, see [6]). *If GCH holds and there is a non-reflecting stationary subset of S_0^2 , then there is an \aleph_1 -branching \aleph_2 -Souslin tree.*

6. DIRECTED SET BETWEEN $[\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega}$ AND $[\omega_2]^{<\omega}$

The standing assumption in this section will be that GCH holds, and that S is a non-reflecting stationary subset of S_0^2 . GCH implies that there is a collection $\mathcal{C} = \{C_\alpha : \alpha \in S\}$ of sets of order type ω , such that $C_\alpha \subseteq \alpha$ for each $\alpha \in S$, and that for each set $X \subseteq \omega_2$ of size \aleph_2 there is some $\alpha \in S$ such that $C_\alpha \subseteq X$. Define

$$D_{\mathcal{C}} = \{Y \in [\omega_2]^{\aleph_0} : (\forall \alpha \in S) |Y \cap C_\alpha| < \aleph_0\},$$

and consider $D_{\mathcal{C}}$ directed by inclusion. In this section we prove that $[\omega_1]^{<\omega} \times D_{\mathcal{C}}$ is the directed set strictly between $[\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega}$ and $[\omega_2]^{<\omega}$ in the Tukey ordering.

Since GCH implies that $\aleph_2^{\aleph_0} = \aleph_2$, we have $|D_{\mathcal{C}}| = \aleph_2$. Thus $\text{cf}(D_{\mathcal{C}}) = \aleph_2$, and consequently $D_{\mathcal{C}} \leq_T [\omega_2]^{<\omega}$. Together with $[\omega_1]^{<\omega} \leq_T [\omega_2]^{<\omega}$ and Lemma 2.1, this implies that $[\omega_1]^{<\omega} \times D_{\mathcal{C}} \leq_T [\omega_2]^{<\omega}$.

Since GCH implies $|D_{\mathcal{C}}| = |[\omega_2]^{\leq\omega}| = \aleph_2$, there is a 1-1 function $\phi : D_{\mathcal{C}} \rightarrow \omega_2$. Denote $X = \{x \cup \{\phi(x)\} : x \in D_{\mathcal{C}}\}$. Clearly, X is a cofinal subset of $D_{\mathcal{C}}$. Since ϕ is 1-1, union of every uncountable subset of X contains an uncountable subset of \aleph_2 as a subset. Hence, every uncountable subset of X is unbounded in $D_{\mathcal{C}}$. This means that any 1-1 function $g : [\omega_2]^{\leq\omega} \rightarrow X$ witnesses that $[\omega_2]^{\leq\omega} \leq_T D_{\mathcal{C}}$ holds. Now f mapping $[\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega}$ to $[\omega_1]^{<\omega} \times D_{\mathcal{C}}$, defined by $f(F, Y) = (F, g(Y))$,

is Tukey. So we finally have

$$[\omega_1]^{<\omega} \times [\omega_2]^{\leq\omega} \leq_T [\omega_1]^{<\omega} \times D_C \leq_T [\omega_2]^{<\omega}$$

In the remainder of this section we show that these inequalities are strict.

Lemma 6.1. *There is no Tukey map from $D_C \times [\omega_1]^{<\omega}$ into $[\omega_2]^{\leq\omega} \times [\omega_1]^{<\omega}$.*

Proof. Suppose that $f : D_C \times [\omega_1]^{<\omega} \rightarrow [\omega_2]^{\leq\omega} \times [\omega_1]^{<\omega}$ is Tukey. For each $\xi < \omega_2$ denote $(x_\xi, s_\xi) = f(\{\xi\}, \emptyset)$. Consider the set $\{(x_\xi, s_\xi) : \xi < \omega_2\}$. Since $[\omega_1]^{<\omega}$ is of size \aleph_1 , there is some $X \subseteq \omega_2$ of size \aleph_2 , and $s \in [\omega_1]^{<\omega}$ such that $s_\xi = s$ for each $\xi \in X$. By the assumption on \mathcal{C} , there is some $\alpha \in S$ such that $C_\alpha \subseteq X$. Now, the set $A = \{(\{\xi\}, \emptyset) : \xi \in C_\alpha\}$ is unbounded in $D_C \times [\omega_1]^{<\omega}$. In particular the set $\pi''_{D_C} A$ is unbounded in D_C , as witnessed by the infinite intersection $\pi''_{D_C} A \cap C_\alpha$. But the set $f''A = \{(x_\xi, s) : \xi \in C_\alpha\}$ is bounded in $[\omega_2]^{\leq\omega} \times [\omega_1]^{<\omega}$ contradicting the assumption that f is a Tukey map. To see that $f''A$ is bounded, note that C_α is countable so the bound for $f''A$ is $(\bigcup_{\xi \in C_\alpha} x_\xi, s)$. This is because $\bigcup_{\xi \in C_\alpha} x_\xi$ is in $[\omega_2]^{\leq\omega}$ (being a countable union of countable sets). \square

