MAXIMAL CHAINS OF COPIES OF FREELY GENERATED ULTRAHOMOGENEOUS DIGRAPHS

BORIŠA KUZELJEVIĆ AND STEPAN MILOŠEVIĆ

Abstract. The purpose of this note is to extend previous known results about maximal chains of copies of countable ultrahomogeneous relational structures. In particular, we completely describe order types of maximal chains in posets of the form $\mathbb{P}(\Gamma) = \{A \subseteq \Gamma : A \cong \Gamma\}$, where Γ is a countable ultrahomogeneous digraph belonging to Cherlin's fourth class, freely generated countable ultrahomogeneous directed graphs. Analogous characterizatons were known for countable ultrahomogeneous posets and graphs.

1. Introduction

1.1. **Background.** The purpose of this paper is to extend some known results about the classification of maximal chains in the posets of the form $(\mathbb{P}(\mathbb{X}), \subseteq)$, where $\mathbb{P}(\mathbb{X}) = \{A \subseteq \mathbb{X} : A \cong \mathbb{X}\}$ and $\mathbb{X}$ is a countable ultrahomogeneous structure. All the notions are defined in Section 1.3, however, let us mention that we denote the class of maximal chains in a poset P by $\mathcal{M}_P$. Since we only deal with suborders of the powerset $\mathcal{P}(\omega)$, we aways assume that the order relation is the inclusion $\subseteq$.

Previously, there has been work on this and similar topics. Probably the first related result is due to Kuratowski in [12] where he showed that

$$\mathcal{M}_{\mathcal{P}(\omega)} = \{\mathrm{Init}(L) : \ L \text{ is a countable linear order}\},$$

where $\mathrm{Init}(L)$ is the set of initial segments of the linear order L. Afterwords, Day in [3] and Koppelberg in [8] obtained characterizations of maximal chains in more general Boolean algebras. Recently, Kurilić initiated the investigation of orders $\mathbb{P}(\mathbb{X}) \subseteq \mathcal{P}(\omega)$ where $\mathbb{X}$ is a relational structure. One aspect of this was exploration of forcing equivalence of these posets for various structures $\mathbb{X}$. This line of research was done, for example, in [20, 15, 21, 22]. The other aspect was characterization of maximal chains in similar posets, and Kurilić gave characterizations of maximal chains in any positive family on ω in [13] as well as the characterization of the class $\mathcal{M}_{\mathbb{P}(\mathbb{Q})}$ in [14]. In a joint work with the first author, Kurilić obtained analogous characterizations for the case of countable ultrahomogeneous posets in [17] and countable ultrahomogeneous graphs in [16]. Kurilić and the first author even investigated antichains in these posets, this can be found in [19].

Let us define two classes of order that will be relevant in the paper. The notation is consistent with the one in [18]. Let $\mathcal{C}_{\mathbb{R}}$ denote the class of order types of sets of the form $K \setminus \{\min K\}$ where $K \subseteq \mathbb{R}$ is a compact set of reals such that $\min K$ is an accumulation point of K. Let $\mathcal{B}_{\mathbb{R}}$ be the subclass of $\mathcal{C}_{\mathbb{R}}$ for which the corresponding compact set K is, in addition, nowhere dense.

In this paper, we deal with the case of countable ultrahomogeneous directed graphs. We will be shortly saying *digraphs* instead of *directed graphs*. These were completely classified in a long program by Cherlin, see [2]. For differently arranged lists see also [23, pp. 1604] and [1, pp. 47]. We will focus on the fourth group in Cherlin's classification - freely generated countable ultrahomogeneous digraphs.

Date: April 11, 2025.

Note that this is not exactly the same as those whose age has the free amalgamation property, in a sense that is used today. However, it is very close to it. As an ilustration, in this group only $\mathrm{Age}(\Gamma_n)$, for all n, does not have the free amalgamation property, and for trivial reasons. According to [2, pp. 74], there are two subclasses of the class of freely generated countable ultrahomogeneous digraphs:

- One contains, for $n > 0$, digraphs Γ_n - this is the countable ultrahomogeneous digraph whose age consists of all finite digraphs not embedding I_{n+1}, the empty digraph with $n+1$ vertices.
- The other contains, for any family of finite tournaments $\mathcal{T}$ not containing the one-element tournament, digraphs $\Gamma(\mathcal{T})$ - this is the countable ultrahomogeneous digraph whose age consists of all finite digraphs not embedding any tournament from $\mathcal{T}$.

The restriction on $\mathcal{T}$ of not containing the one-element tournament is needed only for the existence of the required digraph to make sense. We will call such families non-trivial. Note that infinite digraphs have been investigated in various other contexts, for just one example one may take a look at [4]. There, certain partition properties of infinite digraphs have been obtained, for example, for $\Gamma(\emptyset)$.

1.2. **Results.** For the start, note that the age of each digraph in the mentioned two subclasses satisfies the strong amalgamation property (see Definition 1.6), so by [18, Theorem 1] we have:

Theorem 1.1. *If Γ is a freely generated countable ultrahomogeneous digraph, then*

$$\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mathbb{P}(\Gamma)} \subseteq \mathcal{C}_{\mathbb{R}}.$$

Next, in Section 2 we deal with the first subclass, countable ultrahomogeneous digraphs not embedding a particular empty digraph. Observe that for $n = 1$, the digraph Γ_1 is the countable ultrahomogeneous digraph universal for all finite digraphs not embedding I_2, i.e. there is an edge between every two vertices. This means that Γ_1 is exactly the random tournament T^∞, so by the result of Kurilić and Todorčević from [22, Theorem 2.3], we have

$$\mathbb{P}(\Gamma_1) \cong \mathbb{P}(T^\infty) \cong \mathbb{P}(\mathbb{G}_{\mathrm{Rado}}),$$

where $\mathbb{G}_{\mathrm{Rado}}$ is the countable Rado graph. By [16, Theorem 1.2], this implies that $\mathcal{M}_{\mathbb{P}(\Gamma_1)} = \mathcal{C}_{\mathbb{R}}$. So in the mentioned section, we complete the proof of the following.

Theorem 1.2. *Let $n > 0$. Then $\mathcal{M}_{\mathbb{P}(\Gamma_n)} = \mathcal{C}_{\mathbb{R}}$.*

In Section 3, we deal with the other class, digraphs $\Gamma(\mathcal{T})$ for a non-trivial family of finite tournaments. Note that whenever a non-trivial family of finite tournaments $\mathcal{T}$ contains the two element tournament, then the graph $\Gamma(\mathcal{T})$ is the empty digraph. Clearly, in this case $\mathbb{P}(\Gamma(\mathcal{T})) \cong [\omega]^\omega$, so by [13, Theorem 1] we have the following.

Theorem 1.3. *Let $\mathcal{T}$ be any non-trivial family of finite tournaments containing the two element tournament. Then $\mathcal{M}_{\mathbb{P}(\Gamma(\mathcal{T}))} = \mathcal{B}_{\mathbb{R}}$.*

To complete the picture for digraphs $\Gamma(\mathcal{T})$, in the last section we analyze the remaining case, and prove the following theorem, thus completing the picture for all countable freely generated ultrahomogeneous digraphs.

