

VARIETIES OF INVOLUTION SEMIGROUPS AND INVOLUTION SEMIRINGS: A SURVEY

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Dedicated to Professor Veselin Perić on the occasion of his 70th birthday.

INTRODUCTION

A function $f : A \rightarrow A$ (that is, a unary operation on the set A) is called an *involution* if $f(f(x)) = x$ holds for all $x \in A$. Involutions appear in many different areas of mathematics, and their role seems to be most transparent in geometry (euclidian, projective, differential, etc.). They occur quite frequently in topology and functional analysis, and especially in algebra. When one is concerned with involutions on algebraic structures, it is usual to require that the considered involution is also an antiautomorphism of the underlying algebraic system (it is clear that any involution is a permutation). In that way, involutions indicate a certain kind of internal symmetry of such systems. Maybe the simplest example of an algebraic involution is the transposition of matrices in the algebra of matrices over a ring.

Further, an involution can be considered as a fundamental operation, and therefore, a part of the algebra on which it acts. For example, an *involution semigroup* is a triple $(S, \cdot, *)$ such that (S, \cdot) is a semigroup, while $*$ is an involution on S such that

$$(xy)^* = y^*x^*$$

holds for all $x, y \in S$. Similarly, if $(S, +, \cdot)$ is a semiring (i.e. a structure in which $(S, +)$ is a commutative semigroup, (S, \cdot) is a semigroup, and \cdot distributes over $+$), then $(S, +, \cdot, *)$ is called an *involution semiring*, provided that $*$ is an involution of S satisfying the identities

$$\begin{aligned}(x + y)^* &= x^* + y^*, \\ (xy)^* &= y^*x^*.\end{aligned}$$

Sometimes, semirings may be equipped with a zero 0 , and/or an identity 1 . In such a case we require that the involution satisfies $0^* = 0$ and $1^* = 1$.

The present survey concentrates primarily to universal-algebraic and related structural features of involution semigroups and involution semirings. To be more precise, our aim is to present an overview of results which concern varieties (equational classes) formed by these structures. Of course, this overview is by no means exhaustive: it represents a subjective selection of topics in this rich and extensive

Both authors are supported by Grant No.1227 of the Ministry of Science, Technologies and Development of Republic of Serbia.

theory. For example, involution semigroups include groups, and more generally, inverse semigroups; however, having a number of excellent survey texts on varieties of groups and inverse semigroups in the literature, e.g. [62, 77, 86], the first section of this survey focusses on *other* parts of the theory of involution semigroup varieties. Some basic results on inverse semigroups will be occasionally mentioned, but mostly in the wider context of regular $*$ -semigroups.

On the other hand, the second part is devoted to selected problems in the growing subject of involution semirings and their varieties. Involution semirings are the natural companions of involution semigroups in the study of the interaction between associative phenomena and involution. Yet, they appear to have a theory of their own, with some interesting connections to theoretical computer science (e.g. complete semirings of formal languages and algebras of binary relations). We present here some of those connections. Finally, some universal-algebraic aspects of the theory of involution rings are also included in this paper.

1. VARIETIES OF INVOLUTION SEMIGROUPS

1.1. Some General Properties. As it was defined above, *involution semigroups* (or $*$ -semigroups) are unary semigroups which satisfy $(xy)^* = y^*x^*$ and $(x^*)^* = x$. The motivation for investigation of involution semigroups for their own, came from the most important class of involution algebras, *involution rings*, by omitting the additive structure. The paper of Nordahl and Scheiblich [78] is commonly established as the pioneering paper on involution semigroups from the standpoint of theory of varieties (though there are several earlier papers dealing with similar subjects, e.g. [37, 39, 54, 102]).

Nordahl and Scheiblich define *regular $*$ -semigroups* as involution semigroups satisfying

$$xx^*x = x.$$

In other words, x^* is always an inverse of x , since the above identity implies

$$x^*xx^* = (xx^*x)^* = x^*.$$

Note that regular $*$ -semigroups are distinguished from (but closely related to) regular involution semigroups in the sense of Drazin [32] and Nambooripad and Pastijn [76], see also [11, 12, 18, 93], as involution semigroups in which every element has a Moore-Penrose inverse [83].

Throughout the section, we adopt the following notation. If \mathcal{V} is a semigroup variety, by \mathcal{V}^* we denote the variety consisting of all involution semigroups whose semigroup reducts belong to \mathcal{V} (it is clear that \mathcal{V}^* is axiomatized by the identities of \mathcal{V} and involution axioms). The subvariety of \mathcal{V}^* consisting of all of its regular $*$ -semigroup members is denoted by \mathcal{V}^{reg} . Finally, if S is an involution semigroup and $a \in S$ is such that $a^* = a^2 = a$, then a is called a *projection*.

