MAXIMAL CHAINS OF ISOMORPHIC SUBGRAPHS OF THE RADO GRAPH

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Abstract

The partial order $\langle E(R) \cup \{\emptyset\}, \subset\rangle$, where $E(R)$ is the set of isomorphic subgraphs of the Rado graph R, is investigated. The order types of maximal chains in this poset are characterized as the order types of compact sets of reals having the minimum non-isolated.
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1 Preliminaries

The countable random graph (the Rado graph) introduced by Erdös and Rényi [3] (see also [1]) is, up to isomorphism, the unique countable graph $\langle R, \rho\rangle$ such that for arbitrary finite disjoint subsets H and K of R the set

$$R_H^{H\cup K} = \{r \in R \setminus (H \cup K) : \forall h \in H \ \{r, h\} \in \rho \wedge \forall k \in K \ \{r, k\} \notin \rho\}$$

is non-empty. By $E(R, \rho)$, or $E(R)$, we denote the collection of all sets $A \subset R$ such that the structure $\langle A, \rho \cap [A]^2\rangle$, shortly denoted by $\langle A, \rho\rangle$, is a random graph, which, by the uniqueness of the Rado graph, means that $\langle A, \rho\rangle \cong \langle R, \rho\rangle$.

The object of our study is the partial order $\langle E(R), \subset\rangle$. It is easy to see that it is a chain complete non-atomic suborder of the partial order $\langle [R]^\omega, \subset\rangle$ and the aim of the paper is to find one of its order-invariants - the class of order types of maximal chains in the poset $\langle E(R), \subset\rangle$. When, instead of the Rado graph, the rational line is in question, the corresponding class is the class of order types of linear orders of the form $K \setminus \{\min K\}$, where K is a compact set of reals with $\min K$ non-isolated [7]. Our main result, Theorem 2, shows that the same holds for the Rado graph.

We note that analogous characterizations were obtained for: the interval algebra $\mathrm{Intalg}[0, 1)_{\mathbb{R}}$ (dense σ-compact subsets of $[0, 1]_{\mathbb{R}}$ containing 0 and 1; Koppelberg [4]), the power set algebra $P(\kappa)$ (the orders of initial segments $\langle \mathrm{Init}(L), \subset\rangle$, for linear orders L of size κ; Kuratowski [5]), $<\kappa$-complete atomic Boolean algebras ($<\kappa$-complete linear orders having 0,1 and dense jumps; Day [2]). We will use the following characterization provided by Kuratowski's and Day's results (see [6]).

Fact 1. An infinite linear order L is isomorphic to a maximal chain in $P(\omega)$ iff L is $\mathbb{R}$-embeddable and Boolean.

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We will also use the following characterization from [7]. We recall that a family $\mathcal{P} \subset P(\omega)$ is called a *positive family* iff: (P1) $\emptyset \notin \mathcal{P}$; (P2) $\mathcal{P} \ni A \subset B \subset \omega \Rightarrow B \in \mathcal{P}$; (P3) $A \in \mathcal{P} \wedge |F| < \omega \Rightarrow A \setminus F \in \mathcal{P}$; (P4) $\exists A \in \mathcal{P}\ \ |\omega \setminus A| = \omega$.

Theorem 1. Let $\mathcal{P} \subset P(\omega)$ be a positive family. A linear order L is isomorphic to a maximal chain $\mathcal{L}$ in the poset $\langle \mathcal{P} \cup \{\emptyset\}, \subset\rangle$ satisfying $\bigcap(\mathcal{L} \setminus \{\emptyset\}) = \emptyset$ iff L is an $\mathbb{R}$-embeddable Boolean linear order with 0_L non-isolated.

A few words on notation and terminology. If $\langle P, \leq\rangle$ is a partial order, then the *smallest* and the *largest element* of P are denoted by 0_P and 1_P; the *intervals* $(x,y)_P$, $[x,y]_P$, $(-\infty,x)_P$ etc. are defined in the usual way. A set $D \subset P$ is *dense* iff for each $p \in P$ there is $q \in D$ such that $q \leq p$. $G \subset P$ is a filter iff (F1) for each $p, q \in G$ there is $r \in G$ such that $r \leq p, q$ and (F2) $G \ni p \leq q$ implies $q \in G$.

A pair $\langle \mathcal{A}, \mathcal{B}\rangle$ is a *cut* in a linear order $\langle L, <\rangle$ iff $L = \mathcal{A} \,\dot\cup\, \mathcal{B}$, $\mathcal{A}, \mathcal{B} \neq \emptyset$ and $a < b$, for each $a \in \mathcal{A}$ and $b \in \mathcal{B}$. A cut $\langle \mathcal{A}, \mathcal{B}\rangle$ is a *gap* iff neither $\max \mathcal{A}$ nor $\min \mathcal{B}$ exist. $\langle L, <\rangle$ is called: *complete* iff it has 0 and 1 and has no gaps; $\mathbb{R}$-*embeddable* iff it is isomorphic to a subset of $\mathbb{R}$; *Boolean* iff it is complete and *has dense jumps* (if $x < y$, then there are a, b such that $x \leq a < b \leq y$ and $(a,b)_L = \emptyset$). If $\langle I, <_I\rangle$ and $\langle L_i, <_i\rangle$, $i \in I$, are linear orders and $L_i \cap L_j = \emptyset$, whenever $i \neq j$, the corresponding *lexicographic sum* $\sum_{i\in I} L_i$ is the linear order $\langle \bigcup_{i\in I} L_i, <\rangle$, where $x < y \Leftrightarrow \exists i \in I\ (x, y \in L_i \wedge x <_i y) \vee \exists i, j \in I\ (i <_I j \wedge x \in L_i \wedge y \in L_j)$.