Theorem 1.4. *Let $\mathcal{T}$ be a non-trivial family of finite tournaments. Then*

$$\mathcal{M}_{\mathbb{P}(\Gamma(\mathcal{T}))} = \begin{cases} \mathcal{B}_{\mathbb{R}}, & \textit{if } \mathcal{T} \textit{ contains the two element tournament,} \\ \mathcal{C}_{\mathbb{R}}, & \textit{otherwise.} \end{cases}$$

Let us also mention that, as graphs Γ_n, for $n > 0$, have many features in common with $\mathbb{K}_n$-free graphs from the symmetric world, the proofs in Section 2 follow closely the proof of the main result in [16]. In Section 3, there is a technical difficulty that since we do not know the family $\mathcal{T}$ in advance, we do not have a nice presentation of the Fraïssé limit in question. So the approach taken there, although similar in spirit, is significantly more general, and possibly can be used in many other cases. On the other hand, in Section 2 we had to develop methods similar to the ones in [7], in order to get a nice presentation of graphs Γ_n.

1.3. **Preliminaries.** As usual in set theory, we identify a positive integer n with the set $\{0, 1, \ldots, n-1\}$, $\mathbb{Q}$ denotes the set of rational numbers, and $\mathbb{Z}$ is the set of integers. For a set X and $n > 0$, we denote $[X]^n = \{A \subseteq X : A \text{ is of size } n\}$. For a function $f : X \to Y$ and $A \subseteq X$, we denote $f[A] = \{f(x) : x \in A\}$. Formally, a *relational structure* consists of a non empty set, say X, and a family of relations on X, say $\{\rho_i : i \in I\}$. Typically, we will write $(X, \bar{\rho})$ for such structure. Note that we sometimes use just X if the family of relations is clear from the context. For a relation ρ, its arity is denoted $\operatorname{ar}(\rho)$. The set $\{\operatorname{ar}(\rho_i) : i \in I\}$ is the *signature* of the structure $(X, \bar{\rho})$. A relational structure $(Y, \bar{\sigma})$ is a *substructure* of $(X, \bar{\rho})$ if $Y \subseteq X$, both have the same signature, and $\sigma_i = Y^{\operatorname{ar}(\rho_i)} \cap \rho_i$ for each $i \in I$. A function $f : X \to Y$ is an *embedding* between structures $(X, \bar{\rho})$ and $(Y, \bar{\sigma})$, if they both have the same signature, and for each $i \in I$ and $(x_j : j < \operatorname{ar}(\rho_i)) \in X^{\operatorname{ar}(\rho_i)}$, we have

$$(x_j : j < \operatorname{ar}(\rho_i)) \in \rho_i \Leftrightarrow (f(x_j) : j < \operatorname{ar}(\rho_i)) \in \sigma_i.$$

In this paper, we will deal mostly with digraphs. By a digraph, we assume a relational structure (D, ρ) such that:

(1) $\operatorname{ar}(\rho) = 2$, i.e. $\rho \subseteq D^2$,
(2) there are no loops in D, i.e. $(\forall d \in D)\ (d, d) \notin \rho$,
(3) There are no two-way edges in D, i.e. $(\forall d, e \in D)\ (d, e) \in \rho \Rightarrow (e, d) \notin \rho$.

So, if (X, ρ) and (Y, σ) are digraphs, f is an embedding from X to Y if:

$$(\forall a, b \in X)\ (a, b) \in \rho \Leftrightarrow (f(a), f(b)) \in \sigma.$$

Similarly, a digraph (Y, σ) is a substructure of a digraph (X, ρ) if $Y \subseteq X$ and $\sigma = Y^2 \cap \rho$. For a digraph (X, ρ) and $Y \subseteq X$, we will typically, when there is no danger of confusion, write simply (Y, ρ) instead of $(Y, Y^2 \cap \rho)$ for the substructure of (X, ρ) given by the subset Y. For a digraph (D, ρ), we say it is a *tournament* if for any two different vertices a and b in D, either (a, b) or (b, a) belong to ρ.

Example 1.5. We give an example of two digraphs. The first one we will call Ω. It has three vertices, say u, v, and w, together with one edge (u, v). Thus, formally,

$$\Omega = (\{u, v, w\}, \{(u, v)\}).$$

Note that no empty digraph with more than two vertices embeds into Ω.

The other digraph will be called Λ. It also has three vertices, say again u, v, and w. This one has exactly two edges, (u, w) and (v, w). So, formally,

$$\Lambda = (\{u, v, w\}, \{(u, w), (v, w)\}).$$

Note that no tournament with more than two vertices embeds into Λ.

One direction in general investigations of relational structures is through Fraïssé's theory. It focuses on countable ultrahomogeneous structures, although it is possible to put it in a much more general framework, see for example [9, 10, 11]. Recall that a relational structure $(X, \bar{\rho})$ is *ultrahomogeneous* if any isomorphism between finite substructures of $(X, \bar{\rho})$ can be extended to an automorphism of $(X, \bar{\rho})$. Note that some authors call this notion *homogeneity*.

There are many examples of ultrahomogeneous structures, just to name a few: the rational line $(\mathbb{Q}, <)$, the countable random graph $\mathbb{G}_{\mathrm{Rado}}$, or the countable random tournament T^∞. To get more general theory, one considers the concept of age. For a relational structure $(X, \bar{\rho})$, its *age*, denoted $\mathrm{Age}(X, \bar{\rho})$, is the class of all finite structures embeddable in $(X, \bar{\rho})$. A class $\mathcal{K}$ of finite structures in the same signature can have various properties, let us mention four properties relevant for this paper:

Definition 1.6. For a class of structures in the same signature, we define:

(HP) *Hereditary property*: If $A \in \mathcal{K}$ and B is a substructure of A, then B is isomorphic to some structure in $\mathcal{K}$.

(JEP) *Joint embedding property*: If A, B are in $\mathcal{K}$, then there is $C \in \mathcal{K}$ such that both A and B are embeddable into C.

(AP) *Amalgamation property*: If A, B, C are in $\mathcal{K}$ and $e : A \to B$ and $f : A \to C$ are embeddings, then there are D in $\mathcal{K}$, and embeddings $g : B \to D$ and $h : C \to D$ such that $g \circ e = h \circ f$.

(SAP) *Strong amalgamation property*: If A, B, C are structures in $\mathcal{K}$ and $e : A \to B$ and $f : A \to C$ are embeddings, then there are D in $\mathcal{K}$, and embeddings $g : B \to D$ and $h : C \to D$ such that $g \circ e = h \circ f$ and $g[B] \cap h[C] = g[e[A]]$.

The next theorem is the key for all the further development of Fraïssé theory.

Theorem 1.7 (Fraïssé [5], see page 333 in [6])**.** *Let $\mathcal{K}$ be a class of finite structures satisfying HP, JEP, and AP. Then there is an ultrahomogeneous structure $(X, \bar{\rho})$ such that $\mathrm{Age}(X, \bar{\rho}) = \mathcal{K}$. Moreover, every two ultrahomogeneous structures with the same age are isomorphic.*

We call the ultrahomogeneous structure from the previous theorem the Fraïssé limit of the class $\mathcal{K}$. Note that Fraïssé also developed a useful criterion to check whether a given countable structure is ultrahomogeneous.

Theorem 1.8 (Fraïssé [5], see page 332 in [6])**.** *A countable relational structure $(X, \bar{\rho})$ is ultrahomogeneous iff for every finite subset $F \subseteq X$, every embedding $f : (F, \bar{\rho}) \to (X, \bar{\rho})$ and $a \in X \setminus F$, there is an embedding $\Phi : (F \cup \{a\}, \bar{\rho}) \to (X, \bar{\rho})$ such that $f \subseteq \Phi$.*

We will take a 'generic' approach to building ultrahomogeneous structures. This approach had already been used in [16], for example, with the same purpose. This means that we will partially order finite approximations to our desired ultrahomogeneous structure, and then a sufficiently generic filter in this poset will provide the required relation on the rational line $\mathbb{Q}$. Recall that $(P, \leq)$ is a *partial order* if P is a non-empty set and $\leq$ is reflexive, transitive, and antisymmetric relation on P. A set $D \subseteq P$ is *dense* in P if for each $q \in P$ there is some $p \in D$ such that $p \leq q$. A set $E \subseteq P$ is *open* in P if for each $q \in E$ and each $p \leq q$, we have that $p \in E$ as well. A set $\mathcal{G} \subseteq P$ is a *filter in P* if it is upward closed, meaning that for each $p \in \mathcal{G}$ and $q \geq p$ we have $q \in \mathcal{G}$, and downwards directed, meaning that for each $p, q \in \mathcal{G}$ there is some $r \in \mathcal{G}$ such that $r \leq p$ and $r \leq q$. In this setup, we will only still need the following Rasiowa-Sikorski lemma.