Lemma 6.2. *There is no Tukey map from $[\omega_2]^{<\aleph_0}$ into $[\omega_1]^{<\omega} \times D_C$.*

Proof. Suppose that the lemma fails. This means that $[\omega_2]^{<\aleph_0} \equiv_T [\omega_1]^{<\omega} \times D_C$, i.e. that there is $Y \subseteq [\omega_1]^{<\omega} \times D_C$ of size \aleph_2 such that every infinite subset of Y is unbounded in D_C . Since Y is of size \aleph_2 , there is a finite $F \subseteq \omega_1$, and $X \subseteq D_C$ of size \aleph_2 such that $(F, x) \in Y$ for each $x \in X$. Let $X = \{x_\alpha : \alpha < \omega_2\}$. Clearly, every infinite subset of X is unbounded in D_C .

Using GCH we may assume that X forms an increasing Δ -system with the root X_0 . In other words, for $\alpha < \beta < \omega_2$ we have $x_\alpha \cap x_\beta = X_0$ and $x_\alpha \setminus X_0 < x_\beta \setminus X_0$. Take some large enough θ so that all relevant object belong to $H(\theta)$. Pick a continuous increasing sequence $\langle M_\xi : \xi < \omega_2 \rangle$ of elementary submodels of $H(\theta)$ of cardinality \aleph_1 , such that $X_0 \in M_0$. Now the set $C = \{\delta < \omega_2 : M_\delta \cap \omega_2 = \delta\}$ is a club in ω_2 . Let $\gamma \in C$ be such that $\text{cf}(\gamma) = \omega_1$ and $\text{sup}(C \cap \gamma) = \gamma$. Such a γ exists because C is a club. Since S is non-reflecting, there is $E \subseteq C \cap \gamma$, club in $C \cap \gamma$ such that $\text{otp}(E) = \omega_1$ and $E \cap S = \emptyset$. Let $E = \{\delta_\alpha : \alpha < \omega_1\}$ be an increasing enumeration of E . For each $\alpha < \omega_1$ let $\delta_\alpha^+ = \min(E \setminus \delta_\alpha)$. By elementarity, there is a sequence $\langle x_{\xi_\alpha} : \alpha < \omega_1 \rangle$ such that $x_{\xi_\alpha} \setminus X_0 \subseteq M_{\delta_\alpha^+} \setminus M_{\delta_\alpha}$. Consider the set $x = \bigcup_{n < \omega} x_{\xi_n}$ and take $\alpha \in S$. Then $\alpha \notin E$, so either $\alpha > \sup_{n < \omega} \delta_n$ or $\alpha < \sup_{n < \omega} \delta_n$. If $\alpha > \sup_{n < \omega} \delta_n$, then $C_\alpha \cap x$ is a finite set. If $\alpha < \sup_{n < \omega} \delta_n$, then there is n_0 so that $\delta_{n_0+1} > \alpha$. Since $C_\alpha \cap x_{\xi_m}$ is finite for each $m \leq n_0$, and $C_\alpha \cap x = \bigcup_{m \leq n_0} (C_\alpha \cap x_{\xi_m})$ it must be that $C_\alpha \cap x$ is finite in this case also. Thus $x \in D_C$ is a bound for a countable set $\{x_{\xi_n} : n < \omega\} \subseteq X$, which is in contradiction with the choice of X . So the lemma is proved. \square

This concludes the proof of Theorem 1.2.

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