The following facts will be used in our construction as well.

Fact 2. If $\langle L, <\rangle$ is an at most countable complete linear order, it is Boolean.

Proof. Let $x, y \in L$ and $x < y$. Suppose that for each $a, b \in [x,y]_L$ satisfying $a < b$ we have $(a,b)_L \neq \emptyset$. Then $[x,y]_L$ would be a dense complete linear order, which is impossible because L is countable. Thus L has dense jumps. □

Fact 3. Let $\langle R, \rho\rangle$ be a countable random graph. Then:

(a) $R \setminus F \in E(R)$, for each finite subset F of R;

(b) If $R = X_1 \cup X_2 \cup \ldots \cup X_k$ is a partition, then $X_i \in E(R)$ for some $i \leq k$;

(c) Each countable graph can be embedded in R;

(d) $E(R)$ contains a positive subfamily of $P(R)$;

(e) If $\mathcal{L}$ is a chain in $E(R)$, then $\bigcup \mathcal{L} \in E(R)$.

Proof. Proofs of (a)-(c) can be found in [1].

(d) By (c) R contains a copy of the countable complete graph, $K_{\aleph_0}$. Let $\mathcal{P} = \{A \subset R : R \setminus K_{\aleph_0} \subset^* A\}$ (where $X \subset^* Y \Leftrightarrow |X \setminus Y| < \aleph_0$). If $A \in \mathcal{P}$, then $R \setminus A \subset^* K_{\aleph_0}$ and, hence $R \setminus A \notin E(R)$, which by (b) implies $A \in E(R)$. Thus $\mathcal{P} \subset E(R)$. $\mathcal{P}$ is a filter in $P(R)$ containing all cofinite subsets of R and the coinfinite set $R \setminus K_{\aleph_0}$ so it is a positive family.

(e) If H and K are disjoint finite subsets of $\bigcup \mathcal{L}$, then $H, K \subset L$, for some $L \in \mathcal{L}$ and, hence, $R_H^{H\cup K}$ intersects L and $\bigcup \mathcal{L}$ as well. □

Lemma 1. Let L be an at most countable complete linear order, $A, B \in E(R)$, $A \subset B$, $|B \setminus A| = |L| - 1$ and $[A, B]_{E(R)} = [A, B]_{P(B)}$. Then there is a chain $\mathcal{L}$ in $[A, B]_{E(R)}$ satisfying $A, B \in \mathcal{L} \cong L$ and such that $\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}$ and $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$, for each cut $\langle \mathcal{A}, \mathcal{B} \rangle$ in $\mathcal{L}$.

Proof. If $|B \setminus A|$ is a finite set, say $B = A \cup \{a_1, \dots a_n\}$, then $|L| + 1$ and $\mathcal{L} = \{A, A \cup \{a_1\}, A \cup \{a_1, a_2\}, \dots, B\}$ is a chain with the desired properties.

If $|B \setminus A| = \aleph_0$, then L is a countable and, hence, $\mathbb{R}$-embeddable complete linear order. By Fact 2 L is a Boolean order and, by Fact 1, there is a maximal chain $\mathcal{L}_1$ in $P(B \setminus A)$ isomorphic to L. Let $\mathcal{L} = \{A \cup C : C \in \mathcal{L}_1\}$. Since $\emptyset, B \setminus A \in \mathcal{L}_1$ we have $A, B \in \mathcal{L}$ and the function $f : \mathcal{L}_1 \to \mathcal{L}$, defined by $f(C) = A \cup C$, witnesses that $\langle \mathcal{L}_1, \subsetneq \rangle \cong \langle \mathcal{L}, \subsetneq \rangle$ so $\mathcal{L}$ is isomorphic to L. For each cut $\langle \mathcal{A}, \mathcal{B} \rangle$ in $\mathcal{L}_1$ we have $\bigcup \mathcal{A} \subset \bigcap \mathcal{B}$ and, by the maximality of $\mathcal{L}_1$, $\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_1$ and $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$. Clearly, the same is true for each cut in $\mathcal{L}$. □

2 Maximal chains of copies of the Rado graph

Theorem 2. If R is a random graph, then for each linear order L the following conditions are equivalent:

(a) L is isomorphic to a maximal chain in the poset $\langle E(R) \cup \{\emptyset\}, \subset \rangle$;

(b) L is an $\mathbb{R}$-embeddable complete linear order with 0_L non-isolated;

(c) L is isomorphic to a compact set $K \subset [0, 1]_{\mathbb{R}}$ such that $0 \in K'$ and $1 \in K$.

Proof. The equivalence (b) $\Leftrightarrow$ (c) is proved in Theorem 6 of [7].