Lemma 1.9. *If P is a partial order, and $\mathcal{C}$ a countable family of sets dense in P, then there is a filter $\mathcal{G}$ in P such that $\mathcal{G} \cap D \neq \emptyset$ for each $D \in \mathcal{C}$*

The main tool used in the proofs is the following result.

Theorem 1.10 (Reformulation of Theorem 3.2 from [16])**.** *Let $(\mathbb{X}, \bar{\rho})$ be a countable relational structure. Suppose that there is a structure $(\mathbb{Q}, \bar{\sigma})$ of the same signature as $\mathbb{X}$, such that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mathbb{P}(\mathbb{X})}$ and that the following two conditions hold:*

(1) $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, x) \subseteq A \subseteq \mathbb{Q} \cap (-\infty, x) \Rightarrow (A, \bar{\sigma}) \cong \mathbb{X}$*, for all* $x \in \mathbb{R} \cup \{\infty\}$*;*

(2) $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, y] \subseteq C \subseteq \mathbb{Q} \cap (-\infty, y] \Rightarrow (C, \bar{\sigma}) \ncong \mathbb{X}$*, for all* $y \in \mathbb{Q} \setminus \mathbb{Z}$*.*

Then $\mathcal{M}_{\mathbb{P}(\mathbb{X})} = \mathcal{C}_{\mathbb{R}}$*.*

2. Omitting the empty digraph

In this section we prove Theorem 1.2 for $n \geq 2$. Note that we have already explained the case $n = 1$. So fix an integer $n \geq 2$ in this section. Since finite digraphs not embedding I_{n+1} clearly have amalgamation property, there is a Fraïssé limit of this class, and we denote it (Γ_n, ρ) in this section. We take the announced generic approach in order to construct the copy of Γ_n living on $\mathbb{Q}$, and satisfying the conditions of Theorem 1.10. Then, by Theorem 1.10 the required result follows.

Before we continue with the generic construction, we develop a particular characterization of the graph Γ_n. This is very useful as it allows us to avoid more abstract construction, as needed in Section 3.

If (D, τ) is a digraph and $n \geq 2$, let $C_n(D, \tau)$ denote the set of all triples (F, K, H) of finite subsets of D such that:

(C1) $F \cap K = K \cap H = F \cap H = \emptyset$ and
(C2) I_n does not embed in (H, τ).

For $(F, K, H) \in C_n(D, \tau)$ we define the set $D^H_{F,K}$ as the set of all $d \in D \setminus (F \cup K \cup H)$ such that:

(1) $(\forall a \in F)\ (d, a) \in \tau$,
(2) $(\forall a \in K)\ (a, d) \in \tau$,
(3) $(\forall a \in H)\ (a, d) \notin \tau\ \wedge\ (d, a) \notin \tau$.

The next lemma shows why the introduced notions are useful.

Lemma 2.1. *Let $n \geq 2$. A countable digraph (D, τ) is isomorphic to Γ_n if and only if I_{n+1} does not embed into (D, τ) and $D^H_{F,K} \neq \emptyset$ for each $(F, K, H) \in C_n(D, \tau)$.*

Proof. Note that by Theorem 1.7, it is enough to prove that $\mathrm{Age}(D, \tau) = \mathrm{Age}(\Gamma_n)$ and that (D, τ) is ultrahomogeneous.

First we prove that $\mathrm{Age}(D, \tau) = \mathrm{Age}(\Gamma_n)$. Note that $\mathrm{Age}(D, \tau) \subseteq \mathrm{Age}(\Gamma_n)$ follows from the facts that Γ_n embeds every finite digraph not embedding I_{n+1} and that I_{n+1} does not embed into (D, τ). Now we show that $\mathrm{Age}(\Gamma_n) \subseteq \mathrm{Age}(D)$. Take any $(G, \rho_G) \in \mathrm{Age}(\Gamma_n)$. Then $G = \{a_1, \dots, a_m\}$, for some $m > 0$. We will prove that for $k < m$, if $(\{a_1, \dots, a_k\}, \rho_G) \in \mathrm{Age}(D, \tau)$, then $(\{a_1, \dots, a_k, a_{k+1}\}, \rho_G)$ is an element of $\mathrm{Age}(D, \tau)$. As $\{a_1\}$ clearly embeds into (D, τ), this is, by induction, enough to show that $G \in \mathrm{Age}(D, \tau)$. So suppose that for some $k < m$, there is an embedding $f : (\{a_1, \dots, a_k\}, \rho_G) \to (D, \tau)$. Consider the following three sets:

$$F = \{x \in \{a_1, \dots, a_k\} : (a_{k+1}, x) \in \rho_G\},$$
$$K = \{x \in \{a_1, \dots, a_k\} : (x, a_{k+1}) \in \rho_G\},$$
$$H = \{a_1, \dots, a_k\} \setminus (F \cup K).$$

Note that $\{a_1, \dots, a_k\} = F \cup K \cup H$, and that $F \cap K = K \cap H = F \cap H = \emptyset$. Since the map f is an embedding, the sets $f[F]$, $f[K]$, and $f[H]$ form a partition of the set $f[\{a_1, \dots, a_k\}]$. Since $(G, \rho_G) \in \mathrm{Age}(\Gamma_n)$, the digraph I_{n+1} does not embed into G. This means that I_{n+1} does not embed into $\{a_1, \dots, a_{k+1}\}$ as well. Consequently, I_n cannot embed into H. Since f is an embedding, this means that I_n does not embed into $f[H]$. Hence $(f[F], f[K], f[H]) \in C_n(D, \tau)$. By the assumption, there is some $b \in D^{f[H]}_{f[F], f[K]}$. Then $g : \{a_1, \dots, a_{k+1}\} \to D$, defined by $g(x) = f(x)$ for $x \in \{a_1, \dots, a_k\}$ and $g(a_{k+1}) = b$, is an embedding of $(\{a_1, \dots, a_{k+1}\}, \rho_G)$ into (D, τ). This completes the proof that G embeds into D, and that $\mathrm{Age}(\Gamma_n) \subseteq \mathrm{Age}(D, \tau)$.

Now we prove that (D, τ) is ultrahomogeneous. We use the characterization from Theorem 1.8. So let $X \subseteq D$ be finite, let $f : (X, \tau) \to (D, \tau)$ be an embedding, and let $a \in D \setminus X$. Consider sets: $F = \{x \in X : (a, x) \in \tau\}$, $K = \{x \in X : (x, a) \in \tau\}$, and $H = X \setminus (F \cup K)$. Clearly $X = F \cup K \cup H$ and $F \cap K = F \cap H = K \cap H = \emptyset$.

