P-IDEAL DICHOTOMY AND A STRONG FORM OF THE SOUSLIN HYPOTHESIS

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ABSTRACT. We introduce a forcing notion which forces the P-ideal dichotomy, while every almost Souslin tree from the ground model remains non-special. Thus, while the P-ideal dichotomy implies the Souslin Hypothesis, or equivalently that every Aronszajn tree has an *uncountable* antichain, it does not imply that every Aronszajn tree has a *stationary* antichain.

1. INTRODUCTION

Recall that Souslin's problem asks if in the Cantor characterization of the unit interval the *separability* can be replaced by the *countable chain condition* ([13]). This problem has played a major role in the development of set theory in the twentieth century especially after it was reformulated as a problem about trees (asking if every uncountable tree has an uncountable chain or an uncountable antichain). For example, the notion of an Aronszajn tree (a tree of height ω_1 with countable levels but without uncountable chains) came as a byproduct of an analysis of this problem (see [8]). Martin's Axiom (MA), the first forcing axiom of set theory, is another byproduct of a work on Souslin's problem ([12], [9]).

The P-ideal dichotomy (PID) is a set-theoretic combinatorial principle which involves simple and natural notion of a P-ideal of countable subsets of some index-set S and is easy to understand and use. As such PID is a set-theoretic principle that has found applications to different areas of mathematics where the notion of a P-ideal naturally shows up (see, for example, [16]). Even at the early stages of analysis of this principle it has been discovered that PID is independent of the Continuum Hypothesis while it does imply the Souslin Hypothesis ([1], [15]).

In this paper we analyze the relationship between PID and the standard strengthening of the Souslin Hypothesis which is also independent of CH and which asserts that every Aronszajn tree T admits a *stationary* antichain, an antichain that meets a stationary set of levels of T ([3], [4]). We show that while PID implies that every Aronszajn tree T has an uncountable antichain it does not imply that such a T must have also a stationary antichain. In other words, we show that PID is consistent with the existence of an Aronszajn tree T which contains no antichains that intersect a *stationary set* of levels of T. Such trees are in the literature called *almost Souslin trees*. (Recall that the negation of the Souslin Hypothesis is equivalent with the existence of a *Souslin tree*, an uncountable tree with no uncountable chains nor antichains, or equivalently an Aronszajn tree with no *uncountable* antichain (see [8]).) This should be compared with the case $\mathrm{MA}_{\aleph_1}$ an important set-theoretic

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principle that came as a byproduct of the solution of the Souslin problem ([12], [9]). This principle has consequences that have been analyzed long before $\mathrm{MA}_{\aleph_1}$ was formulated ([6]). One of them is Knaster's principle (K) asserting that every topological space X satisfying the Souslin condition satisfies the stronger condition requiring that every uncountable family $\mathcal{F}$ of open subsets of X has an uncountable subfamily $\mathcal{F}_0$ such that every two elements of $\mathcal{F}_0$ have nonempty intersection. While it is obvious that the principle K implies the Souslin Hypothesis it is much less obvious that it also implies that every Aronszajn tree admits a strictly increasing function into the rationals ([14]). So we have here an interesting discrepancy between PID and K.

2. Preliminaries

We use mostly standard set-theoretic notation. For a set X, the powerset of X is denoted $\mathcal{P}(X)$, while for a cardinal κ, $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$. If M and N are two models of some theory, then $M \prec N$ means that M is an elementary submodel of N. The relation $\subseteq^*$ denotes containment modulo a finite set, i.e. $a \subseteq^* b$ if and only if $|a \setminus b| < \omega$. We say that a partially ordered set $\langle T, <\rangle$ is a *tree* if for every $t \in T$, the set $\{x \in T : x < t\}$ of all predecessors of t is well ordered by $<$. When the ordering relation is clear from the context, we do not write it. The γth level of the tree T is denoted T_γ, while the height of an element $t \in T$ is denoted $\mathrm{ht}(t)$. Height of a tree is the supremum of the heights of its elements. If $A \subseteq T$ is a subset of a tree T, we say that it is a *level set* if $|A \cap T_\gamma| \leq 1$ for each $\gamma < \mathrm{ht}(T)$, We say that a tree of height ω_1 is *special* if it can be embedded into the rational line, or, equivalently, if it can be partitioned into countably many antichains. A tree T of height ω_1 is called an *almost Souslin* tree if, for every antichain A of T, the set $L(A) = \{\gamma < \omega_1 : T_\gamma \cap A \neq \emptyset\}$ is not stationary in ω_1. Clearly every almost Souslin tree is non-special.

Recall that a collection $\mathfrak{I} \subseteq \mathcal{P}(S)$ is an *ideal* on a set S if $S \notin \mathfrak{I}$, if $[S]^{<\omega} \subseteq \mathfrak{I}$, if $A \cup B \in \mathfrak{I}$ for $A, B \in \mathfrak{I}$, and if $A \subseteq B \in \mathfrak{I}$ implies $A \in \mathfrak{I}$. We say that an ideal $\mathfrak{I}$ on a set S is a *P-ideal* if for any countable collection $\mathscr{A} \subseteq \mathfrak{I}$ there is $B \in \mathfrak{I}$ such that $A \subseteq^* B$ for every $A \in \mathscr{A}$. Given a collection $\mathscr{A} \subseteq \mathcal{P}(S)$, we say that the set $B \subseteq S$ is *orthogonal* to $\mathscr{A}$ if the set $B \cap A$ is finite for each $A \in \mathscr{A}$. For $\mathscr{A} \subseteq \mathcal{P}(S)$, the set of all subsets of S orthogonal to $\mathscr{A}$ is denoted $\mathscr{A}^\perp$.

P-ideal Dichotomy. For every P-ideal $\mathfrak{I}$ of countable subsets of some set S, either

(1) there is an uncountable $X \subseteq S$ such that $[X]^{\aleph_0} \subseteq \mathfrak{I}$, or
(2) S can be decomposed into countably many sets orthogonal to $\mathfrak{I}$.

The P-ideal Dichotomy was introduced in [15], and the version restricted to sets of cardinality $\aleph_1$ was introduced in [1]. It is well known that the Proper Forcing Axiom (PFA) implies the P-ideal dichotomy. Recall that PFA implies the negation of CH. As already mentioned in the introduction, PID is consistent with CH, and PID implies the Souslin Hypothesis.

Next, we recall the notions of properness and strong properness. Suppose that $\mathbb{P}$ is a forcing notion, $p \in \mathbb{P}$ is a condition and X is a set. We say that p is $(X, \mathbb{P})$-*generic* if for every $q \leq p$ and every $D \in X$ which is dense in $\mathbb{P}$, there is an element of $D \cap X$ compatible with q. If p is such that for every $q \leq p$ and every D which is a dense subset of $\mathbb{P} \cap X$, there is an element of D compatible with q, then we say that p is *strongly $(X, \mathbb{P})$-generic* condition. The poset $\mathbb{P}$ is *proper for* *a class* $\mathcal{X}$ if for every $X \in \mathcal{X}$, each condition $p \in \mathbb{P} \cap X$ can be extended to an $(X, \mathbb{P})$-generic condition. It is *strongly proper for* $\mathcal{X}$ if for every $X \in \mathcal{X}$, every $p \in \mathbb{P} \cap X$ can be extended to a strongly $(X, \mathbb{P})$-generic condition. Clearly, every strongly $(X, \mathbb{P})$-generic condition is $(X, \mathbb{P})$-generic, hence if $\mathbb{P}$ is strongly proper for $\mathcal{X}$, then it is also proper for $\mathcal{X}$.

3. CONSISTENCY

In this section we will give the proof of the main theorem. Fix an almost Souslin Aronszajn tree T. First we prove a lemma we will be using in the proof.