(a) $\Rightarrow$ (b) Let $\mathcal{L}$ be a maximal chain in $\langle E(R) \cup \{\emptyset\}, \subset \rangle$ and $R = \{q_n : n \in \omega\}$ an enumeration. Since $\mathcal{L} \subset [R]^\omega \cup \{\emptyset\}$, the function $f : \mathcal{L} \to \mathbb{R}$ defined by $f(A) = \sum_{n \in \omega} 2^{-n} \cdot \chi_A(q_n)$ (where $\chi_A : R \to \{0, 1\}$ is the characteristic function of the set $A \subset R$) is an embedding of $\mathcal{L}$ into $\mathbb{R}$. Thus $\langle \mathcal{L}, \subset \rangle$ is $\mathbb{R}$-embeddable.

Clearly, $\min \mathcal{L} = \emptyset$ and $\max \mathcal{L} = R$. Let $\langle \mathcal{A}, \mathcal{B} \rangle$ be a cut in $\mathcal{L}$. If $\mathcal{A} = \{\emptyset\}$ then $\max \mathcal{A} = \emptyset$. If $\mathcal{A} \neq \{\emptyset\}$, by Fact 3(e) we have $\bigcup \mathcal{A} \in E(R)$ and, since $A \subset \bigcup \mathcal{A} \subset B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the maximality of $\mathcal{L}$ implies $\bigcup \mathcal{A} \in \mathcal{L}$. So, if $\bigcup \mathcal{A} \in \mathcal{A}$ then $\max \mathcal{A} = \bigcup \mathcal{A}$. Otherwise $\bigcup \mathcal{A} \in \mathcal{B}$ and $\min \mathcal{B} = \bigcup \mathcal{A}$. Thus $\langle \mathcal{L}, \subset \rangle$ is complete. Suppose that A is the successor of $\emptyset$ in $\mathcal{L}$. By Fact 3(b) there is $B \in E(R)$ such that $B \subsetneq A$. A contradiction to the maximality of $\mathcal{L}$.

(b) $\Rightarrow$ (a) First we prove this implication for countable L. Let $\mathcal{P} \subset E(R)$ be a positive family in $P(R)$ (see Fact 3(d)). Then, by Fact 2, L is a Boolean order and, by Theorem 1, in the poset $\langle \mathcal{P} \cup \{\emptyset\}, \subset \rangle$ there is a maximal chain $\mathcal{L}$ isomorphic to L and such that $\bigcap(\mathcal{L} \setminus \{\emptyset\}) = \emptyset$. Since $\mathcal{P} \subset E(R)$, $\mathcal{L}$ is a chain in the poset $\langle E(R) \cup \{\emptyset\}, \subset \rangle$ and it remains to be proved that it is maximal. Suppose that $\mathcal{L} \cup \{A\}$ is a chain, where $A \in E(R) \setminus \mathcal{L}$. Then $A \subsetneq S$ or $S \subsetneq A$, for each $S \in \mathcal{L} \setminus \{\emptyset\}$ and, since $\bigcap(\mathcal{L} \setminus \{\emptyset\}) = \emptyset$, there is $S \in \mathcal{L} \setminus \{\emptyset\}$ such that $S \subset A$, which implies $A \in \mathcal{P}$. But $\mathcal{L} \setminus \{\emptyset\}$ is a maximal chain in $\mathcal{P}$. A contradiction.

In the sequel we prove (b) $\Rightarrow$ (a) for uncountable L.

Claim 1. $L \cong \sum_{x\in[-\infty,\infty]} L_x$, where
(i) L_x, $x \in [-\infty,\infty]$, are at most countable complete linear orders,
(ii) The set $M = \{x \in [-\infty,\infty] : |L_x| > 1\}$ is at most countable,
(iii) $|L_{-\infty}| = 1$ or $0_{L_{-\infty}}$ is non-isolated.

Proof. $L = \sum_{i\in I} L_i$, where L_i are the equivalence classes corresponding to the condensation relation $\sim$ on L given by: $x \sim y \Leftrightarrow |[\min\{x,y\},\max\{x,y\}]| \leq \aleph_0$ (see [8]). Since L is complete and $\mathbb{R}$-embeddable I is too and, since the cofinalities and coinitialities of L_i's are countable, I is a dense linear order; so $I \cong [0,1] \cong [-\infty,\infty]$. Hence L_i's are complete and, since $\min L_i \sim \max L_i$, countable. If $|L_i| > 1$, then L_i is not dense (otherwise we would have $|L_i| = \mathfrak{c}$) and, hence, L_i contains a jump so, $L \hookrightarrow \mathbb{R}$ implies $|M| \leq \aleph_0$. □

(I): $|L_{-\infty}| = 1$. First we construct a set $\rho \subset [\mathbb{Q}]^2$ such that $\langle \mathbb{Q}, \rho\rangle$ is a random graph. Let $(0,1)\cap\mathbb{Q} = J \cup \bigcup_{y\in M} J_y$ be a partition of the set $(0,1)\cap\mathbb{Q}$ into $|M|+1$ disjoint sets, dense in $(0,1)\cap\mathbb{Q}$. (See e.g. [9], p. 216.) Let $\mathbb{Z}$ denote the set of integers and let us define $I = \{q+m : q \in J \wedge m \in \mathbb{Z}\} \cup \mathbb{Z}$ and $J_y^{\mathbb{Z}} = \{q+m : q \in J_y \wedge m \in \mathbb{Z}\}$, for $y \in M$. Then, clearly, we have

Claim 2. $\{I\} \cup \{J_y^{\mathbb{Z}} : y \in M\}$ is a partition of $\mathbb{Q}$ consisting of dense subsets of $\mathbb{Q}$.