Again, since f is an embedding, sets $f[F]$, $f[K]$, $f[H]$ form a partition of the set $f[X]$. Observe that I_n does not embed into $f[H]$. To see this, note that if I_n embeds into $f[H]$, then I_n embeds into H (as f is an embedding). But then I_{n+1} would be embeddable into $H \cup \{a\} \subseteq D$ contradicting that $\operatorname{Age}(D, \tau) = \operatorname{Age}(\Gamma_n)$. Since I_n does not embed into $f[H]$, we have $(f[F], f[K], f[H]) \in C_n(D, \tau)$. By the assumption of the lemma, there is some $b \in D^{f[H]}_{f[F],f[K]}$. This exactly means that $g : X \cup \{a\} \to D$, given by $g(x) = f(x)$ for $x \in X$, and $g(a) = b$ is an embedding. Moreover, $f \subseteq g$, so the conclusion of Theorem 1.8 is satisfied. Hence, (D, τ) is ultrahomogeneous. □

Now we proceed with the generic construction. Recall that we have fixed an integer $n \geq 2$ in this section. Let $\mathbb{P}_n$ be the set of all finite digraphs $p = (D_p, \sigma_p)$ such that

(R1) D_p is a finite subset of $\mathbb{Q}$,
(R2) the empty digraph I_{n+1} does not embed into p,
(R3) if $a, b \in \mathbb{Q}$, $(b, a) \notin \sigma_p$, and $(b, a+1) \notin \sigma_p$, then $a + 1 < b$,
(R4) if $a, a+1 \in D_p$, then $(a, a+1) \in \sigma_p$.

For $p, q \in \mathbb{P}_n$, let the relation $\leq$ on $\mathbb{P}_n$ be given by

$$p \leq q \Leftrightarrow D_p \supseteq D_q \wedge \sigma_p \cap D_q^2 = \sigma_q,$$

i.e. $p \leq q$ iff q is a substructure of p.

Clearly, $(\mathbb{P}_n, \leq)$ is a partial order. For each $a \in \mathbb{Q}$, consider the set

$$\mathcal{E}_a = \{p \in \mathbb{P}_n : a \in D_p\}.$$

To see that each $\mathcal{E}_a$ is dense open in $\mathbb{P}_n$, take any $q \in \mathbb{P}_n$. If $a \in D_q$, then $q \in \mathcal{E}_a$, and so there is nothing to prove. If $a \notin D_q$, then

$$p = (D_q \cup \{a\}, \sigma_q \cup ((D_q \setminus \{a+1\}) \times \{a\}) \cup (\{a\} \times (D_q \cap \{a+1\})))$$

is clearly a condition satisfying all (R1)-(R4). Also, such p satisfies both $p \leq q$ and $p \in \mathcal{E}_a$, so $\mathcal{E}_a$ is dense open in $\mathbb{P}_n$. Note that this directly implies that the set $\mathcal{E}_F = \{p \in \mathbb{P}_n : F \subseteq D_p\}$ is also dense open for any finite $F \subseteq \mathbb{Q}$.

We now define another family of dense open sets in $\mathbb{P}_n$. Suppose that F, K, and H are pairwise disjoint finite subsets of $\mathbb{Q}$, and that $m > 0$ is an integer. We define the set $\mathcal{E}_{F,K,H,m}$ as the set of all $p \in \mathbb{P}_n$ such that:

(G1) $F \cup K \cup H \subseteq D_p$,
(G2) if $(F, K, H) \in C_n(p)$, then there is $u \in D_p \cap (\mathbb{Q} \setminus \mathbb{Z})$ such that
   (a) $\max(F \cup K \cup H) < u < \max(F \cup K \cup H) + \frac{1}{m}$,
   (b) $u \in (D_p, \sigma_p)^H_{F,K}$.

We will prove that $\mathcal{E}_{F,K,H,m}$ is dense open in $\mathbb{P}_n$. Take any $q \in \mathbb{P}_n$. If $q \in \mathcal{E}_{F,K,H,m}$ there is nothing to prove, so assume not. By density of the set $D_{F \cup K \cup H}$ there is some $q' \leq q$ such that $F \cup K \cup H \subseteq D_{q'}$. If $(F, K, H) \notin C_n(q')$, then $q' \in \mathcal{E}_{F,K,H,m}$ and the proof is finished. Thus, we can assume that $(F, K, H) \in C_n(q')$. Take any $u \in (\mathbb{Q} \setminus \mathbb{Z}) \setminus \bigcup_{a \in D_{q'}} \{a, a-1, a+1\}$ such that

$$\max(F \cup K \cup H) < u < \max(F \cup K \cup H) + \frac{1}{m}.$$

This is possible as $D_{q'}$ is a finite set. Finally, define

$$p = \{D_{q'} \cup \{u\}, \sigma_{q'} \cup (\{u\} \times (D_{q'} \setminus (K \cup H))) \cup (K \times \{u\})\}.$$

It is clear that p satisfies (G1) and (G2). It is also clear that p satisfies (R1). We still have to show that it satisfies (R2),(R3), and (R4). To see that (R2) holds in p, note that there is an edge between u and all elements of D_p except elements from H. Since I_n does not embed into H (as (F, K, H) is in $C_n(q')$, and consequently in

$C(p)$), it must be that I_{n+1} does not embed into p. Hence (R2) is true in p. Next we show (R3). Take any $a, b \in \mathbb{Q}$, and suppose that $(b, a) \notin \sigma_p$, that $(b, a+1) \notin \sigma_p$, and that $b \le a+1$. Since $q' \in \mathbb{P}_n$, it must be that $u = a$, $u = a+1$, or $u = b$. So let us consider these three cases.

(1) $u = a$. Then $u + 1 \in D_{q'}$ contradicting the choice of u.
(2) $u = a + 1$. Then $u - 1 \in D_{q'}$ contradicting the choice of u.
(3) $u = b$. Then $(u, a) \notin \sigma_p$ and $(u, a + 1) \notin \sigma_p$. Be definition of p, this means that $a + 1 \in K \cup H$. From the choice of u, we have then $a + 1 < u$, contradicting the assumption that $u = b \le a + 1$.

Since all three cases are not possible, we conclude that p satisfies (R3) as well. Finally, we prove (R4). Suppose that $a, a+1 \in D_p$ are such that $(a, a+1) \notin \sigma_p$. As q' satisfies (R4), it must be that $u = a$ or $u = a+1$. In either case we get contradiction with the choice of u, simply because then either $u+1$ or $u-1$ would belong to $D_{q'}$ which is not possible. Hence p is a well defined condition in $\mathbb{P}_n$. Obviously, $p \le q' \le q$ and $p \in \mathcal{E}_{F,K,H,m}$, so the latter set is dense open in $\mathbb{P}_n$.

Now we proceed with the generic construction. By Lemma 1.9, there is a filter $\mathcal{G}_n$ in $\mathbb{P}_n$ intersecting every element of the following family of dense sets:

$$\left\{D_F : F \in [\mathbb{Q}]^{<\omega}\right\} \cup \left\{D_{F,K,H,m} : F, K, H \in [\mathbb{Q}]^{<\omega} \text{ pairwise disjoint}, m > 0\right\}.$$

This is saying that $\mathcal{G}_n$ is *sufficiently generic*. Define

$$\sigma = \bigcup\nolimits_{p \in \mathcal{G}_n} \sigma_p.$$

Now we will prove that $(\mathbb{Q}, \sigma)$ satisfies all conditions of Theorem 1.10 which will imply that $\mathcal{M}_{\mathbb{P}(\Gamma_n)} = \mathcal{C}_{\mathbb{R}}$.