Now, inverse semigroups (considered as unary semigroups) are clearly included in the class of regular $*$ -semigroups, and moreover, they can be equationally characterized within that class. Hence, they form a variety, which follows from the well-known result of Schein.

Theorem 1.1.1. (Schein, [102]) *The class of all inverse semigroups is defined within the variety of regular $*$ -semigroups by the identity*

$$xx^*x^*x = x^*xx^*.$$

Another approach to characterization of inverse semigroups by using involution semigroups can be found in Easdown and Munn [33].

On the other hand, one can define *orthodox $*$ -semigroups* as regular $*$ -semigroups in which idempotents form a subsemigroup (it is obvious that idempotency is preserved by $*$). It is interesting that orthodox $*$ -semigroups form yet another subvariety of the variety \mathcal{S}^{reg} of regular $*$ -semigroups.

Theorem 1.1.2. (Nordahl and Scheiblich, [78]) *The class of orthodox $*$ -semigroups is defined within the variety of regular $*$ -semigroups by the identity*

$$(xx^*yy^*zz^*)^2 = xx^*yy^*zz^*.$$

It is worth noting that the semigroup structure of regular $*$ -semigroups is well preserved by the involution. For example, some of the results of Nordahl and Scheiblich [78] can be summarized as follows.

Theorem 1.1.3. *In any regular $*$ -semigroup, the involution induces a bijection between \mathcal{R} -classes and \mathcal{L} -classes. Also, it preserves \mathcal{H} -classes. Consequently, each \mathcal{R} -class contains exactly one projection, and \mathcal{D} -classes are square.*

Further, every idempotent-separating congruence of (the semigroup reduct of) a regular $*$ -semigroup is a $*$ -congruence, which means that it preserves $*$ as well. Moreover, we have the following characterization of $*$ -congruences due to Nordahl and Scheiblich.

Theorem 1.1.4. *Let S be a regular $*$ -semigroup and let ϱ be a congruence of its semigroup reduct. Then ϱ is a $*$ -congruence if and only if for any idempotents $e, f \in S$ such that $(e, f) \in \varrho$ we have $(e^*, f^*) \in \varrho$.*

Finally, let us mention that quite recently Polák [91] gave a simple and elegant solution of the word problem for free regular $*$ -semigroups (i.e. of the problem of algorithmic decision of the equational theory of \mathcal{S}^{reg}), by using term rewriting. He proved that the reduction on the set of involution semigroup words (cf. 1.3. below) arising from the identity $xx^*x = x$ is locally confluent, which implies that each word has a unique normal form. Moreover, the corresponding algorithm works in polynomial time.

1.2. Minimal Varieties of Involution Semigroups. A variety of algebras is *minimal* if it has only trivial variety as a proper subvariety. It is not difficult to see that for a variety this is just equivalent to being *equationally complete*, which means that its equational theory is maximal. Equationally complete (minimal) varieties have been extensively studied since the fifties, when the topic was conceived by Kalicki and Scott (see, e.g. [60]). We refer to [109] for a contemporary overview of results on minimal varieties.

Since an involution semigroup variety satisfying $x^* = x$ (we say that it is equipped with a *trivial* involution) must be commutative, it follows from [60]

that minimal varieties of this kind are exhausted by \mathcal{SL}^{id} , the variety of semilattices, \mathcal{N}^{id} , the variety of null semigroups, and for each prime p , the variety $\mathcal{A}_p^{\text{id}}$ of Abelian groups of exponent p . Now let \mathcal{RB}^* denote the variety of *rectangular bands* with involution, that is, involution semigroups satisfying

$$xyz = xz, \quad x^2 = x,$$

where the latter two identities can be replaced by $xyx = x$. Finally, let \mathcal{SL}^0 denote the variety of involution semilattices defined by

$$xx^*y = xx^*.$$

In other words, xx^* is always the zero in any member of \mathcal{SL}^0 .

Theorem 1.2.1. (Fajtlowicz, [37]) *An involution semigroup variety is minimal if and only if it is one of the following:*

- (1) $\mathcal{SL}^{\text{id}}, \mathcal{SL}^0, \mathcal{RB}^*, \mathcal{N}^{\text{id}}$,
- (2) $\mathcal{A}_p^{\text{id}}, \mathcal{A}_p$, for some prime p ,

where \mathcal{A}_p denotes the variety of Abelian groups of exponent p with group inverse as the involution.

We refer to varieties from (1) and (2) of the above theorem as the *nongroup* and the *group atoms*, respectively.