In our construction of ρ we will use the poset $\langle \mathbb{P}, \supset\rangle$, where $\mathbb{P}$ is the set of finite partial functions p from $[\mathbb{Q}]^2$ to $2 = \{0,1\}$ such that for each $a, b \in \mathbb{Q}$

$$\langle\{a,b\},1\rangle, \langle\{a+1,b\},1\rangle \in p \;\Rightarrow\; b > a+1. \tag{1}$$

Claim 3. $\mathcal{D}_{\{q,r\}} = \{p \in \mathbb{P} : \{q,r\} \in \mathrm{dom}(p)\}$, $\{q,r\} \in [\mathbb{Q}]^2$, are dense sets in $\mathbb{P}$.

Proof. If $p \in \mathbb{P} \setminus \mathcal{D}_{\{q,r\}}$, then $p_1 = p \cup \{\langle\{q,r\},0\rangle\} \in \mathrm{Fn}([\mathbb{Q}]^2, 2)$ and we check (1). If $\langle\{a,b\},1\rangle, \langle\{a+1,b\},1\rangle \in p_1$, then, clearly, both pairs belong to p and, since $p \in \mathbb{P}$, we have $b > a+1$. So $p_1 \in \mathcal{D}_{\{q,r\}}$ and $p_1 \supset p$. □

Let $C = \{\langle K, L\rangle : K, L \in [\mathbb{Q}]^{<\omega} \wedge K \cap L = \emptyset\}$ and, for each $\langle K, L\rangle \in C$, let $m_{K,L} = \max(K \cup L)$.

Claim 4. For each $\langle K, L\rangle \in C$ and $m \in \omega$ the set $\mathcal{D}_{K,L,m}$ is dense in $\mathbb{P}$, where

$$\begin{aligned}\mathcal{D}_{K,L,m} = \{p \in \mathbb{P} \;\;:\;\; & \exists q \in I \cap (m_{K,L}, m_{K,L} + \tfrac{1}{m}) \\ & \forall r \in K\; \forall s \in L\;\; \langle\{q,r\},1\rangle, \langle\{q,s\},0\rangle \in p\}.\end{aligned}$$

Proof. Let $\langle K, L\rangle \in C$, $m \in \omega$ and $p = \{\langle\{p_i,q_i\},k_i\rangle : i < n\} \in \mathbb{P}$. Since the set $S = K \cup L \cup \bigcup_{i<n}\{p_i, q_i, p_i+1, q_i+1, p_i-1, q_i-1\}$ is finite and, by Claim 2, I is a dense subset of $\mathbb{Q}$, there is $q \in I \cap (m_{K,L}, m_{K,L}+\frac{1}{m})) \setminus S$. We show that $p_1 = p \cup \{\langle\{q,r\},1\rangle : r \in K\} \cup \{\langle\{q,s\},0\rangle : s \in L\} \in \mathbb{P}$. Since $q \notin K \cup L$ we have $p_1 \subset [\mathbb{Q}]^2 \times 2$. Suppose that $\langle\{a,b\},0\rangle, \langle\{a,b\},1\rangle \in p_1$. Then, since p is a function, either one of the pairs is new, which is impossible because

$q \notin \bigcup \mathrm{dom}(p)$, or both of them are new, and, hence, there are $r \in K$ and $s \in L$ such that $\{q, r\} = \{q, s\}$, which implies $r = s$. But this is impossible, because $K \cap L = \emptyset$. Thus p_1 is a function and we check that it satisfies (1). Suppose that

$$\langle\{a, b\}, 1\rangle, \langle\{a + 1, b\}, 1\rangle \in p_1 \ \wedge \ b \leq a + 1. \tag{2}$$

Then, since $p \in \mathbb{P}$, at least one of the two pairs does not belong to p and, hence $q \in \{a, a + 1, b\}$ so we have the following three cases.

$q = a$. Then by (2) we have $b \neq q$ and $\langle\{q + 1, b\}, 1\rangle \in p_1$, which implies $\langle\{q + 1, b\}, 1\rangle \in p$ and, hence, $q = (q + 1) - 1 \in S$, a contradiction.

$q = a + 1$. Then by (2) we have $b \neq q$ and $\langle\{q - 1, b\}, 1\rangle \in p_1$, which implies $\langle\{q - 1, b\}, 1\rangle \in p$ and, hence, $q = (q - 1) + 1 \in S$, a contradiction.

$q = b$. Then by (2) we have $\langle\{a, q\}, 1\rangle, \langle\{a + 1, q\}, 1\rangle \in p_1 \setminus p$, which implies that $a, a + 1 \in K$. Since $q > m_{K,L}$ we have $q > a + 1$, that is $b > a + 1$. A contradiction again. □

Since $|[\mathbb{Q}]^2| = |C| = \aleph_0$, by the Rasiowa-Sikorski theorem there is a filter G in $\mathbb{P}$ intersecting the sets $\mathcal{D}_{\{q,r\}}$, $\{q, r\} \in [\mathbb{Q}]^2$, and $\mathcal{D}_{K,L,m}$, $\langle K, L\rangle \in C$, $m \in \omega$.