Lemma 2.2. *$(\mathbb{Q}, \sigma)$ is a digraph not embedding I_{n+1}.*

Proof. As $\sigma \subseteq \mathbb{Q}^2$, it is enough to prove that σ does not contain loops and that there are no edges in both directions. To see that there are no loops in σ, suppose that $(a, a) \in \sigma$ for some $a \in \mathbb{Q}$. Then there is some $p \in \mathcal{G}_n$ so that $(a, a) \in \sigma_p$, but this is not possible as (D_p, σ_p) is a digraph. To see that there are no edges in both directions, suppose that $(a, b), (b, a) \in \sigma$ for some $a, b \in \mathbb{Q}$. Then there is $p_1 \in \mathcal{G}_n$ such that $(a, b) \in \sigma_{p_1}$ and there is $p_2 \in \mathcal{G}_n$ such that $(b, a) \in \sigma_{p_2}$. Since $\mathcal{G}_n$ is a filter, there is some $p \in \mathcal{G}_n$ such that $p \le p_1, p_2$. Then $(a, b), (b, a) \in \sigma_p$ which is again not possible since p is a digraph.

We still have to show that $(\mathbb{Q}, \sigma)$ does not embed I_{n+1}. Suppose on the contrary that $\varphi : I_{n+1} \to (\mathbb{Q}, \sigma)$ is an embedding. By density of the set $\mathcal{E}_{\varphi[I_{n+1}]}$ and sufficient genericity of $\mathcal{G}_n$, there is some $p \in \mathcal{G}_n \cap \mathcal{E}_{\varphi[I_{n+1}]}$. This means that $\varphi[I_{n+1}] \subseteq D_p$. Since $\sigma_p = \sigma \cap D_p^2$, this means that φ is an embedding from I_{n+1} into (D_p, σ_p) which is not possible by the definition of $\mathbb{P}_n$. Hence, I_{n+1} does not embed into $(\mathbb{Q}, \sigma)$. □

Lemma 2.3. *For each $x \in \mathbb{R} \cup \{\infty\}$, and each $A \subseteq \mathbb{Q}$ such that*

$$(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, x) \subseteq A \subseteq (-\infty, x),$$

we have $(A, \sigma) \cong \Gamma_n$.

Proof. First fix $x \in \mathbb{R} \cup \{\infty\}$ and $A \subseteq \mathbb{Q}$ such that $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, x) \subseteq A \subseteq (-\infty, x)$. To prove the statement of the lemma, by Lemma 2.1, it is enough to prove that (A, σ) does not embed I_{n+1} and that $(A, \sigma)^H_{F,K} \neq \emptyset$ for each $(F, K, H) \in C_n(A, \sigma)$. By Lemma 2.2, I_{n+1} does not embed into (A, σ). So take $(F, K, H) \in C_n(A, \sigma)$. Take $m > 0$ such that

$$\max(F \cup K \cup H) + \frac{1}{m} < x.$$

Such an m exists because $F \cup K \cup H$ is a finite set. By density of the set $\mathcal{E}_{F,K,H,m}$ and sufficient genericity of $\mathcal{G}_n$, there is some $p \in \mathcal{G}_n \cap \mathcal{E}_{F,K,H,m}$. Then we have $F \cup K \cup H \subseteq D_p$. As σ_p is just a restriction of σ to D_p, we have $(F, K, H) \in C_n(p)$. Hence, by the definition of $\mathcal{E}_{F,K,H,m}$, there is some $u \in D_p \cap (\mathbb{Q} \setminus \mathbb{Z})$ such that $\max(F \cup K \cup H) < u < \max(F \cup K \cup H) + \frac{1}{m}$ and $u \in (D_p, \sigma_p)^H_{F,K}$. In particular, this means that $u \in A$, and since σ_p is just a restriction of σ, that $u \in (A, \sigma)^H_{F,K}$ as well. Hence $(A, \sigma)^H_{F,K}$ is non-empty, and $(A, \sigma) \cong \Gamma_n$. $\square$

Lemma 2.4. *For each $y \in \mathbb{Q}\setminus\mathbb{Z}$, and each $C \subseteq \mathbb{Q}$ such that $(\mathbb{Q}\setminus\mathbb{Z}) \cap (-\infty, y] \subseteq C$ and $C \subseteq (-\infty, y]$, we have that $(C, \sigma) \ncong \Gamma_n$.*

Proof. Suppose that $(C, \sigma) \cong \Gamma_n$ for some $y \in \mathbb{Q} \setminus \mathbb{Z}$ and some $C \subseteq \mathbb{Q}$ such that $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, y] \subseteq C \subseteq (-\infty, y]$. Note that since $y \in \mathbb{Q} \setminus \mathbb{Z}$, we know that both y and $y - 1$ are in $\mathbb{Q} \setminus \mathbb{Z}$. Note also that $y = \max(C)$. By density of the set $D_{\{y,y-1\}}$ and sufficient genericity of $\mathcal{G}_n$, we know that there is some $p \in \mathcal{G}_n \cap D_{\{y,y-1\}}$. Then $y, y - 1 \in D_p$. Since $\mathcal{G}_n$ is a filter, and since $(y - 1, y) \in \sigma_p$ by property (R4), we know that $(y - 1, y)$ is in σ as well. Since (C, σ) is universal for all digraphs not embedding I_{n+1}, it contains a substructure isomorphic to Ω from Example 1.5, i.e. we can assume that $(\{u, v, w\}, \{(u, v)\})$ is actually a substructure of (C, σ). This, in particular means that $(u, v) \in \sigma$ and there are no other edges between u, v, and w in σ. Since we assumed that $(C, \sigma) \cong \Gamma_n$, by Lemma 2.1, there is some $b \in (C, \sigma)^{\{y-1,y\}}_{\emptyset,\emptyset}$ (note that $(\emptyset, \emptyset, \{y - 1, y\}) \in C_n(C, \sigma)$ as I_{n+1} does not embed into $(\{y - 1, y\}, \sigma)$ for any $n \geq 2$). As $y = \max(C)$, this means $b < y$. By density of the set $\mathcal{E}_{y-1,y,b}$ there is some $q \in \mathcal{G}_n$ such that $\{y - 1, y, b\} \subseteq D_q$. This is not possible as σ_q is a restriction of σ to D_q, and we have a situation that $b < y$, $(b, y - 1) \notin \sigma_q$, and $(b, y) \notin \sigma_q$ contradicting (R3) for q. $\square$

Lemma 2.5. $\mathcal{B}_\mathbb{R} \subseteq \mathcal{M}_{\mathbb{P}(\mathbb{Q},\sigma)}$.

Proof. Directly follows from Theorem 1.1, as $(\mathbb{Q}, \sigma)$ is isomorphic to Γ_n and the age of the latter digraph has strong amalgamation. $\square$

Theorem 1.2 follows by direct application of Theorem 1.10, using Lemma 2.2, Lemma 2.3, Lemma 2.4, and Lemma 2.5 (case $n = 1$ is showed in Subsection 1.2).

3. Omitting tournaments

In this section we complete the proof of Theorem 1.4. So, for the remaining of the section, fix a non-trivial family of finite tournaments $\mathcal{T}$ which does not contain the two-element tournament. For clarity, let us denote by ρ the digraph relation on $\Gamma(\mathcal{T})$, i.e. $(\Gamma(\mathcal{T}), \rho)$ is the unique countable ultrahomogeneous digraph whose age consists of all finite digraphs not embedding any tournament from $\mathcal{T}$. Note that both existence and uniqueness follow from Fraïssé's Theorem 1.7. We will prove that $\mathcal{M}_{\mathbb{P}(\Gamma(\mathcal{T}))} = \mathcal{C}_\mathbb{R}$. We do this in a series of lemmas, each of which shows that conditions of Theorem 1.10 are satisfied. The proof is generic, in the same sense as in Section 2. We also try to keep the same structure of the proof.