Lemma 3.1. *Suppose that T is an almost Souslin tree, and that $A \subseteq T$ is a level set. If $M \prec H(\theta)$ is countable, θ large enough, $A, T \in M$, and $A \cap T_{M\cap\omega_1} = \{t_0\}$, then there is $t_1 \in M \cap T$ such that $t_1 \in A$ and $t_1 < t_0$.*

Proof. Suppose that there is no such t_1. Consider the set

$$X = \left\{t \in A : (\forall t' \in A)\ \mathrm{ht}(t') < \mathrm{ht}(t) \Rightarrow t' \perp t\right\}.$$

Clearly, $X \in M$, and since $t_0 \in X$, X is a stationary antichain in T which contradicts the fact that T is an almost Souslin tree. □

In the rest of the proof we rely heavily on the Neeman's proof of the consistency of PFA using the finite supports iteration of proper posets (as presented in [11]). So let θ be a supercompact cardinal, and let $F : \theta \to H(\theta)$ be a Laver function. Fix a well ordering $<_w$ of $H(\theta)$ in such a way that for every $x, y \in H(\theta)$, if $|\mathrm{trcl}(x)| < |\mathrm{trcl}(y)|$, then $x <_w y$. Denote

$$\Sigma = \{\alpha < \theta : (H(\alpha), \in, F \restriction \alpha, <_w\restriction H(\alpha)) \prec (H(\theta), \in, F, <_w)\}.$$

For an ordinal $\alpha \in \Sigma$, let α^+ be a minimal cardinal in Σ above α. In the central definition we will refer to two classes of models. The first is the class $\mathcal{S}$ of all countable elementary submodels of the structure $(H(\theta), \in, F, <_w)$. The second is the class $\mathcal{T}$ of all models $H(\alpha)$ where $\alpha \in \Sigma$ and $H(\alpha)^\omega \subseteq H(\alpha)$. Neeman refers to the elements of $\mathcal{S}$ and $\mathcal{T}$ as small nodes and transitive nodes (respectively), so we will follow his abbrevations. In particular, note that $\mathcal{S}$ and $\mathcal{T}$ satisfy conditions of Definition 2.2 in [11] for $\kappa = \omega$, $\lambda = \omega_1$ and $K = H(\theta)$. If κ is a regular uncountable cardinal and $X \in H(\kappa)$, then $\mathrm{Sk}^{H(\kappa)}(X)$ denotes the Skolem closure of a set X in $H(\kappa)$. When κ is clear from the context, and $M \prec H(\kappa)$, the Skolem closure of $M \cup \{M\}$ in $H(\kappa)$ will be denoted M^+.

Lemma 3.2. *Suppose that M and N belong to $\mathcal{S}$, and that κ is an uncountable regular cardinal such that $H(\kappa) \in M \in N$. Then $(M \cap H(\kappa))^+ \in N \cap H(\kappa)$, where the Skolem closure is taken in $H(\kappa)$.*

We will need a slight modification of the notion of an $\in$-chain of countable elementary submodels of $H(\kappa)$.

Definition 3.3. Suppose that κ is a regular uncountable cardinal and that $\mathcal{M}$ is a collection of elementary submodels of $H(\kappa)$. We say that $\{M_0, \dots, M_k\} \subseteq \mathcal{M}$ is an $\in$-*path* in $\mathcal{M}$ if $M_i \in M_{i+1}$ for $i < k$. If $\{M_0, \dots, M_k\}$ is an $\in$-path in $\mathcal{M}$, and $M_i \in M_i^+ \in M_{i+1}$ for $i < k$, then we say that it is a *strong* $\in$-path in $\mathcal{M}$.

Now we can define the poset we will be using in the proof. To simplify the notation let s/α denote the set $\{M \cap H(\alpha^+) : M \in s \cap \mathcal{S}\ \&\ \alpha \in M\}$, for α an ordinal, and s an $\in$-path in $\mathcal{S} \cup \mathcal{T}$. Suppose that s is an $\in$-path in $\mathcal{S} \cup \mathcal{T}$. We say that s is *closed under intersections* if $M \cap N \in s$ whenever $M, N \in s$. A *side condition* is an $\in$-path in $\mathcal{S} \cup \mathcal{T}$ closed under intersections.

Definition 3.4. $\mathbb{A}$ is the poset of all pairs $\langle s, p\rangle$ such that:

(1) s is a side condition.
(2) p is a finite set of ordinals such that for each $\alpha \in p$, $H(\alpha) \in s$ and

$$\Vdash_{\mathbb{A}\cap H(\alpha)} \text{"}F(\alpha) \text{ is a P-ideal on some ordinal } \check{\gamma} \text{ which is not a countable union of sets orthogonal to } F(\alpha)\text{"}.$$

Let $\mathcal{N}$ be the set of all $\mathbb{A} \cap H(\alpha)$-names. For a countable $M \prec H(\alpha^+)$, let $\dot{x}^\alpha_M \in \mathcal{N}$ be a name for a minimal ordinal in γ which is not in $\tau[G_\alpha]$ for any $\tau \in \mathcal{N} \cap M^+$ such that $\tau[G_\alpha]$ belongs to $F(\alpha)[G_\alpha]^\perp$, and let $\dot{b}^\alpha_M \in \mathcal{N}$ be a name for a $<_w$-minimal set in $F(\alpha)[G_\alpha]$ contained in $M[G_\alpha] \cap \gamma$ and such that $\tau[G_\alpha] \subseteq^* \dot{b}^\alpha_M[G_\alpha]$ for each $\tau \in \mathcal{N} \cap M$ such that $\tau[G_\alpha] \in F(\alpha)[G_\alpha]$, where G_α is the $\mathbb{A} \cap H(\alpha)$ generic filter.
(3) $\langle s^*, p^*\rangle \le \langle s, p\rangle$ iff $s^* \supseteq s$, and for each $\alpha \in p$, and every $M \in s/\alpha$, if $N \in (s^*/\alpha \cap M) \setminus (s/\alpha)$, then

$$\langle s^* \cap H(\alpha), p^* \cap \alpha\rangle \Vdash_{\mathbb{A}\cap H(\alpha)} \dot{x}^\alpha_N \in \dot{b}^\alpha_M.$$

For $\beta \in \Sigma \cup \{\theta\}$, let $\mathbb{A}_\beta$ be the restriction of $\mathbb{A}$ to the set of all pairs $\langle s, p\rangle$ satisfying $p \subseteq \beta$. Clearly $\mathbb{A}_\theta = \mathbb{A}$. In order to simplify the notation, for a condition $\langle s, p\rangle \in \mathbb{A}$ and $\alpha \in \Sigma$, let $\langle s, p\rangle \restriction \alpha$ denote the condition $\langle s \cap H(\alpha), p \cap \alpha\rangle$.

Remark 3.5. If $\langle s, p\rangle \in \mathbb{A}$ and $\alpha \in p$, then Lemma 3.2 implies that the set s/α is a strong $\in$-path in the collection of all countable elementary submodels of $H(\alpha^+)$. Moreover, if W is the first transitive node of s above $H(\alpha)$ (assume $W = H(\theta)$ if there is no such W), then

$$s/\alpha = \left\{M \cap H(\alpha^+) : M \in s \,\&\, H(\alpha) \in M \in W\right\}.$$

Remark 3.6. For a side condition s and $X \subseteq s$, we say that X is an *interval of* s, if it does not contain transitive nodes, and $X = \{M \in s \cap \mathcal{S} : W_1 \in M \in W_2\}$ for some $W_1, W_2 \in s \cap \mathcal{T}$.

Remark 3.7. Condition (2) of Definition 3.4 says that $\dot{x^\alpha_M}$ is a name for a minimal ordinal with the listed properties. Notice that by definition of $<_w$, it is also minimal for those properties in the $<_w$ ordering.

Since $\mathcal{S}$ and $\mathcal{T}$ satisfy Definition 2.2 of [11], the set of side conditions satisfies conditions in Definition 2.4 of [11], for $\kappa = \omega$ and $\lambda = \omega_1$. The next three lemmas list the properties of side conditions which we will use, as presented in [11]. For a side condition s and a node $Q \in s$, denote $\mathrm{res}_Q(s) = s \cap Q$. If s and t are side conditions, we say that they are *directly compatible* if the closure of $s \cup t$ under intersections is a side condition.

Lemma 3.8 ([11], Lemma 2.20)**.** *Let s be a side condition, and let $Q \in s$ be a transitive node. Suppose that t is a side condition that belongs to Q and extends $\mathrm{res}_Q(s)$. Then $s \cup t$ is a side condition, and in particular s and t are directly compatible.*

Lemma 3.9 ([11], Corollary 2.32)**.** *Let $M \in \mathcal{S} \cup \mathcal{T}$, and let t be a side condition that belongs to M. Then there is a side condition $r \supseteq t$ with $M \in r$. Moreover r can be taken to be the closure of $t \cup \{M\}$ under intersections.*

Lemma 3.10 ([11], Corollary 2.31)**.** *Let s be a side condition and let Q be a node of s. Suppose t is a side condition that belongs to Q and extends $\mathrm{res}_Q(s)$. Then s and t are directly compatible, and if r witnesses this, then $\mathrm{res}_Q(r) = t$ and all the small nodes of r outside Q are of the form N or $N \cap W$ where N is a small node of s and W is a transitive node of t.*

Now we start with the proof of the main theorem. First we prove a few simple lemmas that will be used in the sequel.