Claim 5. (a) $f = \bigcup_{p \in G} p$ is a function from $[\mathbb{Q}]^2$ to 2.

(b) Let $\rho = f^{-1}[\{1\}]$. If $I \subset A \subset \mathbb{Q}$ then $\langle A, \rho \cap [A]^2\rangle$ is a random graph. In particular, $\langle \mathbb{Q}, \rho\rangle$ is a random graph and $A \in E(\mathbb{Q}, \rho)$.

(c) If $C \subset \mathbb{Q}$, $\max C = a$ and $a - 1 \in C$, then $C \notin E(\mathbb{Q}, \rho)$.

Proof. (a) Clearly we have $f \subset [\mathbb{Q}]^2 \times 2$ and, since G is a filter, its elements are compatible thus f is a function. If $\{q, r\} \in [\mathbb{Q}]^2$, then there is $p \in G \cap \mathcal{D}_{\{q,r\}}$ and, hence, $\{q, r\} \in \mathrm{dom}(p) \subset \mathrm{dom}(f)$. So $\mathrm{dom}(f) = [\mathbb{Q}]^2$.

(b) Let $I \subset A \subset \mathbb{Q}$ and let K and L be finite disjoint subsets of A. Then $\langle K, L\rangle \in C$ and, by the choice of G, there is $p \in G \cap \mathcal{D}_{K,L,1}$. Hence there exists $q \in I \cap (m_{K,L}, m_{K,L} + 1) \subset A$ such that $\langle\{q, r\}, 1\rangle \in p \subset f$, that is $\{q, r\} \in \rho$, for each $r \in K$ and $\langle\{q, s\}, 0\rangle \in p \subset f$, that is $\{q, s\} \notin \rho$, for each $s \in L$.

(c) Suppose that $b \in C$ and $\{a - 1, b\}, \{a, b\} \in \rho$, that is $\langle\{a - 1, b\}, 1\rangle$, $\langle\{a, b\}, 1\rangle \in f$. Then these pairs are in some $p_1, p_2 \in G$ and, since G is a filter, there is $p \in G$ such that $p_1, p_2 \subset p$. Consequently, p contains these pairs, which, by (1) implies $b > a$. But this is impossible, because $a = \max C$ and $b \in C$. □

For $y \in M$ let us take $I_y \in [J_y^{\mathbb{Z}} \cap (-\infty, y)]^{|L_y|-1}$ and define $A_{-\infty} = \emptyset$ and

$$A_x = (I \cap (-\infty, x)) \cup \bigcup\nolimits_{y \in M \cap (-\infty, x)} I_y, \quad \text{for } x \in (-\infty, \infty];$$

$$A_x^+ = A_x \cup I_x, \quad \text{for } x \in M.$$

We split the proof for the case (I) considering two subcases: $\infty \in M$ and $\infty \notin M$.

(I.I): $\infty \in M$. Since $I \subset A_\infty^+ \subset \mathbb{Q}$, by Claim 5(b) $\langle A_\infty^+, \rho\rangle$ is a random graph.

Claim 6. The sets A_x, $x \in [-\infty, \infty]$ and A_x^+, $x \in M$ are subsets of the set A_∞^+ and of $\mathbb{Q}$. In addition, for each $x, x_1, x_2 \in [-\infty, \infty]$ we have

(a) $A_x \subset (-\infty, x)$;
(b) $A_x^+ \subset (-\infty, x)$, if $x \in M$;
(c) $x_1 < x_2 \Rightarrow A_{x_1} \varsubsetneq A_{x_2}$;
(d) $M \ni x_1 < x_2 \Rightarrow A_{x_1}^+ \varsubsetneq A_{x_2}$;
(e) $|A_x^+ \setminus A_x| = |L_x| - 1$, if $x \in M$.

Proof. (c) and (d) are true since I is a dense subset of $\mathbb{Q}$ (Claim 2). The rest follows from the definitions of A_x and A_x^+ and the choice of the sets I_y. □

Claim 7. (a) $A_x \in E(A_\infty^+, \rho)$, for each $x \in (-\infty, \infty]$. Moreover, $A \in E(A_\infty^+, \rho)$, whenever $I \cap (-\infty, x) \subset A \subset A_x$.

(b) $A_x^+ \in E(A_\infty^+, \rho)$ and $[A_x, A_x^+]_{E(A_\infty^+,\rho)} = [A_x, A_x^+]_{P(A_x^+)}$, for each $x \in M$. Moreover, $A \in E(A_\infty^+, \rho)$, whenever $I \cap (-\infty, x) \subset A \subset A_x^+$.

Proof. (a) Let K and L be finite and disjoint subsets of A. By Claim 6(a) we have $K, L \subset A_x \subset (-\infty, x)$, which implies $m_{K,L} < x$ and, clearly, there is $m > 0$ such that $(m_{K,L}, m_{K,L} + \frac{1}{m}) \subset (-\infty, x)$ and, hence,

$$I \cap (m_{K,L}, m_{K,L} + \tfrac{1}{m}) \subset I \cap (-\infty, x) \subset A. \tag{3}$$

Let $p \in G \cap \mathcal{D}_{K,L,m}$. Then there is $q \in I \cap (m_{K,L}, m_{K,L} + \frac{1}{m})$ such that for each $r \in K$ we have $\langle \{q, r\}, 1\rangle \in p \subset f$ and, hence, $\{q, r\} \in \rho$ and for each $s \in L$ we have $\langle \{q, s\}, 0\rangle \in p \subset f$ and, hence, $\{q, s\} \notin \rho$. By (3) we have $q \in A$.