Let $\mathbb{P}$ be the set of all finite digraphs $p = (D_p, \sigma_p)$ such that:

(P1) D_p is a finite subset of $\mathbb{Q}$;
(P2) (D_p, σ_p) embeds into $\Gamma(\mathcal{T})$;
(P3) for each $a, b \in \mathbb{Q}$, if $(a, b) \in \sigma_p$ and $(a + 1, b) \in \sigma_p$, then $a + 1 < b$;
(P4) $(a, a + 1) \notin \sigma_p$ and $(a + 1, a) \notin \sigma_p$ for each $a \in \mathbb{Q}$.

We order $\mathbb{P}$ by the relation $\leq$ as follows: $p \leq q$ if q is a substructure of p, i.e. if $D_p \supseteq D_q$ and $\sigma_p \cap D_q^2 = \sigma_p$. Clearly, $(\mathbb{P}, \leq)$ defined this way is a partial order.

Observe next, that for each finite set $K \subseteq \mathbb{Q}$, the set $\mathcal{C}_K = \{p \in \mathbb{P} : K \subseteq D_p\}$ is dense open in $\mathbb{P}$. To see this, note that for any $q \in \mathbb{P}$ and any $a \in \mathbb{Q}$, the condition $q' = (D_q \cup \{a\}, \sigma_q)$ also belongs to $\mathbb{P}$. The consecutive application of this observation then provides $p \leq q$ such that $K \subseteq D_p$.

We will need another family of dense open sets in $\mathbb{P}$. For this, consider two integers $m, n > 0$, a set $K \in [\mathbb{Q}]^n$, a digraph (D, ρ) of size $n+1$ which is a substructure of $\Gamma(\mathcal{T})$, and a 1-1 function $\psi : K \to D$. Now, let us define $\mathcal{C}_{m,n,K,D,\psi}$ as the set of all $p \in \mathbb{P}$ satisfying the following two conditions:

(d1) $K \subseteq D_p$;
(d2) if $\psi : (K, \sigma_p) \to (D, \rho)$ is an embedding, then there are $u \in D_p \cap (\mathbb{Q} \setminus \mathbb{Z})$ and a map η such that $\max(K) < u < \max(K) + \frac{1}{m}$, that $\psi \subseteq \eta$, and that $\eta : (K \cup \{u\}, \sigma_p) \to (D, \rho)$ is an isomorphism.

We will now explan why each set $\mathcal{C}_{m,n,K,D,\psi}$ is dense open in $\mathbb{P}$. Take any $q \in \mathbb{P}$. By the density of the set $\mathcal{C}_K$, we can extend q to some q' such that $K \subseteq D_{q'}$. Now, if $\psi : (K, \sigma_{q'}) \to (D, \rho)$ is not an embedding, then $q' \leq q$ and $q' \in \mathcal{C}_{m,n,K,D,\psi}$, so we showed the density of the latter set. So assume that $\psi : (K, \sigma_{q'}) \to (D, \rho)$ is an embedding. Take $u \in \left((\mathbb{Q} \setminus \mathbb{Z}) \cap (\max(K), \max(K) + \frac{1}{m})\right) \setminus \bigcup_{y \in D_{q'}} \{y-1, y, y+1\}$. This is possible as intervals in $\mathbb{Q} \setminus \mathbb{Z}$ are infinite and $D_{q'}$ is a finite set. Now define a map η and a digraph relation σ' on $K \cup \{u\}$ so that $\sigma_{q'} \cap K^2 \subseteq \sigma'$, that $\psi \subseteq \eta$, and that $\eta : (K \cup \{u\}, \sigma') \to (D, \rho)$ is an isomorphism. This is possible as ψ is an embedding from $(K, \sigma_{q'})$ into (D, ρ) and D has exactly one element more than K. Now we are in the situation that K is the intersection of domains of digraphs $(D_{q'}, \sigma_{q'})$ and $(K \cup \{u\}, \sigma')$, and that $\sigma_{q'}$ and σ' coincide on K. We define a digraph $p = (D_p, \sigma_p)$ so that $D_p = D_{q'} \cup \{u\}$ and $\sigma_p = \sigma_{q'} \cup \sigma'$. We will now prove that $p \in \mathbb{P}$. Condition (P1) is clearly satisfied, as well as (P2) - note however, that for (P2) we use the fact that no tournament from $\mathcal{T}$ embeds neither into $(K \cup \{u\}, \sigma')$ nor into $(D_{q'}, \sigma_{q'})$ and there are no other edges in σ_p. Next, we prove (P3). So take any $a, b \in \mathbb{Q}$ and suppose on the contrary, that both (a, b) and $(a+1, b)$ are in σ_p, and that $b \leq a+1$. Since q' satisfies (P3), it must be that u is equal to either a, $a+1$, or b. So we consider all three cases.

(1) $u = a$. Then $u + 1 = a + 1 \in D_{q'}$, contradicting the choice of u.
(2) $u = a + 1$. Then $u - 1 = a \in D_{q'}$, again contradicting the choice of u.
(3) $u = b$. Then $(a, u) \in \sigma_p$ and $(a+1, u) \in \sigma_p$. Since there are no edges between u and elements of $D_{q'} \setminus (K \cup \{u\})$, this means that $a, a+1 \in K$. But then, $b \leq a + 1 \leq \max(K)$ contradicting the choice of u.

Since all three cases are impossible, we conclude that p satisfies (P3). We still have to show that p satisfies (P4). So suppose that for some $a \in \mathbb{Q}$, we have $(a, a+1) \in \sigma_p$. Again, since q' satisfies (P4), it must be that $u = a$ or $u = a+1$. If $u = a$, then $u + 1 = a + 1 \in D_{q'}$ contradicting the choice of u. In the same way, it can be seen that $u \neq a + 1$, as well. Hence, p satisfies (P4). Since obviously $p \leq q' \leq q$ and $p \in \mathcal{C}_{m,n,K,D,\psi}$, we conclude that $\mathcal{C}_{m,n,K,D,\psi}$ is dense open in $\mathbb{P}$.

Since the intersection of finitely many dense open sets is dense open, and there are only finitely many 1-1 functions from K into D, for two fixed finite sets K and D, the following set is also dense open (for $m, n > 0$, $K \in [\mathbb{Q}]^n$, and $D \in [\Gamma(\mathcal{T})]^{n+1}$): $\mathcal{C}_{m,n,K,D}$ is the set of all $p \in \mathbb{P}$ such that for all $\psi : K \xrightarrow{1\text{-}1} D$, if ψ is an embedding, then there are u and η satisfying (d1) and (d2).

Consider now the family

$$\{\mathcal{C}_K : K \subseteq \mathbb{Q} \text{ finite}\} \cup \left\{\mathcal{C}_{m,n,K,D} : m, n > 0, K \in [\mathbb{Q}]^n, D \in [\Gamma(\mathcal{T})]^{n+1}\right\}.$$

This is a countable family of dense sets in $\mathbb{P}$, so by Lemma 1.9, we know that there is a filter $\mathcal{G}$ in $\mathbb{P}$ intersecting each member of this family. We call this property *sufficient genericity* of $\mathcal{G}$. Define

$$\sigma = \bigcup_{p \in \mathcal{G}} \sigma_p.$$

Clearly, $\sigma \subseteq Q^2$, so the relational structure $(\mathbb{Q}, \sigma)$ is in the same signature as $(\Gamma(\mathcal{T}), \rho)$. We proceed to prove that it satisfies all the conditions of Theorem 1.10.