Lemma 3.11. *Let $\alpha < \beta$ belong to $\Sigma \cup \{\theta\}$. Let $\langle s, p\rangle \in \mathbb{A}_\beta$ with $H(\alpha) \in s$. Let $\langle t, q\rangle \in \mathbb{A} \cap H(\alpha)$ extend $\langle s, p\rangle \restriction \alpha$. Then $\langle s, p\rangle$ and $\langle t, q\rangle$ are compatible in $\mathbb{A}_\beta$, and this is witnessed by the condition $\langle u, h\rangle$ with $u = t \cup s$, $h = p \cup q$.*

Proof. We will prove that $\langle u, h\rangle$, as defined in the statement of the lemma, works. First, by Lemma 3.8 we know that $u = t \cup s$ is a side condition. Condition (2) of Definition 3.4 is clearly satisfied. We still have to show that $\langle u, h\rangle \leq \langle s, p\rangle, \langle t, q\rangle$ in $\mathbb{A}_\beta$. First we prove that $\langle u, h\rangle \leq \langle t, q\rangle$. Let $\gamma \in q$ and $M \in t \cap \mathcal{S}$ such that $\gamma \in M$. Since $M \in t \in H(\alpha)$, it must be that $M \in H(\alpha)$. But $u \cap H(\alpha) = t \cap H(\alpha)$, so the set $(u/\gamma \cap M \cap H(\alpha^+)) \setminus (t/\gamma)$ is empty, and hence $\langle u, h\rangle$ and $\langle t, q\rangle$ satisfy (3) of Definition 3.4. Now we will prove that $\langle u, h\rangle \leq \langle s, p\rangle$. Let $\gamma \in p$ and $M \in s/\gamma$. Take $N \in (u/\gamma \cap M) \setminus (s/\gamma)$. Since $N \in (u/\gamma) \setminus (s/\gamma)$, it must be that $N \in t/\gamma$. Let $M' \in s \cap \mathcal{S}$ be such that $M = M' \cap H(\gamma^+)$. Since s is closed under intersections $M' \cap H(\alpha) \in s$. This means that $M \in (s \cap H(\alpha))/\gamma$ because $M' \cap H(\alpha) \cap H(\gamma^+) = M' \cap H(\gamma^+) = M$. Now $\gamma \in q$ and $\langle t, q\rangle \leq \langle s, p\rangle \restriction \alpha$ together imply that $\langle t, q\rangle \restriction \gamma \Vdash_{\mathbb{A} \cap H(\gamma)} \dot{x}^\gamma_N \in \dot{b}^\gamma_M$. But $\gamma < \alpha$ and the form of $\langle u, h\rangle$ imply that $\langle t, q\rangle \restriction \gamma = \langle u, h\rangle \restriction \gamma$, which proves the claim. $\square$

Lemma 3.12. *Suppose that $\beta \in \Sigma \cup \{\theta\}$.*

(1) If $\langle s, p\rangle \in \mathbb{A}_\beta$ and $H(\alpha) \in s$, then $\langle s, p\rangle$ is a strongly $(H(\alpha), A_\beta)$-generic condition.
(2) If $\langle s, p\rangle \in \mathbb{A}_\beta$, $W \in \mathcal{T}$, and $\langle s, p\rangle \in W$, then $\langle s \cup \{W\}, p\rangle \in \mathbb{A}_\beta$.
(3) $\mathbb{A}_\beta$ is strongly proper for $\mathcal{T}$.

Proof. Condition (2) follows directly from Lemma 3.9 and the definition of $\mathbb{A}$. Condition (3) follows from (1) and (2). So we prove (1). We distinguish two cases, $\alpha < \beta$ and $\alpha \geq \beta$. First let $\alpha < \beta$ and let D be a dense subset of $\mathbb{A}_\beta \cap H(\alpha)$. We will prove that every $\langle s, p\rangle$ with $H(\alpha) \in s$ is compatible with an element of D. This is clearly enough. Since D is dense in $\mathbb{A}_\beta \cap H(\alpha)$, there is $\langle t, q\rangle \in D$ extending $\langle s, p\rangle \restriction \alpha$. By Lemma 3.11, there is $\langle u, h\rangle \leq \langle s, p\rangle, \langle t, q\rangle$ which belongs to $\mathbb{A}_\beta$ because $\alpha < \beta$. Suppose now that $\alpha \geq \beta$, that D is dense in $\mathbb{A}_\beta \cap H(\alpha)$, that $H(\alpha) \in s$, and that $\langle s, p\rangle \in \mathbb{A}_\beta$. By density of D, there is again $\langle t, q\rangle \in D$ extending $\langle s, p\rangle \restriction \alpha$. This means that $s \cap H(\alpha) \subseteq t \subseteq H(\alpha) \in s$. We will prove that $\langle s \cup t, q\rangle \in \mathbb{A}_\beta$ and $\langle s \cup t, q\rangle \leq \langle s, p\rangle, \langle t, q\rangle$. This clearly proves the lemma. By Lemma 3.8, $s \cup t$ is a side condition so (1) of Definition 3.4 is satisfied. Since $\langle t, q\rangle \in \mathbb{A}_\beta$, it must be that $\langle s \cup t, q\rangle$ satisfies (2) of Definition 3.4, and that $q \subset \beta$, which implies $\langle s \cup t, q\rangle \in \mathbb{A}_\beta$. We still have to show that $\langle s \cup t, q\rangle \leq \langle s, p\rangle, \langle t, q\rangle$. First we prove that $\langle s \cup t, q\rangle$ is stronger then $\langle s, p\rangle$. Take any $\gamma \in p$, any $M \in s/\gamma$, and any $N \in ((s \cup t)/\gamma \cap M) \setminus (s/\gamma)$. Let $M' \in s \cap \mathcal{S}$ be such that $M' \cap H(\alpha) = M$. Since $M', H(\alpha)$ both belong to s, $M' \cap H(\alpha)$ also belongs to s. Moreover, $M' \cap H(\alpha)$ belongs to $s \cap H(\alpha)$. Now in the same way as in the previous proof $\gamma < \alpha$, $\langle t, q\rangle \restriction \gamma = \langle s \cup t, q\rangle \restriction \gamma$, $\langle t, q\rangle \leq \langle s, p\rangle \restriction \alpha$, and

$M' \cap H(\alpha) \cap H(\gamma^+) = M' \cap H(\gamma^+) = M$ together imply that (3) of Definition 3.4 for $\langle s \cup t, q\rangle$ and $\langle s, p\rangle$ is fullfiled. We still have to prove that $\langle s \cup t, q\rangle \leq \langle t, q\rangle$. Notice that t is an initial segment of $s \cup t$, so the set $((s \cup t)/\gamma \cap M) \setminus (t/\gamma)$ is empty for any choice of $\gamma \in q$ and $M \in t/\gamma$. Hence condition (3) of Definition 3.4 is satisfied by $\langle s \cup t, q\rangle$ and $\langle t, q\rangle$ as well. $\square$

Lemma 3.13. *Let $\langle s, p\rangle \in \mathbb{A}$. Suppose that $H(\alpha) \in s$ and $\alpha \notin p$. Let $M \in s \cap \mathcal{S}$ and let $\langle t, q\rangle \in \mathbb{A} \cap M$ satisfy $\alpha \in \operatorname{dom}(q)$ and $\langle s, p\rangle \leq \langle t, q \setminus \{\alpha\}\rangle$. Suppose further that $\operatorname{res}_M(s) \setminus H(\alpha) \subseteq t$. Then $\langle s, p \cup \{\alpha\}\rangle \leq \langle t, q\rangle$.*

Proof. First we explain why $\langle s, p \cup \{\alpha\}\rangle$ is in $\mathbb{A}$. (1) of Definition 3.4 is satisfied. For (2) notice that since $\langle t, q\rangle \in \mathbb{A}$ and $\alpha \in q$, it must be that $\Vdash_{\mathbb{A} \cap H(\alpha)}$"$F(\alpha)$ is a P-ideal on some ordinal $\check{\gamma}$ which is not a countable union of sets orthogonal to $F(\alpha)$". We still have to show that $\langle s, p \cup \{\alpha\}\rangle$ and $\langle t, q\rangle$ satisfy (3) of Definition 3.4. It is clear that $t \subseteq s$, so take any $\gamma \in q$, any $M_1 \in t/\gamma$, and any $N \in (s/\gamma \cap M_1) \setminus (t/\gamma)$. Now we distinguish two cases: either $\gamma < \alpha$, or $\gamma \geq \alpha$.

If $\gamma < \alpha$, then $\langle s, p\rangle \leq \langle t, q \setminus \{\alpha\}\rangle$ implies $\langle s, p\rangle \restriction \gamma \Vdash_{\mathbb{A} \cap H(\gamma)} \dot{x}^\gamma_N \in \dot{b}^\gamma_{M_1}$. Now the fact that $\langle s, p\rangle \restriction \gamma = \langle s, p \cup \{\alpha\}\rangle \restriction \gamma$ finishes the proof in this case.