(b) By Claim 6(b) $A_x^+ \subset (-\infty, x)$ and we proceed as in the proof of (a). □

Now we define chains $\mathcal{L}_x \subset E(A_\infty^+, \rho) \cup \{\emptyset\}$, $x \in [-\infty, \infty]$ as follows.

For $x \in [-\infty, \infty] \setminus M$ we define $\mathcal{L}_x = \{A_x\}$. In particular, $\mathcal{L}_{-\infty} = \{\emptyset\}$.

For $x \in M$, by (a) and (b) of Claim 7 we have $A_x, A_x^+ \in E(A_\infty^+, \rho)$, by Claim 6(e), $|A_x^+ \setminus A_x| = |L_x| - 1$ and, by Claim 7(b), $[A_x, A_x^+]_{E(A_\infty^+,\rho)} = [A_x, A_x^+]_{P(A_x^+)}$. So, since L_x is a complete linear order, by Lemma 1, there is a set $\mathcal{L}_x \subset E(A_\infty^+, \rho)$ such that $\langle \mathcal{L}_x, \varsubsetneq\rangle \cong \langle L_x, <_x\rangle$ and

$$A_x, A_x^+ \in \mathcal{L}_x \subset [A_x, A_x^+]_{E(A_\infty^+,\rho)}, \tag{4}$$

$$\bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_x \text{ and } |\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1, \text{ for each cut } \langle \mathcal{A}, \mathcal{B}\rangle \text{ in } \mathcal{L}_x. \tag{5}$$

For $\mathcal{A}, \mathcal{B} \subset E(A_\infty^+)$ we will write $\mathcal{A} \lhd \mathcal{B}$ iff $A \varsubsetneq B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Claim 8. Let $\mathcal{L} = \bigcup_{x \in [-\infty,\infty]} \mathcal{L}_x$. Then

(a) If $-\infty \leq x_1 < x_2 \leq \infty$, then $\mathcal{L}_{x_1} \lhd \mathcal{L}_{x_2}$ and $\bigcup \mathcal{L}_{x_1} \subset A_{x_2} \subset \bigcup \mathcal{L}_{x_2}$.
(b) $\mathcal{L}$ is a chain in $\langle E(A_\infty^+) \cup \{\emptyset\}, \subset\rangle$ isomorphic to $L = \sum_{x \in [-\infty,\infty]} L_x$.

Proof. (a) Let $A \in \mathcal{L}_{x_1}$ and $B \in \mathcal{L}_{x_2}$. If $x_1 \in (-\infty, \infty] \setminus M$, then, by (4) and Claim 6(c) we have $A = A_{x_1} \varsubsetneq A_{x_2} \subset B$. If $x_1 \in M$, then, by (4) and Claim 6(d), $A \subset A_{x_1}^+ \varsubsetneq A_{x_2} \subset B$. The second statement follows from $A_{x_2} \in \mathcal{L}_{x_2}$.

(b) By (a), $\langle [-\infty, \infty], <\rangle \cong \langle \{\mathcal{L}_x : x \in [-\infty, \infty]\}, \lhd\rangle$. Since $\mathcal{L}_x \cong L_x$, for $x \in [-\infty, \infty]$, we have $\langle \mathcal{L}, \varsubsetneq\rangle \cong \sum_{x \in [-\infty,\infty]} \langle \mathcal{L}_x, \varsubsetneq\rangle \cong \sum_{x \in [-\infty,\infty]} L_x = L$. □

Claim 9. $\mathcal{L}$ is a maximal chain in $\langle E(A^+_\infty, \rho) \cup \{\emptyset\}, \subset\rangle$.

Proof. Suppose that $C \in E(A^+_\infty, \rho) \cup \{\emptyset\}$ witnesses that $\mathcal{L}$ is not maximal. Clearly $\mathcal{L} = \mathcal{A} \dot\cup \mathcal{B}$ and $\mathcal{A} \lhd \mathcal{B}$, where $\mathcal{A} = \{A \in \mathcal{L} : A \subsetneq C\}$ and $\mathcal{B} = \{B \in \mathcal{L} : C \subsetneq B\}$. Now $\emptyset \in \mathcal{L}_{-\infty}$ and, since $\infty \in M$, by (4) we have $A^+_\infty \in \mathcal{L}_\infty$. Thus $\emptyset, A^+_\infty \in \mathcal{L}$, which implies $\mathcal{A}, \mathcal{B} \neq \emptyset$ and, hence, $\langle \mathcal{A}, \mathcal{B}\rangle$ is a cut in $\langle \mathcal{L}, \subsetneq\rangle$. By (4) we have $\{A_x : x \in (-\infty, \infty]\} \subset \mathcal{L} \setminus \{\emptyset\}$ and, by Claim 6(a), $\bigcap(\mathcal{L} \setminus \{\emptyset\}) \subset \bigcap_{x\in(-\infty,\infty]} A_x \subset \bigcap_{x\in(-\infty,\infty]}(-\infty, x) = \emptyset$, which implies $\mathcal{A} \neq \{\emptyset\}$. Clearly,