Lemma 3.1. *$(\mathbb{Q}, \sigma)$ is a digraph not embedding any element from $\mathcal{T}$.*

Proof. As $\sigma \subseteq \mathbb{Q}^2$, it is enough to prove that σ does not contain loops and that there are no edges in both directions. To see that there are no loops in σ, suppose that $(a, a) \in \sigma$ for some $a \in \mathbb{Q}$. Then there is some $p \in \mathcal{G}$ so that $(a, a) \in \sigma_p$, but this is not possible as (D_p, σ_p) is a digraph. To see that there are no edges in both directions, suppose that $(a, b), (b, a) \in \sigma$ for some $a, b \in \mathbb{Q}$. Then there is $p_1 \in \mathcal{G}$ such that $(a, b) \in \sigma_{p_1}$ and there is $p_2 \in \mathcal{G}$ such that $(b, a) \in \sigma_{p_2}$. Since $\mathcal{G}$ is a filter, there is some $p \in \mathcal{G}$ such that $p \leq p_1, p_2$. Then $(a, b), (b, a) \in \sigma_p$ which is again not possible since p is a digraph.

We still have to show that $(\mathbb{Q}, \sigma)$ does not embed any element from $\mathcal{T}$. Suppose that $(T, \rho') \in \mathcal{T}$ is such that there is an embedding $\varphi : (T, \rho') \to (\mathbb{Q}, \sigma)$. By density of the set $\mathcal{C}_{\varphi[T]}$, there is some $p \in \mathcal{G}$ such that $\varphi[T] \subseteq D_p$. Since $\sigma_p = \sigma \cap D_p^2$, this means that φ is an embedding from (T, ρ') into (D_p, σ_p) which is not possible by the definition of $\mathbb{P}$. Hence, no element from $\mathcal{T}$ embeds into $(\mathbb{Q}, \sigma)$. □

Lemma 3.2. *For $x \in \mathbb{R} \cup \{\infty\}$, and $A \subseteq \mathbb{Q}$ such that $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, x) \subseteq A$ and $A \subseteq (-\infty, x)$, we have $(A, \sigma) \cong \Gamma(\mathcal{T})$.*

Proof. Fix $x \in \mathbb{R} \cup \{\infty\}$ and $A \subseteq \mathbb{Q}$ such that $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, x) \subseteq A \subseteq (-\infty, x)$. To show that $(A, \sigma) \cong \Gamma(\mathcal{T})$, by Theorem 1.7, it is enough to prove that (A, σ) is countable, ultrahomogeneous, and embeds all digraphs not embedding any tournament from $\mathcal{T}$. Clearly, A is a countable set.

First, we show that (A, σ) embeds all finite digraphs not embedding any element from $\mathcal{T}$. So let (D, ρ') be any finite digraph not embedding any element from $\mathcal{T}$. Let $D = \{d_1, \dots, d_m\}$. By induction on $k < m$, we prove that if $(\{d_1, \dots, d_k\}, \rho')$ embeds into (A, σ), then $(\{d_1, \dots, d_k, d_{k+1}\}, \rho')$ also embeds into (A, σ). Since $(\{d_1\}, \rho')$ clearly embeds into (A, σ), this will complete this part of the proof. So suppose that $k < m$ and that $\varphi : (\{d_1, \dots, d_k\}, \rho') \to (A, \sigma)$ is an embedding. To simplify notation, we can assume that $D \subseteq \Gamma(\mathcal{T})$ and that $\rho' = \rho \cap D^2$. Denote $E = \{d_1, \dots, d_{k+1}\}$ and $G = \{d_1, \dots, d_k\}$. Let $m > 0$ be such that $(\max(\varphi[G]), \max(\varphi[G]) + \frac{1}{m}) \subseteq (-\infty, x)$. Let $\psi : (\varphi[G], \sigma) \to (E, \rho)$ be defined by $\psi(a) = b$ iff $\varphi(b) = a$. Since φ is an isomorphism between (G, ρ) and $(\varphi[G], \sigma)$, ψ is an embedding. By density of the set $\mathcal{C}_{m,k,\varphi[G],E}$ and sufficient genericity of $\mathcal{G}$, there is $p \in \mathcal{G} \cap \mathcal{C}_{m,k,\varphi[G],E}$. This means that:

(1) $\varphi[G] \subseteq D_p$,
(2) since ψ is an embedding, there is $u \in D_p \cap (\mathbb{Q} \setminus \mathbb{Z})$ and a map η such that
 (a) $\max(\varphi[G]) < u < \max(\varphi[G]) + \frac{1}{m}$,
 (b) $\eta : (\varphi[G] \cup \{u\}, \sigma_p) \to (E, \rho)$ is an isomorphism and $\psi \subseteq \eta$.

Since $p \in \mathcal{G}$, in particular $\sigma_p \subseteq \sigma$, this means that η^{-1} is an embedding of (E, ρ') into (A, σ) (note that $\varphi[G] \cup \{u\} \subseteq A$ as $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, x) \subseteq A$). Hence, the proof of induction is complete, which in turn finishes the proof of universality of (A, σ).

Now we prove that (A, σ) is ultrahomogeneous. We use the characterization of ultrahomogeneity given in Theorem 1.8. So let $F \subseteq A$ be a finite substructure, let f be an embedding from (F, σ) into (A, σ), and take any $a \in A \setminus F$. Consider the digraph $(F \cup \{a\}, \sigma)$. This is a finite digraph not embedding any element from $\mathcal{T}$ (by Lemma 3.1), thus it embeds into $\Gamma(\mathcal{T})$ by the result of the previous paragraph. Let $\psi : F \cup \{a\} \to \Gamma(\mathcal{T})$ be this embedding. Since $A \subseteq (-\infty, x)$ and F is a finite set, there is some $m > 0$ such that $(\max(F), \max(F) + \frac{1}{m}) \subseteq (-\infty, x)$. Denote the cardinality of F by n. By density of the set $\mathcal{C}_{m,n,f[F],\psi[F \cup \{a\}]}$ and sufficient genericity of $\mathcal{G}$, there is $p \in \mathcal{G} \cap \mathcal{C}_{m,n,F,\psi[F \cup \{a\}]}$. This means that:

(1) $F \subseteq D_p$,
(2) since $\psi \circ f^{-1} : (f[F], \sigma) \to (\psi[F \cup \{a\}], \rho)$ is an embedding, there is an element $u \in D_p \cap (\mathbb{Q} \setminus \mathbb{Z})$ and a map η such that
  (a) $\max(F) < u < \max(F) + \frac{1}{m}$, in particular $u \in A$,
  (b) $\eta : (f[F] \cup \{u\}, \sigma_p) \to (\psi[F \cup \{u\}], \rho)$ is an isomorphism such that $\psi \circ f^{-1} \subseteq \eta$.

Since $p \in \mathcal{G}$, we can assume that all conditions above hold for σ. So, finally, define $\Phi = \eta^{-1} \circ \psi : (F \cup \{a\}, \sigma) \to (f[F] \cup \{u\}, \sigma)$. Since η is an isomorphism and ψ is an embedding, Φ is also an embedding. Note that since $u \in A$, the image of Φ is a subset of A, as required. To see that $f \subseteq \Phi$, take some $v \in F$. Then

$$\Phi(v) = \eta^{-1}(\psi(v)) = (\psi \circ f^{-1})^{-1}(\psi(v)) = f(\psi^{-1}(\psi(v))) = f(v).$$

Hence, $f \subseteq \Phi$, and this completes the proof of ultrahomogeneity of (A, σ) using Theorem 1.8. □

Lemma 3.3. *For each $y \in \mathbb{Q} \setminus \mathbb{Z}$, and $C \subseteq \mathbb{Q}$ such that $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, y] \subseteq C$ and $C \subseteq (-\infty, y]$, we have that $(C, \sigma) \ncong \Gamma(\mathcal{T})$.*