Suppose now that $\gamma \geq \alpha$. Because $N \in s/\gamma$, there is some $N' \in s \cap \mathcal{S}$ such that $N' \cap H(\gamma^+) = N$ and $\gamma \in N'$. Let W be the first transitive node of s above $H(\gamma)$ if there is such, and $H(\theta)$ if not. Note that $N' \cap M \cap W \in M$ and $N' \cap M \cap W \in s$. Now the assumption $\operatorname{res}_M(s) \setminus H(\alpha) \subseteq t$, together with $\gamma \in N' \cap M \cap W$ and $\gamma \geq \alpha$, imply that $N' \cap M \cap W \in t$. But then $N = N' \cap M \cap W \cap H(\gamma^+) \in t/\gamma$, which contradicts the choice of N. Hence, this case is not possible. $\square$

Lemma 3.14. *Let $\langle s, p\rangle$, $\langle t, q\rangle \in \mathbb{A}$. Suppose that $M \in s \cap \mathcal{S}$, and that $\langle t, q\rangle \in M$. Suppose also that for some $\gamma < \theta$, $\langle s, p\rangle$ extends $\langle t, q \cap \gamma\rangle$ and that $q \setminus \gamma$ is disjoint from p. Suppose further that $\operatorname{res}_M(s) \setminus H(\gamma) \subseteq t$. Then for $p' = p \cup (q \setminus \gamma)$, $\langle s, p'\rangle$ is a condition in $\mathbb{A}$ extending $\langle t, q\rangle$.*

Proof. The proof follows by successive applications of Lemma 3.13 for each ordinal $\alpha \in q \setminus \gamma$. $\square$

Lemma 3.15. *Let M be a small node and let $\langle t, q\rangle \in \mathbb{A} \cap M$. Then there is $\langle s, p\rangle \leq \langle t, q\rangle$ with $M \in s$.*

Proof. Let $s \supseteq t$ be as given by Lemma 3.9, i.e. $M \in s$ and $s \supseteq t$. Now apply Lemma 3.14 with $\gamma = 0$ to $\langle s, \emptyset\rangle$ and $\langle t, q\rangle$ $\square$

Lemma 3.16. *Let $\beta \in \Sigma \cup \{\theta\}$ and $\langle s, p\rangle \in \mathbb{A}_\beta$. Let $\theta^* > \theta$ and let $M^* \prec H(\theta^*)$ be countable so that $\theta, F, \beta \in M^*$. Denote $M = M^* \cap H(\theta)$. If $M \in s$, then:*

1. *(1) For each $D \in M^*$ dense in $\mathbb{A}_\beta$, there is $\langle t, q\rangle \in M^* \cap D$ compatible with $\langle s, p\rangle$. Moreover, there is $\langle s^*, p^*\rangle \in \mathbb{A}_\beta$ extending both $\langle s, p\rangle$ and $\langle t, q\rangle$ in such a way that $\operatorname{res}_M(s^*) \setminus H(\beta) \subseteq t$, and all small nodes of s^* above β and outside M are either in s or of the form $N' \cap W$ where $N' \in s \cap \mathcal{S}$ and $W \in \mathcal{T}$.*
2. *(2) $\langle s, p\rangle$ is an $(M^*, \mathbb{A}_\beta)$-generic condition.*

Proof. Condition (2) follows immediately from (1) so we prove (1) by induction on β. If β is the first element of Σ, the proof follows exactly in the same way as in the proof of Lemma 6.12 in [11]. In the rest of the proof we also follow Neeman's presentation. So suppose that β is a limit point of $\Sigma \cup \{\theta\}$, and denote $\bar{\beta} = \sup(M^* \cap \beta)$. Let $\delta < \bar{\beta}$ in $\Sigma \cap M^*$ be large enough that $p \cap \bar{\beta} \subseteq \delta$. This is possible because $\bar{\beta}$ is a limit point of $\Sigma \cap M^*$ and p is finite. Let E be the set of conditions $\langle t, \bar{q} \rangle \in \mathbb{A}_\delta$ which extend to conditions $\langle t, q \rangle \in D$ with $q \cap \delta = \bar{q}$. The set E belongs to M^*. To see that E is dense in $\mathbb{A}_\delta$, take any $(\bar{s}, \bar{p}) \in \mathbb{A}_\delta \subseteq \mathbb{A}_\beta$. Since D is dense in $\mathbb{A}_\beta$, there is $\langle t, q \rangle \in D$ such that $\langle t, q \rangle \leq \langle \bar{s}, \bar{p} \rangle$. Then $\langle t, q \cap \delta \rangle$ witnesses that there is an element of E below $\langle \bar{s}, \bar{p} \rangle$ in $\mathbb{A}_\delta$. By induction, there is $\langle t, \bar{q} \rangle \in E \cap M^*$ compatible with $\langle s, p \cap \delta \rangle$ and such that there is $\langle s^*, p_1 \rangle \in \mathbb{A}_\delta$ which extends both $\langle s, p \cap \delta \rangle$ and $\langle t, \bar{q} \rangle$, with $\mathrm{res}_M(s^*) \setminus H(\delta) \subseteq t$, and so that all the small nodes of s^* above $H(\delta)$ and outside M are either small nodes of s or of the form $N \cap W$ where N is a small node of s and $W \in \mathcal{T}$. Let $\langle t, q \rangle \in D$ witness that $\langle t, \bar{q} \rangle \in E$. By elementarity of M^*, q can be chosen to be in M^*. By Lemma 3.14, for $p_2 = p_1 \cup (q \setminus \delta)$, $\langle s^*, p_2 \rangle$ extends $\langle t, q \rangle$. Finally, define $p^* = p_2 \cup (p \setminus \bar{\beta})$. We have to show that $\langle s^*, p^* \rangle$ is a condition in $\mathbb{A}_\beta$ and that $\langle s^*, p^* \rangle \leq \langle t, q \rangle, \langle s, p \rangle$.

Condition (1) of Definition 3.4 is clear because $\langle s^*, p_2 \rangle \in \mathbb{A}_\beta$. For (2) of Definition 3.4, there are two cases: either $\alpha \in p_2$ or $\alpha \in p \setminus \bar{\beta}$. If $\alpha \in p_2$, condition is satisfied because $\langle s^*, p_2 \rangle$ is in $\mathbb{A}_\beta$. If $\alpha \in p \setminus \bar{\beta}$, condition is satisfied because $\langle s, p \rangle \in \mathbb{A}_\beta$ and $s \subseteq s^*$. Since p_2 is an initial segment of p^* and $\langle s^*, p_2 \rangle \leq \langle t, q \rangle$, it is clear that $\langle s^*, p^* \rangle \leq \langle t, q \rangle$. Finally, we check $\langle s^*, p^* \rangle \leq \langle s, p \rangle$. Take any $\alpha \in p$, and any $M_1 \in s/\alpha$. We distinguish two cases: either $\alpha < \bar{\beta}$ or $\alpha \geq \bar{\beta}$. If $\alpha < \bar{\beta}$, then $\alpha < \delta$. So for $N \in (s^*/\alpha \cap M_1) \setminus (s/\alpha)$, since $\langle s^*, p_2 \rangle \leq \langle s, p \cap \delta \rangle$, it must be that $\langle s^*, p_2 \rangle \restriction \alpha \Vdash_{\mathbb{A} \cap H(\alpha)} \dot{x}^\alpha_N \in \dot{b}^\alpha_{M_1}$. Now the fact that $\langle s^*, p_2 \rangle \restriction \alpha = \langle s^*, p^* \rangle \restriction \alpha$ finishes the proof in this case. Suppose now that $\alpha \geq \bar{\beta} > \delta$. If $N \in (s^*/\alpha \cap M_1) \setminus (s/\alpha)$, then there is some $N' \in s*$ such that $N = N' \cap H(\alpha^+)$. By the hypothesis, N' is either a small node of s or an intersection of a small node of s, say N_1, and some $W \in \mathcal{T}$. If $N' \in s$, then $N \in s/\alpha$, contradicting the choice of N. The other option implies that $N = N_1 \cap W \cap H(\alpha^+) = N_1 \cap H(\alpha^+) \in s/\alpha$ which is again in contradiction with the choice of N. Hence, $\langle s^*, p^* \rangle \leq \langle s, p \rangle$ as required.