Fajtlowicz's plan of the proof was the following. First he proved, in a straightforward way, that \mathcal{SL}^0 and \mathcal{RB}^* are equationally complete (the other varieties involved in the above theorem were previously known to be minimal). Then he proved that if an involution semigroup S belongs to a minimal variety and has at least two involution fixed points (also called *Hermitian elements*), then S must be a rectangular band (with involution), and consequently, the considered variety is \mathcal{RB}^* . On the other hand, if S has exactly one Hermitian element (and it must have at least one, for aa^* is such for any $a \in S$), then it is either the zero, or the identity of S . From this he obtained that S is either a group, or contains an involution sub-semigroup which can be homomorphically mapped onto the following involution semilattice:

$$\begin{array}{c|ccc} & 0 & 1 & 1^* \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1^* & 0 & 0 & 1^* \end{array}$$

This involution semilattice we denote by Σ_3 . It is not too difficult to see that Σ_3 generates \mathcal{SL}^0 , whence the required result follows.

The paper of Fajtlowicz inspired further investigations of the lattice of varieties of involution semigroups. For example, sublattices of this lattice generated by nongroup and group atoms, respectively, were explored in [24]. These and other related results will be reviewed in the following subsections. But first we need to take a look at some previous 'classical' results concerning the fragments of the lattice of involution semigroup varieties.

1.3. Varieties of Regular $*$ -Bands. Starting from the result of Theorem 1.1.2, Adair [1] initiated a systematic study of varieties of orthodox $*$ -semigroups. In particular, she managed in [2] to fully describe the lattice of all varieties of regular $*$ -bands (idempotent $*$ -semigroups). Adair's proof is considerably involved, and it is based mainly on the methods and techniques which were used by Biryukov [7], Fennemore [38] and Gerhard [40] in describing all varieties of bands. The strategy is to find a complete list of all possible nonequivalent identities of regular $*$ -bands, i.e. to classify all involution semigroup identities according to their equivalence in the presence of the axioms of regular $*$ -bands.

Let us introduce some notation. First of all, note that due to the involution axioms, any term in the signature $\{\cdot, *\}$ (of involution semigroups) over the set of variables X can be equivalently expressed so that the star acts to the variables only. Terms of the latter form can be considered as words in the extended alphabet $X \cup X^*$, where $X^* = \{x^* : x \in X\}$. In the sequel, we shall assume this is always the case, so that we deal with *involution semigroup words*.

Now, if w is such a word, we define $c(w)$, the *content* of w , as the set of all variables from X which occur in w . However, if w is considered just as a word in the extended alphabet $X \cup X^*$ (the symbols from X^* being irreducible and independent from those from X), we define $c^*(w)$, the *$*$ -content* of w , as the set of all members of $X \cup X^*$ having an occurrence in w (for example, $c^*(x^*yz y^*x^*) = \{x^*, y, y^*, z\}$, while $c(x^*yz y^*x^*) = \{x, y, z\}$). An identity $u = v$ is called *homotypical* if $c(u) = c(v)$, otherwise it is called *heterotypical*.

We define two sequences of words, assuming that $X = \{x_1, x_2, x_3, \dots\}$. Let $U_1 = x_1$, $V_1 = x_1^*x_1$, $U_2 = x_1x_2$ and $V_2 = x_1x_2^*x_1x_2$. If $n \geq 3$ and n is even, we let

$$\begin{aligned} U_n &= U_{n-2}^R x_{\frac{n+2}{2}}, \\ V_n &= V_{n-2}^R x_{\frac{n+2}{2}} U_{n-2}^R x_{\frac{n+2}{2}}, \end{aligned}$$

where w^R denotes the *reverse* (*mirror-image*) of the word w . On the other hand, if $n \geq 3$ and n is odd, we put

$$\begin{aligned} U_n &= U_{n-2}^R x_{\frac{n+1}{2}}, \\ V_n &= V_{n-2}^R x_{\frac{n+1}{2}} U_{n-2}^R x_{\frac{n+1}{2}}. \end{aligned}$$

The key result of Adair [2] now reads as follows.

Proposition 1.3.1. *In the class of regular $*$ -bands, every homotypical involution semigroup identity is either equivalent to $U_n = V_n$ for some $n \geq 1$, or it follows from the axioms of regular $*$ -bands.*

It is a folklore (see e.g. [87]) that any regular $*$ -band which satisfies a heterotypical identity must be a rectangular band. Bearing in mind the fact that \mathcal{RB}^* is minimal, this means that within \mathcal{B}^{reg} , the variety of regular $*$ -bands, every heterotypical identity defines either \mathcal{RB}^* , or the trivial variety. This, along with the above proposition, and some further work on establishing the implications between the identities $U_n = V_n$, yields the main achievement of [2].