$$\bigcup \mathcal{A} \subset C \subset \bigcap \mathcal{B}. \tag{6}$$

Case 1: $\mathcal{A} \cap \mathcal{L}_{x_0} \neq \emptyset$ and $\mathcal{B} \cap \mathcal{L}_{x_0} \neq \emptyset$, for some $x_0 \in (-\infty, \infty]$. Then $|\mathcal{L}_{x_0}| > 1$, $x_0 \in M$ and $\langle \mathcal{A} \cap \mathcal{L}_{x_0}, \mathcal{B} \cap \mathcal{L}_{x_0}\rangle$ is a cut in $\mathcal{L}_{x_0}$ satisfying (5). By Claim 8(a), $\mathcal{A} = \bigcup_{x<x_0} \mathcal{L}_x \cup (\mathcal{A} \cap \mathcal{L}_{x_0})$ and, consequently, $\bigcup \mathcal{A} = \bigcup(\mathcal{A} \cap \mathcal{L}_{x_0}) \in \mathcal{L}$. Similarly, $\bigcap \mathcal{B} = \bigcap(\mathcal{B} \cap \mathcal{L}_{x_0}) \in \mathcal{L}$ and, since $|\bigcap \mathcal{B} \setminus \bigcup \mathcal{A}| \leq 1$, by (6) we have $C \in \mathcal{L}$. A contradiction.

Case 2: $\neg$ Case 1. Then for each $x \in (-\infty, \infty]$ we have $\mathcal{L}_x \subset \mathcal{A}$ or $\mathcal{L}_x \subset \mathcal{B}$. Since $\mathcal{L} = \mathcal{A} \dot\cup \mathcal{B}$, $\mathcal{A} \neq \{\emptyset\}$ and $\mathcal{A}, \mathcal{B} \neq \emptyset$, the sets $\mathcal{A}' = \{x \in (-\infty, \infty] : \mathcal{L}_x \subset \mathcal{A}\}$ and $\mathcal{B}' = \{x \in (-\infty, \infty] : \mathcal{L}_x \subset \mathcal{B}\}$ are non-empty and $(-\infty, \infty] = \mathcal{A}' \dot\cup \mathcal{B}'$. Since $\mathcal{A} \lhd \mathcal{B}$, for $x_1 \in \mathcal{A}'$ and $x_2 \in \mathcal{B}'$ we have $\mathcal{L}_{x_1} \lhd \mathcal{L}_{x_2}$ so, by Claim 8(a), $x_1 < x_2$. Thus $\langle \mathcal{A}', \mathcal{B}'\rangle$ is a cut in $(-\infty, \infty]$ and, consequently, there is $x_0 \in (-\infty, \infty]$ such that $x_0 = \max \mathcal{A}'$ or $x_0 = \min \mathcal{B}'$.

Subcase 2.1: $x_0 = \max \mathcal{A}'$. Then $x_0 < \infty$ because $\mathcal{B} \neq \emptyset$ and $\mathcal{A} = \bigcup_{x\leq x_0} \mathcal{L}_x$ so, by Claim 8(a), $\bigcup \mathcal{A} = \bigcup_{x\leq x_0} \bigcup \mathcal{L}_x = \bigcup_{x<x_0} \bigcup \mathcal{L}_x \cup \bigcup \mathcal{L}_{x_0} = \bigcup \mathcal{L}_{x_0}$ which, together with (4) implies

$$\bigcup \mathcal{A} = \begin{cases} A_{x_0} & \text{if } x_0 \notin M, \\ A^+_{x_0} & \text{if } x_0 \in M. \end{cases}$$

Since $\mathcal{B} = \bigcup_{x\in(x_0,\infty]} \mathcal{L}_x$, we have $\bigcap \mathcal{B} = \bigcap_{x\in(x_0,\infty]} \bigcap \mathcal{L}_x$. By (4) $\bigcap \mathcal{L}_x = A_x$, so we have $\bigcap \mathcal{B} = (\bigcap_{x\in(x_0,\infty]}(-\infty, x) \cap I) \cup (\bigcap_{x\in(x_0,\infty]} \bigcup_{y\in M\cap(-\infty,x)} I_y) = ((-\infty, x_0] \cap I) \cup \bigcup_{y\in M\cap(-\infty,x_0]} I_y = A_{x_0} \cup (\{x_0\} \cap I) \cup \bigcup_{y\in M\cap\{x_0\}} I_y$, so

$$\bigcap \mathcal{B} = \begin{cases} A_{x_0} & \text{if } x_0 \notin I \ \wedge \ x_0 \notin M. \\ A_{x_0} \cup \{x_0\} & \text{if } x_0 \in I \ \wedge \ x_0 \notin M, \\ A^+_{x_0} & \text{if } x_0 \notin I \ \wedge \ x_0 \in M, \\ A^+_{x_0} \cup \{x_0\} & \text{if } x_0 \in I \ \wedge \ x_0 \in M. \end{cases}$$

If $x_0 \notin I$, then, by the formulas for $\bigcup \mathcal{A}$ and $\bigcap \mathcal{B}$ we have $\bigcup \mathcal{A} = \bigcap \mathcal{B} \in \mathcal{L}$ and, by (6), $C \in \mathcal{L}$. A contradiction.