Now assume that β is a successor point of Σ, and let α be its predecessor in Σ. Note that by elementarity of M^*, $\alpha \in M^*$. Again we follow the proof of Neeman in [11], so fix G which is generic for $\mathbb{A}_\alpha$ over V, and which contains $\langle s, p \cap \alpha \rangle$. By induction $\langle s, p \cap \alpha \rangle$ is $(M^*, \mathbb{A}_\alpha)$-generic condition, so $M^*[G] \prec H(\theta^*)[G]$ and $M^*[G] \cap V = M^*$. Suppose that $H(\alpha) \in s$. By strong properness (Lemma 3.12), $G \cap H(\alpha)$ is generic for $\mathbb{A} \cap H(\alpha)$ over V. If it is not forced in $\mathbb{A} \cap H(\alpha)$ that $F(\alpha)$ is a P-ideal on some γ which is not a countable union of sets orthogonal to $F(\alpha)$, then $\mathbb{A}_\beta = \mathbb{A}_\alpha$ and there is nothing to do. So suppose that $\Vdash_{\mathbb{A} \cap H(\alpha)}$ "$F(\alpha)$ is a P-ideal on some $\check{\gamma}$ which is not a countable union of sets orthogonal to $F(\alpha)$". Let $\mathbb{Q}$ be the poset of all strong $\in$-paths of countable elementary submodels of $H(\alpha^+)$, ordered as follows: for $s_1, s_2 \in \mathbb{Q}$ set $s_2 \leq s_1$ iff $s_2 \supseteq s_1$ and for every $N_1 \in s_1$, if $N_2 \in (s_2 \cap N_1) \setminus s_1$, then $\langle s, p \rangle \restriction \alpha \Vdash_{\mathbb{A} \cap H(\alpha)} x^{\dot{\alpha}}_{N_2} \in b^{\dot{\alpha}}_{N_1}$. Note that $\mathbb{Q}$ belongs to $M^*[G \cap H(\alpha)]$, and that it is proper for $\mathscr{N} = \{N[G \cap H(\alpha)] : N \prec H(\alpha^+) \ \& \ |N| = \omega\}$. The proof of properness of $\mathbb{Q}$ for $\mathscr{N}$ follows using the same arguments as in the proof of [16, Theorem 20.6].

Let W be the first transitive node of s above $H(\alpha)$ if there is one, and $H(\theta)$ otherwise. Let $N_0 \in \cdots \in N_{l-1}$ list all the small nodes of s between $H(\alpha)$ and W. As in the proof of Lemma 3.13, there is $k < l$ such that $M \cap W = N_k$. Then $\mathrm{res}_M(s) \cap \{N_0, \ldots, N_{l-1}\} = \{N_0, \ldots, N_{k-1}\}$.

Let E be the set of all $u \in \mathbb{Q}$ so that one of the two alternatives hold:

(a) No extension of u contains all the models $N_i \cap H(\alpha^+)$ for $i < k$;

(b) There is $\langle t, q\rangle \in D$ with $\langle t, q\rangle \restriction \alpha \in G$, $t \supseteq s \cap M$, $\alpha \in q$, and $t/\alpha = u$.

Note that if (b) holds for u, then u contains models $N_i \cap H(\alpha^+)$ for each $i < k$.

Claim 3.17. *E is dense in $\mathbb{Q}$ and belongs to $M^*[G \cap H(\alpha)]$.*

Proof of Claim: Since all the parameters in the definition of E belong to $M^*[G \cap H(\alpha)]$, it must be that $E \in M^*[G \cap H(\alpha)]$. To see that E is dense in $\mathbb{Q}$, take any $u \in \mathbb{Q}$, and suppose that it has no extension in E. Let $\langle a, h\rangle \in G \cap H(\alpha)$ force this. By failure of (a) we may assume that $N_i \cap H(\alpha^+) \in u$ for $i < k$. Since $\langle \mathrm{res}_M(s), \emptyset\rangle \in G$, we may also assume that $\mathrm{res}_M(s) \cap H(\alpha) \subseteq a$. Let $a^* = a \cup \mathrm{res}_M(s)$ and $h^* = h \cup \{\alpha\}$. Now $\langle a^*, h^*\rangle$ is a condition in $\mathbb{A}_\beta$ and any $\langle t, q\rangle \in D$ extending it provides a contradiction to the fact that u is forced to have no extension in E. $\square$

Now we use properness of $\mathbb{Q}$ for $\mathscr{N}$ and density of $E \in M^*[G \cap H(\alpha)]$. If $\alpha \in p$, then s/α is an $(M^*[G \cap H(\alpha)], \mathbb{Q})$-generic condition. So there is $u \in E \cap M^*[G \cap H(\alpha)]$, and $u^* \in \mathbb{Q}$ such that $u^* \leq_{\mathbb{Q}} u, s/\alpha$. In case $\alpha \notin \mathrm{dom}(p)$, consider the condition $v = \{N_i \cap H(\alpha^+) : i < l\}$. Clearly, $v \in \mathbb{Q}$. Now we can pick conditions u and u^* in $\mathbb{Q}$ with the same properties as before, only starting from v instead of s/α. So fix these u and u^* for the rest of the proof.

Since $u^* \supseteq u$ contains all the models $N_i \cap H(\alpha^+)$ for $i < l$, the membership of u in E most hold through condition (b) in the definition of E. Let $\langle t, q\rangle$ witness this condition. In particular $t/\alpha = u$. Using the same arguments as in the proof of [16, Theorem 20.6], or in the same way as in the proof of the successor case of (1) of Lemma 3.19, we can assume that $u^* = t/\alpha \cup s/\alpha$. Since $D \in M^*$ we can choose $\langle t, q\rangle \in M^*[G \cap H(\alpha)]$. Moreover it must be that $\langle t, q\rangle \in M^*$ because $M^*[G \cap H(\alpha)] \cap V = M^*$. Now, $\langle t, q\rangle$ belongs to $\mathbb{A}_\beta$. Lemma 3.10 implies that t and s are directly compatible. Let r witness this. Note that in this case $r/\alpha = u^*$, By the same lemma $\mathrm{res}_M(r) = t$, so the small nodes of r inside M are small nodes of t, while the small nodes of r outside M, are either nodes of s or intersections of small nodes of s with transitive nodes of t. Let $\langle a, h\rangle \in G \cap H(\alpha)$ be stronger then both $\langle s, p\rangle \restriction \alpha$ and $\langle t, q\rangle \restriction \alpha$. Note that this implies that $a \in H(\alpha)$ and that $r \cap H(\alpha) \subseteq a$. By Lemma 3.8, a and r are directly compatible, and this is witnessed by $a \cup r$. Define $s^* = a \cup r$ and $p^* = h \cup \{\alpha\}$. Note that $\langle s^*, p^*\rangle \in \mathbb{A}_\beta$ because all the small nodes in s^* above $H(\alpha)$ belong to r, and we have already mentioned that $u^* = r/\alpha$. Note also that all the small nodes of s^* above $H(\beta)$ and outside M are either small nodes of s, or an intersection of a small node of s and a transitive node. The choice of u^* ensures that $\langle s^*, p^*\rangle \leq \langle s, p\rangle, \langle t, q\rangle$ which together with the above properties of a and r shows that (1) holds when $H(\alpha) \in s$.

If $H(\alpha)$ does not belong to s, then in the same way as in [11], using Claim 5.7 and Remark 5.8 from [11], there is an extension s' of s with $H(\alpha) \in s'$, so that the only added nodes in $s' \setminus s$ are transitive nodes and intersections of small nodes of s with transitive nodes. Then $\langle s', p\rangle$ is a condition and the arguments above show that the lemma holds in this case also. $\square$

Corollary 3.18. *Forcing with $\mathbb{A}$ preserves ω_1 and θ as cardinals. All cardinals between ω_1 and θ are collapsed to ω_1.*

Proof. In the same way as in the proof of [11, Corollary 6.13], the preservation of θ follows from stationarity of $\mathcal{T}$ and strong properness for models in $\mathcal{T}$. Preservation of ω_1 follows from Lemma 3.15, stationarity of $\mathcal{S}$, and properness of $\mathbb{A}_\theta$ as proved in Lemma 3.16. Properties of side conditions ensure that all cardinals between ω_1 and θ are collapsed to ω_1 (see [11, Corollary 6.13]). $\square$

Lemma 3.19. *Let $\beta \in \Sigma \cup \{\theta\}$ and $\langle s, p\rangle \in \mathbb{A}_\beta$. Let T be an almost Souslin Aronszajn tree in V, and $\dot{A}$ an $\mathbb{A}_\beta$-name for a level set in T. Let $\theta^* > \theta$ and let $M^* \prec H(\theta^*)$ be countable so that $\theta, F, \beta, \dot{A}, T \in M^*$. Let $M = M^* \cap H(\theta)$. Suppose that $M \in s$ and that $\langle s, p\rangle \Vdash t \in \dot{A}$ for some $t \in T_{M\cap\omega_1}$. Then:*

(1) there is $\bar{t} < t$, and $\langle \bar{s}, \bar{p}\rangle \in M^* \cap \mathbb{A}_\beta$ such that $\langle \bar{s}, \bar{p}\rangle$ is compatible with $\langle s, p\rangle$ and that $\langle \bar{s}, \bar{p}\rangle \Vdash \bar{t} \in \dot{A}$. Moreover, there is $\langle s^*, p^*\rangle \in \mathbb{A}_\beta$ extending both $\langle s, p\rangle$ and $\langle \bar{s}, \bar{p}\rangle$ in such a way that $\mathrm{res}_M(s^*) \setminus H(\beta) \subseteq \bar{s}$, and that all small nodes of s^* above β and outside M are either nodes of s or of the form $N \cap W$ where N is a small node of s and $W \in \mathcal{T}$.