Proof. Suppose that for $y \in \mathbb{Q} \setminus \mathbb{Z}$ and $C \subseteq \mathbb{Q}$ such that $(\mathbb{Q} \setminus \mathbb{Z}) \cap (-\infty, y] \subseteq C$ and $C \subseteq (-\infty, y]$, we have $(C, \sigma) \cong \Gamma(\mathcal{T})$. Note that since $y \in \mathbb{Q} \setminus \mathbb{Z}$, we know that both y and $y - 1$ are in $\mathbb{Q} \setminus \mathbb{Z}$. Note also that $y = \max(C)$. By density of the set $D_{\{y,y-1\}}$ we know that there is some $p \in \mathcal{G}$ such that $y, y - 1 \in D_p$. Since $\mathcal{G}$ is a filter, and by property (P4) both $(y - 1, y) \notin \sigma_p$ and $(y, y - 1) \notin \sigma_p$ hold, we know that $(y, y - 1)$ and $(y - 1, y)$ are not in σ as well. Since (C, σ) embeds all digraphs not embedding any element of $\mathcal{T}$, it contains a substructure isomorphic to Λ - take a look at Example 1.5. Let us assume that Λ is actually a substructure of (C, σ) and that $\Lambda = (\{u, v, w\}, \{(u, w), (v, w)\})$, i.e. $(u, w) \in \sigma$ and $(v, w) \in \sigma$ and there are no other edges between u, v, and w in σ. Consider now two digraphs $\Sigma_1 = (\{u, v\}, \emptyset)$ and $\Sigma_2 = (\{y - 1, y\}, \emptyset)$. These are clearly isomorphic substructures of (C, σ), for example $\varphi : \{u, v\} \to \{y - 1, y\}$ given by $\varphi(u) = y$ and $\varphi(v) = y$ is one isomorphism. As (C, σ) is ultrahomogeneous there is an automorphism $\Phi : (C, \sigma) \to (C, \sigma)$ such that $\varphi \subseteq \Phi$. Consider $\Phi(w)$. As Φ is an isomorphism, it must be that $(\Phi(u), \Phi(w)) \in \sigma$ and $(\Phi(v), \Phi(w)) \in \sigma$. This exactly means that $(y - 1, \Phi(w)) \in \sigma$ and $(y, \Phi(w)) \in \sigma$. By density of the sets $\mathcal{C}_{\{y-1,y,\Phi(w)\}}$ there is some $q \in \mathcal{G}$ such that $y - 1, y, \Phi(w) \in D_q$. Since q satisfies condition (P3), it means that $y < \Phi(w)$. But this is in contradiction with the fact that Φ is an automorphism of C and that $y = \max(C)$. □

Lemma 3.4. $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{M}_{\mathbb{P}(\mathbb{Q},\sigma)}$.

Proof. Directly follows from Theorem 1.1, as $(\mathbb{Q}, \sigma)$ is isomorphic to $\Gamma(\mathcal{T})$ and the age of the latter digraph has strong amalgamation property. □

Theorem 1.4 now follows directly by considering Theorem 1.3 and by application of Theorem 1.10, using Lemma 3.1, Lemma 3.2, Lemma 3.3, and Lemma 3.4.

4. Acknowledgments

The first author gratefully acknowledges the financial support of the Ministry of Science, Technological Development and Innovation of the Republic of Serbia (Grants No. 451-03-137/2025-03/ 200125 & 451-03-136/2025-03/ 200125)

The second author has been supported by the Ministry of Science, Technological Development and Innovation (Contract No. 451-03-137/2025-03/200156) and the Faculty of Technical Sciences, University of Novi Sad throught project "Scientific and Artistic Research Work of Researchers in Teaching and Associate Positions at the Faculty of Technical Sciences, University of Novi Sad 2025" (No. 01-50/295).

Also, the second author is grateful for the support within the project of the Department for general discipline in technology, Faculty of Technical Sciences under the title Soico-technological aspects of improving the teaching process in the English language in fundamental disciplines.

References

[1] N. Ackerman, C. Freer, and R. Patel. Invariant measures concentrated on countable structures. *Forum Math. Sigma*, 4:Paper No. e17, 59, 2016.
[2] G. L. Cherlin. The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments. *Mem. Amer. Math. Soc.*, 131(621):xiv+161, 1998.
[3] G. W. Day. Maximal chains in atomic Boolean algebras. *Fund. Math.*, 67:293–296, 1970.
[4] R. Diestel, I. Leader, A. Scott, and S. Thomassé. Partitions and orientations of the Rado graph. *Trans. Amer. Math. Soc.*, 359(5):2395–2405, 2007.
[5] R. Fraïssé. Sur l'extension aux relations de quelques propriétés des ordres. *Ann. Sci. École Norm. Sup. (3)*, 71:363–388, 1954.
[6] R. Fraïssé. *Theory of relations*, volume 145 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, revised edition, 2000. With an appendix by Norbert Sauer.
[7] C. W. Henson. A family of countable homogeneous graphs. *Pacific J. Math.*, 38:69–83, 1971.
[8] S. Koppelberg. Maximal chains in interval algebras. *Algebra Universalis*, 27(1):32–43, 1990.
[9] W. Kubiś. Fraïssé sequences: category-theoretic approach to universal homogeneous structures. *Ann. Pure Appl. Logic*, 165(11):1755–1811, 2014.
[10] W. Kubiś. Weak Fraïssé categories. *Theory Appl. Categ.*, 38:Paper No. 2, 27–63, 2022.
[11] W. Kubiś and D. Mašulović. Katětov functors. *Appl. Categ. Structures*, 25(4):569–602, 2017.
[12] K. Kuratowski. Sur la notion de l'ordre dans la théorie des ensembles. *Fundamenta mathematicae*, 2(1):161–171, 1921.
[13] M. S. Kurilić. Maximal chains in positive subfamilies of $P(\omega)$. *Order*, 29(1):119–129, 2012.
[14] M. S. Kurilić. Maximal chains of copies of the rational line. *Order*, 30(3):737–748, 2013.
[15] M. S. Kurilić. Forcing with copies of countable ordinals. *Proc. Amer. Math. Soc.*, 143(4):1771–1784, 2015.
[16] M. S. Kurilić and B. Kuzeljević. Maximal chains of isomorphic subgraphs of countable ultrahomogeneous graphs. *Adv. Math.*, 264:762–775, 2014.
[17] M. S. Kurilić and B. Kuzeljević. Maximal chains of isomorphic suborders of countable ultrahomogeneous partial orders. *Order*, 32(1):83–99, 2015.
[18] M. S. Kurilić and B. Kuzeljević. Positive families and Boolean chains of copies of ultrahomogeneous structures. *C. R. Math. Acad. Sci. Paris*, 358(7):791–796, 2020.
[19] M. S. Kurilić and B. Kuzeljević. Antichains of copies of ultrahomogeneous structures. *Arch. Math. Logic*, 61(5-6):867–879, 2022.
[20] M. S. Kurilić and S. Todorčević. Forcing by non-scattered sets. *Ann. Pure Appl. Logic*, 163(9):1299–1308, 2012.
[21] M. S. Kurilić and S. Todorčević. Copies of the random graph. *Adv. Math.*, 317:526–552, 2017.
[22] M. S. Kurilić and S. Todorčević. Posets of copies of countable ultrahomogeneous tournaments. *Ann. Pure Appl. Logic*, 175(10):Paper No. 103486, 12, 2024.
[23] D. Macpherson. A survey of homogeneous structures. *Discrete Math.*, 311(15):1599–1634, 2011.

(Kuzeljević) Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, Serbia.
Email address: `borisha@dmi.uns.ac.rs`
URL: `https://people.dmi.uns.ac.rs/~borisha`

(Milošević) Faculty of Technical Sciences, University of Novi Sad, Serbia.
Email address: `stepanmilosevic@uns.ac.rs`