If $x_0 \in I$ and $x_0 \notin M$, then $\bigcup \mathcal{A} = A_{x_0}$ and $\bigcap \mathcal{B} = A_{x_0} \cup \{x_0\}$. So, by (6) and since $C \notin L$ we have $C = \bigcap \mathcal{B}$. But, by Claim 6(a), $x_0 = \max \bigcap \mathcal{B}$ so, by Claim 5(c), $\bigcap \mathcal{B} \notin E(A^+_\infty, \rho)$. A contradiction.

If $x_0 \in I$ and $x_0 \in M$, then $\bigcup \mathcal{A} = A^+_{x_0}$ and $\bigcap \mathcal{B} = A^+_{x_0} \cup \{x_0\}$. Again, by (6) and since $C \not\in L$ we have $C = \bigcap \mathcal{B}$. By Claim 6(b), $x_0 = \max \bigcap \mathcal{B}$ so, by Claim 5(c), $\bigcap \mathcal{B} \not\in E(A^+_\infty, \rho)$. A contradiction.

Subcase 2.2: $x_0 = \min \mathcal{B}'$. Then, by (4), $A_{x_0} \in \mathcal{L}_{x_0} \subset \mathcal{B}$ which, by Claim 8(a), implies $\bigcap \mathcal{B} = A_{x_0}$. Since $A_x \in \mathcal{L}_x$, for all $x \in (-\infty, \infty]$ and $\mathcal{A} = \bigcup_{x<x_0} \mathcal{L}_x$ we have $\bigcup \mathcal{A} = \bigcup_{x<x_0} \bigcup \mathcal{L}_x \supset \bigcup_{x<x_0} A_x = \bigcup_{x<x_0} ((-\infty, x) \cap I) \cup \bigcup_{x<x_0} \bigcup_{y \in M \cap (-\infty,x)} I_y = ((-\infty, x_0) \cap I) \cup \bigcup_{y \in M \cap (-\infty,x_0)} I_y = A_{x_0}$ so $A_{x_0} \subset \bigcup \mathcal{A} \subset \bigcap \mathcal{B} \subset A_{x_0}$, which implies $C = A_{x_0} \in \mathcal{L}$. A contradiction. □

(I.II): $\infty \not\in M$. Then $L_\infty = \{\max L\}$ and the sum $L + 1$ satisfies condition (I.I). So, there are a maximal chain $\mathcal{L}$ in $\langle E(R) \cup \{\emptyset\}, \subset\rangle$ and an isomorphism $f : \langle L + 1, <\rangle \to \langle \mathcal{L}, \subset\rangle$. Then $A = f(\max L) \in E(R)$ and $\mathcal{L}' = f[L] \cong L$. By the maximality of $\mathcal{L}$, $\mathcal{L}'$ is a maximal chain in $\langle E(A) \cup \{\emptyset\}, \subset\rangle$.

(II): $|L_{-\infty}| > 1$. Then $L = \sum_{x \in [-\infty,\infty]} L_x$, (i) and (ii) of Claim 1 hold and

(iii′) $L_{-\infty}$ is a countable complete linear order with $0_{L_{-\infty}}$ non-isolated.

Clearly $L = L_{-\infty} + L^+$, where $L^+ = \sum_{x \in (-\infty,\infty]} L_x = \sum_{y \in (0,\infty]} L_{\ln y}$ (here $\ln \infty = \infty$). Let L'_y, $y \in [-\infty, \infty]$, be disjoint linear orders such that $L'_y \cong 1$, for $y \in [-\infty, 0]$, and $L'_y \cong L_{\ln y}$, for $y \in (0, \infty]$. Now $\sum_{y \in [-\infty,\infty]} L'_y \cong [-\infty, 0] + L^+$ satisfies (I) and we obtain a maximal chain $\mathcal{L}$ in $E(R) \cup \{\emptyset\}$ and an isomorphism $f : \langle [-\infty, 0] + L^+, <\rangle \to \langle \mathcal{L}, \subset\rangle$. Clearly, for $A_0 = f(0)$ and $\mathcal{L}^+ = f[L^+]$ we have $A_0 \in \mathcal{L}$ and $\mathcal{L}^+ \cong L^+$.

By (iii′) and the fact that (b) ⇒ (a) for countable L's, $E(A_0) \cup \{\emptyset\}$ contains a maximal chain $\mathcal{L}_{-\infty} \cong L_{-\infty}$. Clearly $A_0 \in \mathcal{L}_{-\infty}$ and $\mathcal{L}_{-\infty} \cup \mathcal{L}^+ \cong L_{-\infty} + L^+ = L$. Suppose that a set B witnesses that $\mathcal{L}_{-\infty} \cup \mathcal{L}^+$ is not a maximal chain in $E(R) \cup \{\emptyset\}$. Then either $A_0 \subsetneq B$, which is impossible since $\mathcal{L}$ is maximal in $E(R) \cup \{\emptyset\}$, or $B \subsetneq A_0$, which is impossible since $\mathcal{L}_{-\infty}$ is maximal in $E(A_0) \cup \{\emptyset\}$. □

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