(2) there is $t' \perp t$, and $\langle s^\dagger, p^\dagger\rangle \in M^* \cap \mathbb{A}_\beta$ such that $\langle s^\dagger, p^\dagger\rangle$ is compatible with $\langle s, p\rangle$ and that $\langle s^\dagger, p^\dagger\rangle \Vdash t' \in \dot{A}$. Moreover, there is $\langle s^\ddagger, p^\ddagger\rangle \in \mathbb{A}_\beta$ extending both $\langle s, p\rangle$ and $\langle s^\dagger, p^\dagger\rangle$ in such a way that $\mathrm{res}_M(s^\ddagger) \setminus H(\beta) \subseteq s^\dagger$, and that all small nodes of $s^\ddagger$ above β and outside M are either nodes of s or of the form $N \cap W$ where N is a small node of s and $W \in \mathcal{T}$.

Proof. For (1) we prove the lemma by induction on Σ. The proof for the minimal ordinal $\alpha \in \Sigma$ such that $\Vdash_{\mathbb{A}\cap H(\alpha)}$ "$F(\alpha)$ is a P-ideal on some ordinal γ which is not a countable union of sets orthogonal to $F(\alpha)$" is a simplification of the successor case, so we will only prove the latter one. Hence, suppose that β is the successor of α in Σ, and that the statement of the lemma holds for α. In the same way as in the proof of Lemma 3.16, fix G which is generic for $\mathbb{A}_\alpha$ over V, containing $\langle s, p \cap \alpha\rangle$. By Lemma 3.16 $\langle s, p \cap \alpha\rangle$ is $(M^*, \mathbb{A}_\alpha)$-generic condition, and again we have $M^*[G] \prec H(\theta^*)[G]$ and $M^*[G] \cap V = M^*$. Suppose that $H(\alpha) \in s$. If it is not forced by $\mathbb{A} \cap H(\alpha)$ that $F(\alpha)$ is a P-ideal which does not satisfy (2) of Definition 2, then there is nothing to do, so assume that $\Vdash_{\mathbb{A}\cap H(\alpha)}$ "$F(\alpha)$ is a P-ideal on some ordinal γ which is not a countable union of sets orthogonal to $F(\alpha)$", and let $\mathbb{Q}$ be the same poset as in the proof of Lemma 3.16. So $\mathbb{Q}$ is the poset of all strong $\in$-paths of countable elementary submodels of $H(\alpha^+)$ in V, ordered in such a way that for $s_1, s_2 \in \mathbb{Q}$, $s_2 \le s_1$ iff $s_2 \supseteq s_1$ and for every $N_1 \in s_1$, if $N_2 \in (s_2 \cap N_1) \setminus s_1$, then $\langle s, p\rangle \restriction \alpha \Vdash_{\mathbb{A}\cap H(\alpha)} x^{\dot{\alpha}}_{N_2} \in b^{\dot{\alpha}}_{N_1}$. As before, $\mathbb{Q}$ is proper for the class $\mathscr{N} = \{M[G \cap H(\alpha)] : M \prec H(\theta) \,\&\, |M| = \omega\}$. In order to shorten some statements, let us denote $F(\alpha)[G \cap H(\alpha)] = \mathcal{I}$.

Let W be the first transitive node of s above $H(\alpha)$, or $H(\theta)$ if there is no such W. Let $N_0 \in N_1 \in \cdots \in N_{l-1}$ list all the small nodes of s between $H(\alpha)$ and W, and denote $q = \{N_0, \ldots, N_{l-1}\}$. As before, there is $k < l$ such that $N_k = M \cap W$. Let $\mathbb{Q}^s$ be the set of all sequences $\bar{q} = \{M^{\bar{q}}_0, \ldots, M^{\bar{q}}_{l-1}\}$ of countable elementary submodels of $H(\theta)^V$ of length l such that $\bar{q} \sqsupset \{N_0, \ldots, N_{k-1}\}$, and for which there exists a condition $\langle s^{\bar{q}}, p^{\bar{q}}\rangle \in \mathbb{A}_\beta$ so that $\langle s^{\bar{q}}, p^{\bar{q}} \restriction \alpha\rangle \in G$, $\bar{q}$ is an interval of $s^{\bar{q}}$, $s^{\bar{q}} \supseteq s \cap M$, and moreover that $\langle s^{\bar{q}}, p^{\bar{q}}\rangle \Vdash_{\mathbb{A}_\beta} t^{\bar{q}} \in \dot{A}$, for some $t^{\bar{q}} \in T_{M^{\bar{q}}_k\cap\omega_1}$. Note that $\mathbb{Q}^s \in M^*[G]$ because all the parameters defining it belong to $M^*[G]$. Namely, G, $\dot{A}$, $\{N_0, \ldots, N_{k-1}\}$, $s \cap M$, T, and integers. For $\bar{q} \in \mathbb{Q}^s$, denote

$$x^{\bar{q}} = \left\langle \dot{x}^{\alpha}_{M^{\bar{q}}_i \cap H(\alpha^+)}[G] : k \le i < l \right\rangle,$$

and define

$$\mathcal{F}=\left\{\langle t_0,x\rangle\in T\times\gamma^{l-k}:(\exists\bar q\in\mathbb{Q}^s)\ \left(x=x^{\bar q}\ \&\ t_0=t^{\bar q}\right)\right\}.$$

Since all the parameters in the definition of $\mathcal{F}$ belong to $M^*[G]$, it must be that $\mathcal{F}\in M^*[G]$. Note also that $\langle t,x^q\rangle\in\mathcal{F}$. Let $\mathcal{J}$ be the σ-ideal generated by $\mathcal{I}^\perp$. The choice of names $\dot{x}^\alpha_N$, the fact that it is forced that the set of heights of elements of $\dot{A}$ is stationary in ω_1, the fact that $\{N\cap H(\alpha^+):N\in\bar q\}$ is a *strong* $\in$-path for each $\bar q\in\mathbb{Q}^s$, and the fact that γ is not a countable union of sets in $\mathcal{I}^\perp$, together imply that there is a tree $S\subseteq\{\emptyset\}\cup(T\times\gamma^{\leq l-k})$ such that:

- $\emptyset\in S$;
- $X_0=\{L(\{t_0\}):\langle t_0\rangle\in S\}$ is stationary and in particular $\langle t\rangle\in S$;
- if $\langle t_0,x_0,\dots,x_m\rangle\in S$ for $m<l-k$, then
$$\{x\in\gamma:\langle t_0,x_0,\dots,x_m,x\rangle\in S\}\in\mathcal{J}^+;$$
- $S\cap(T\times\gamma^{l-k})\subseteq\mathcal{F}$.

This means that there is a family $\mathcal{F}_0\subseteq\mathcal{F}$ which forms the set of maximal nodes of S, and which belongs to $M^*[G]$. By Lemma 3.1 we can pick $t_0\in\dot{A}[G]\cap M^*[G]$ such that $t_0<t$ and that there is $x_0\in\mathcal{F}_0$ such that $\langle t_0\rangle\sqsubset x_0$. Note that since $M^*[G]\cap V=M^*$, we in fact have $t_0\in M^*$. Denote

$$X_{t_0}=\{\xi\in\gamma:(\exists x\in\mathcal{F}_0)\ \langle t_0,\xi\rangle\sqsubseteq x\}$$

Then $X_{t_0}\in\mathcal{J}^+\cap M^*[G]$, and consequently there is an infinite set $a_{t_0}\in\mathcal{I}\cap M^*[G]$ such that $a_{t_0}\subseteq X_{t_0}$. Since $a_{t_0}\subseteq^*\dot{b}^\alpha_{N_i\cap H(\alpha^+)}[G\cap H(\alpha)]$ for each $k\leq i<l$, there is

$$\xi_0\in a_{t_0}\cap\textstyle\bigcap_{k\leq i<l}\dot{b}^\alpha_{N_i\cap H(\alpha^+)}[G\cap H(\alpha)].$$

In the same way,

$$X_{t_0,\xi_0}=\{\xi\in\gamma:(\exists x\in\mathcal{F}_0)\ \langle t_0,\xi_0,\xi\rangle\sqsubseteq x\}\in\mathcal{J}^+\cap M^*[G],$$

so there is infinite $a_{t_0,\xi_0}\in\mathcal{I}\cap M^*[G]$ such that $a_{t_0,\xi_0}\subseteq X_{t_0,\xi_0}$. Again, since $a_{t_0,\xi_0}\subseteq^*\dot{b}^\alpha_{N_i\cap H(\alpha^+)}[G\cap H(\alpha)]$ for each $k\leq i<l$, there is

$$\xi_1\in a_{t_0,\xi_0}\cap\textstyle\bigcap_{k\leq i<l}\dot{b}^\alpha_{N_i\cap H(\alpha^+)}[G\cap H(\alpha)].$$

Continuing in this way, we obtain t_0 and $\{\xi_0,\dots,\xi_{l-k-1}\}\subseteq\bigcap_{k\leq i<l}\dot{b}^\alpha_{N_i\cap H(\alpha^+)}$ such that $\langle t,\xi_0,\dots,\xi_{l-k-1}\rangle\in\mathcal{F}\cap M^*[G]$. Finally, since $M^*[G]\cap V=M^*$, we can pick $\langle s^{\bar q},p^{\bar q}\rangle\in M$ in such a way that $\bar q$ is end-extending $q\cap M$, and that $x^{\bar q}=\langle\xi_0,\dots,\xi_{l-k-1}\rangle$. Note that $\langle s^{\bar q},p^{\bar q}\rangle\Vdash t_0\in\dot{A}$, and that

$$u=\left\{M^{\bar q}_0\cap H(\alpha^+),\dots,M^{\bar q}_{l-1}\cap H(\alpha^+),N_k\cap H(\alpha^+),\dots,N_{l-1}\cap H(\alpha^+)\right\}$$

is a strong $\in$-path of countable elementary submodels of $H(\alpha^+)$. This follows from Lemma 3.2, and because $\bar q\in M$, and $M\cap H(\alpha^+)=N_k\cap H(\alpha^+)$.

We still have to show that there is $\langle s^*,p^*\rangle\in\mathbb{A}_\beta$ extending both $\langle s^{\bar q},p^{\bar q}\rangle$ and $\langle s,p\rangle$ in such a way that $\operatorname{res}_M(s^*)\setminus H(\beta)\subseteq s^{\bar q}$, and that all small nodes of s^* above β are either small nodes of s or of the form $N\cap W'$ where N is a small node of s and $W'\in\mathcal{T}$. Since $s^{\bar q}\supseteq\operatorname{res}_M(s)$ and $s^{\bar q}\in M$, the assumptions of Lemma 3.10 are satisfied for $s^{\bar q}$, s, and M. Hence, $s^{\bar q}$ and s are directly compatible, and if r witnesses this, then $\operatorname{res}_M(r)=s^{\bar q}$, and all the small nodes of r outside M are of the form N or $N\cap W'$, where N is a small node of s and W' is a transitive node of $s^{\bar q}$. Let $\langle a,h\rangle\in G\cap H(\alpha)$ be any condition stronger then both

$\langle s, p\rangle \restriction \alpha$ and $\langle s^{\bar{q}}, p^{\bar{q}}\rangle \restriction \alpha$. Since $r \cap H(\alpha) \subseteq a$ and $a \in H(\alpha)$, Lemma 3.8 implies that a and r are directly compatible and that this is witnessed by $a \cup r$. Define $s^* = a \cup r$. Properties of r imply that $\mathrm{res}_M(s^*) \setminus H(\beta) \subseteq s^{\bar{q}}$ and that all the small nodes of s^* above β and outside M are either small nodes of s or of the form $N \cap W'$ where N is a small node of s and W' is a transitive node. We will prove that $\langle s^*, h \cup \{\alpha\}\rangle$ is as required in the statement of the lemma. It is clear that $\langle s^*, h \cup \{\alpha\}\rangle$ is a condition in $\mathbb{A}$, so we only have to prove that $\langle s^*, h \cup \{\alpha\}\rangle \leq \langle s, p\rangle, \langle s^{\bar{q}}, p^{\bar{q}}\rangle$. To see that $\langle s^*, h \cup \{\alpha\}\rangle \leq \langle s^{\bar{q}}, p^{\bar{q}}\rangle$ notice that $s^*/\alpha = s^{\bar{q}}/\alpha = u$. For the same reason, and by the choice of $\langle \xi_0, \ldots, \xi_{l-k-1}\rangle$ (namely it is forced that $x^{\alpha}_{M_{k+j} \cap H(\alpha^+)} = \xi_j \in b^{\alpha}_{N_i \cap H(\alpha^+)}$ for each $j < l-k$ and $k \leq i < l$), it also the case that $\langle s^*, h \cup \{\alpha\}\rangle \leq \langle s, p\rangle$. This finishes the proof in the case when $H(\alpha) \in s$. If not, then in the same way as in the end of the proof of [11, Lemma 6.12], there is $s' \supseteq s$ such that $\langle s', p\rangle$ is a condition in $\mathbb{A}$ and that all the nodes in $s' \setminus s$ are either transitive or intersection of a small node from s and a transitive node. So the above arguments prove the lemma for $\langle s', p\rangle$ which implies that it holds for $\langle s, p\rangle$ as well. So the successor case is done.

Suppose now that β is a limit of cardinals in $\Sigma \cup \{\theta\}$. Let $\bar{\beta} = \sup(\beta \cap M^*)$ and let $\delta < \bar{\beta}$ in $\Sigma \cap M^*$ be such that $p \cap \bar{\beta} \subseteq \delta$. Then $\langle s, p \cap \delta\rangle \in \mathbb{A}_\delta$. Let $\dot{A}_\delta$ be the collection of all pairs $\langle x, \langle s', p'\rangle\rangle$ in $T \times \mathbb{A}_\delta$ for which we can find δ-end-extension of p' to p'' such that $\langle s', p''\rangle \in \mathbb{A}_\beta$ and $\langle s', p''\rangle \Vdash_{\mathbb{A}_\beta} x \in \dot{A}$. We consider $\dot{A}_\delta$ an $\mathbb{A}_\delta$-name for a subset of T. Note that $\dot{A}_\delta \in M^*$ and that $\langle s, p \cap \delta\rangle \Vdash_{\mathbb{A}_\delta} t \in \dot{A}_\delta$. So by the induction hypothesis, there is $\langle \bar{s}, \bar{p}\rangle \in M^* \cap \mathbb{A}_\delta$, and $\bar{t} < t$, such that $\langle \bar{s}, \bar{p}\rangle$ is compatible with $\langle s, p \cap \delta\rangle$, and that $\langle \bar{s}, \bar{p}\rangle \Vdash_{\mathbb{A}_\delta} \bar{t} \in \dot{A}_\delta$. Moreover, there is $\langle s^*, p_1\rangle \in \mathbb{A}_\delta$ extending both $\langle s, p \cap \delta\rangle$ and $\langle \bar{s}, \bar{p}\rangle$ in such a way that $\mathrm{res}_M(s^*) \setminus H(\delta) \subseteq \bar{s}$, and that all small nodes of s^* are either small nodes of s or of the form $N \cap W$ where N is a small node of s and $W \in \mathcal{T}$. Let p_2 be the δ-end-extension of p_1 such that $\langle s^*, p_2\rangle \in \mathbb{A}_\beta$ and $\langle s^*, p_2\rangle \Vdash_{\mathbb{A}_\beta} \bar{t} \in \dot{A}$. Now define $p^* = p_2 \cup (p \cap [\bar{\beta}, \beta))$. It is clear that $\langle s^*, p^*\rangle$ is a condition in $\mathbb{A}_\beta$, so it should only be explained why $\langle s^*, p^*\rangle \leq \langle s, p\rangle, \langle \bar{s}, \bar{p}\rangle$. But this is analogous to the proof of the limit case of (1) in Lemma 3.16.

The proof of (2) is very similar to the proof of (1). $\square$

Lemma 3.20. *The extension of V by $\mathbb{A}$ satisfies the P-ideal dichotomy.*

Proof. The proof is analogous to the proof of [11, Lemma 6.14]. Suppose that the lemma fails, and let $\langle a, h\rangle$ force that $\dot{J}$ is an ideal on some ordinal β which does not satisfy (1) and (2) of Definition 2. Let γ be large enough that $\dot{J}, \beta \in H(\gamma)$. Since F is a Laver function, and θ is supercompact, there is an elementary embedding π with critical point $\bar{\theta} < \theta$, and there are $\bar{\gamma} < \theta, \bar{\beta} < \theta$, and $\dot{I} \in H(\bar{\gamma})$ such that $\pi(\bar{\theta}) > \gamma$, and that $(H(\bar{\gamma}), F \restriction \bar{\theta}, \bar{\beta}, \dot{I}) \prec (H(\gamma), F, \beta, \dot{J})$, where $F(\bar{\theta}) = \dot{I}$. The critical point $\bar{\theta}$ can be chosen so that $\langle a, h\rangle \in \mathbb{A} \cap H(\bar{\theta})$. Then $\left\langle a \cup \left\{H(\bar{\theta})\right\}, h\right\rangle$ is a strongly $(H(\bar{\theta}), \mathbb{A})$-generic condition. Let G be generic filter in $\mathbb{A}$ containing $\left\langle a \cup \left\{H(\bar{\theta})\right\}, h\right\rangle$. By strong properness, $G \cap H(\bar{\theta})$ is generic for $\mathbb{A} \cap H(\bar{\theta})$ over V, and π extends trivially to an embedding of $H(\bar{\gamma})[G \cap H(\bar{\theta})]$ into $H(\gamma)[G]$. Let ψ denote this extension of π. Since $F(\bar{\theta}) = \dot{I}$, the poset $\mathbb{Q}$ of all strong $\in$-chains of countable elementary submodels of $H(\bar{\theta}^+)$, forces that there is an uncountable $X \subseteq \bar{\beta}$ such that $[X]^{\leq\omega} \subseteq \dot{I}[G \cap H(\bar{\theta})]$. Finally, $\psi''X$ is an uncountable subset of β and $[\psi''X]^{\leq\omega} \subseteq \dot{J}[G]$ as required. $\square$

Lemma 3.21. *Let T be an almost Souslin Aronszajn tree in V. Then T is a non-special Aronszajn tree in $V^{\mathbb{A}}$.*

Proof. Suppose that T is an almost Souslin tree in V, and that $\langle a, h\rangle \in \mathbb{A} = \mathbb{A}_\theta$ forces that T is special in $V^{\mathbb{A}}$. This means that there is an $\mathbb{A}$-name $\dot{A}$ for a stationary antichain in T. Let M be a countable elementary submodel of $H(\theta)$ containing $T, \langle a, h\rangle$ and $\dot{A}$. Denote $\delta = M \cap \omega_1$. Then by Lemma 3.15 there is a condition $\langle s, p\rangle \leq \langle a, h\rangle$ such that $M \in s$. Clearly, $\langle s, p\rangle$ also forces that $\dot{A}$ is a stationary antichain of T. So, by going to an extension we may assume that there is $t_\delta \in T_\delta$ such that $\langle s, p\rangle \Vdash t_\delta \in \dot{A}$. By (1) of Lemma 3.19, there is $\langle \bar{s}, \bar{p}\rangle$ compatible with $\langle s, p\rangle$, and $t < t_\delta$ such that $\langle \bar{s}, \bar{p}\rangle \Vdash t \in \dot{A}$. Thus for any generic filter G containing common extension of $\langle s, p\rangle$ and $\langle \bar{s}, \bar{p}\rangle$, we have that $\dot{A}[G]$ is not an antichain of T in $V[G]$. This contradicts the choice of $\dot{A}$ and proves the lemma. To see that T remains Aronszajn, use the same arguments and conclusion (2) of Lemma 3.19. $\square$

4. Concluding remarks

At this point we would like to make a few remarks regarding the material presented in this paper.

The main result in the paper proves that, under some assumptions, there is a class of non-special Aronszajn trees in V all of them remaining non-special in $V^{\mathbb{A}}$. It is the class of all almost Souslin Aronszajn trees in V. We would like to point out that our proof shows more if we restrict ourselves to the preservation of a single almost Souslin tree T, or more generally a single countable family $\mathcal{H}$ of graphs on ω_1 with no stationary anti-cliques. To see this, for $\beta \in \Sigma \cup \{\theta\}$, let $\dot{\mathcal{S}}_\beta$ be the $\mathbb{A}_\beta$-name for the collection of all $M \in \mathcal{S}$ that contain T (or a given countable collection $\mathcal{H}$ of graphs we want to preserve) such that $M[G_\beta] \cap V = M$. Suppose that if we redefine our iteration $\mathbb{A}_\beta$ in such a way that at a given stage β rather than to force the alternative of the P-ideal dichotomy for a P-ideal given to us by the Laver function F we force with a poset $F(\beta)$ given by this function which is $\dot{\mathcal{S}}_\beta$-proper and is T-preserving relative to $\dot{\mathcal{S}}_\beta$ (or more generally $\mathcal{H}$-preserving relative to $\dot{\mathcal{S}}_\beta$). Here by T-preserving, or more generally $\mathcal{H}$-preserving poset, we mean a poset for which the corresponding version of Lemma 3.19 holds[1]. If we take a generic filter G_θ for the resulting poset, we get in $V[G_\theta]$ the conclusion of the Proper Forcing Axiom for all $\mathcal{S}_\theta$-proper T-preserving (or more generally $\mathcal{H}$-preserving) posets. Let us call this axiom $\mathrm{PFA}_{\mathcal{S}_\theta}(\mathcal{H})$. The point is that this forcing axiom implies that the set $\mathcal{S}_\theta$ is *projectively stationary* in the sense that for every stationary set $A \subseteq \omega_1$, the set $\mathcal{S}_\theta(A) = \{M \in \mathcal{S}_\theta : M \cap \omega_1 \in A\}$ is

[1]More precisely, given a stationary set $\mathcal{S}$ of countable subsets of some index set I and a graph $H = (\omega_1, E_H)$, we say that a poset $\mathbb{P}$ preserves H relative to $\mathcal{S}$ if for every large enough regular cardinal κ and every countable elementary submodel M of $H(\kappa)$ containing the sets X and $\mathcal{S}$, the graph H, the poset $\mathbb{P}$ and one of its conditions p such that $M \cap I \in \mathcal{S}$ there is an $(M, \mathbb{P})$-generic condition $q \leq p$ such that for any $r \leq q$ and any $\mathbb{P}$-name $\dot{A}$ for a subset of ω_1 with $\dot{A} \in M$ if r forces $\delta = M \cap \omega_1 \in \dot{A}$ then there is $\gamma < \delta$ such that $\{\gamma, \delta\} \in E_H$ and $\bar{r} \in \mathbb{P} \cap M$ compatible with r and forcing $\gamma \in \dot{A}$. Given an almost Souslin tree T we assume its domain is ω_1 and that its levels are the intervals $[\delta, \delta + \omega)$ for limit ordinals $\delta < \omega_1$, and we associate to it the sequence $H_n = (\omega_1, E_n)$ $(n < \omega)$ of graphs where we put $\{\gamma, \delta\} \in E_n$ whenever γ and δ are limit ordinals and $\gamma + n$ and $\delta + n$ are comparable nodes of T. This way to an almost Souslin tree T, we can associate the family $\mathcal{H}_T$ consisting of the graphs H_n's together with their complementary graphs.

also stationary. This is so, since for every set A for which $\mathcal{S}_\theta(A)$ is not stationary the standard poset that shoots a club through the complement of A is $\mathcal{S}_\theta$-proper and T-preserving (or more generally $\mathcal{H}$-preserving). It follows that in $V[G_\theta]$ our fixed tree T is still Aronszajn and almost Souslin, or more generally, that our fixed countable family $\mathcal{H}$ still consists of graphs on ω_1 with no stationary anti-cliques. We should note, that this gives us also an alternative proof of a result obtained recently by John Krueger in his paper [7] which we learned about after this work was completed. Namely, Kruger [7] proved the consistency of the forcing axiom PFA(T^*), where T^* is a particular almost Souslin Aronszajn tree in V. Forcing axiom PFA(T^*) refers to all proper posets which force that T^* is still an almost Souslin Aronszajn tree. In [7], many applications of this forcing axiom are given. Similarly, our forcing axiom $\mathrm{PFA}_{\mathcal{S}_\theta}(\mathcal{H})$ has PID and $\mathfrak{p} > \omega_1$ as consequences.. It is well-known that this conjunction of axioms has many other interesting consequences (see [16]). Other closely related works can be found in papers by Yorioka [17] and Hirschorn [5], where they obtain similar results using different techniques from ours. In particular, they independently (and in a different way) showed that, modulo the existence of a supercompact cardinal, it is consistent that PID holds, and that there is a non-special Aronszajn tree.

We conclude this paper with the following question which naturally showed up during the course of work on this paper.

Problem 4.1. Assuming CH, does PID imply that every Aronszajn tree can be decomposed into countably many antichains?

Recall that PID is consistent with CH ([15]) but we don't know if the forcing iteration that establishes this preserves almost Souslin trees. Recall also that there are set-theoretic principles that are separately consistent with CH but their conjunction implies its negation (see [2], [10]). Problem 4.1 is asking if the P-ideal dichotomy and the statement that there exist an almost Souslin tree are two such principles.

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