Theorem 1.3.2. (Adair, [2]) *The lattice of all subvarieties of \mathcal{B}^{reg} has the following structure.*

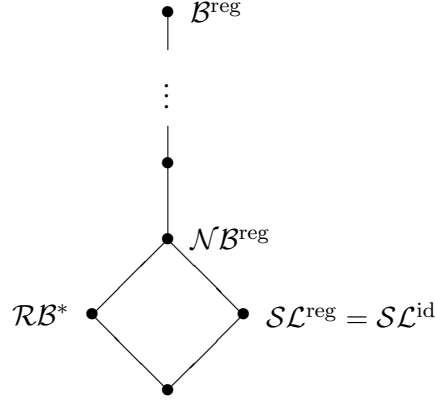


Figure 1. The lattice of all varieties of regular $*$ -bands

Later on, Yamada gave a construction of finitely generated free regular $*$ -bands and determined the free spectrum of \mathcal{B}^{reg} (recall that the free spectrum of a variety is the sequence of the cardinalities of finitely generated free algebras of that variety). Of course, as in the case of bands, $f_n(\mathcal{B}^{\text{reg}})$, the size of the n -generated free regular $*$ -band, is finite.

Theorem 1.3.3. (Yamada, [116])

$$f_n(\mathcal{B}^{\text{reg}}) = \sum_{k=1}^n \binom{n}{k} 4^{2^k-1} \prod_{i=0}^{k-1} (k-i)^{2^{i+1}}.$$

The structure of free regular $*$ -bands was further studied by Gerhard and Petrich in [44]. Related results can be found also in [43].

Finally, we mention a couple of categorical features of regular $*$ -band varieties. If \mathcal{C} is a class of algebras, we say that $I \in \mathcal{C}$ is *injective* in \mathcal{C} if for any $A, B \in \mathcal{C}$, injective homomorphism $f : A \rightarrow B$ and homomorphism $g : A \rightarrow I$, there is a homomorphism $h : B \rightarrow I$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & \swarrow h & \\ I & & \end{array}$$

Dually, an algebra $P \in \mathcal{C}$ is *projective* in \mathcal{C} if for any surjective homomorphism $f : B \rightarrow A$ and any homomorphism $g : P \rightarrow A$ there is a homomorphism $h : P \rightarrow B$ such that

$$\begin{array}{ccc} B & \xrightarrow{f} & A \\ h \uparrow & \swarrow g & \\ P & & \end{array}$$

commutes.

The description of injective and projective semilattices is known from [55], and so it applies to $\mathcal{SL}^{\text{reg}}$. Scheiblich [101] has shown that every rectangular band with involution is injective and projective in \mathcal{RB}^* . Moreover, if N is a regular $*$ -normal band, then it is injective in $\mathcal{NB}^{\text{reg}}$ if and only if the semigroup reduct of N is injective in \mathcal{NB} , and so the characterizations of Gerhard [42] and Schein [107] apply. For the varieties situated above $\mathcal{NB}^{\text{reg}}$ in our Figure 1, the situation is analogous to the case of bands. Namely, we have the following

Theorem 1.3.4. (Scheiblich, [101]) *Let \mathcal{V} be a variety of regular $*$ -bands such that $\mathcal{V} \not\subseteq \mathcal{NB}^{\text{reg}}$. Then I is injective in \mathcal{V} if and only if I is an injective semilattice.*

The information on projective regular $*$ -bands is less complete than the above results on injectives. However, Scheiblich provided a characterization of projective regular $*$ -normal bands.

Theorem 1.3.5. *A regular $*$ -normal band N is projective in $\mathcal{NB}^{\text{reg}}$ if and only if*

- (1) N/\mathcal{D} is a projective semilattice,
- (2) each \mathcal{D} -class is finite,
- (3) for each projection $x \in N$, the set U_x of identities of x has an identity e_x , which is also a projection,
- (4) for each $a, b \in N$, if D_{ab} (the \mathcal{D} -class of ab) contains a projection x , then either $D_a \cap U_x$, or $D_b \cap U_x$ contains a projection.

To the best of our knowledge, a reasonable characterization of projectives in varieties of regular $*$ -bands is still an open problem. Due to Scheiblich, it is known that if a regular $*$ -band P is projective in a subvariety of \mathcal{B}^{reg} , then the semigroup reduct of P is projective in the corresponding band variety. It is not known, however, whether the converse of this result is true.

1.4. Varieties of Completely Regular $*$ -Semigroups. In his fundamental study [87], Petrich considered those regular $*$ -semigroups whose underlying semigroups are completely regular (we shall refer to them as *completely regular $*$ -semigroups*). Of course, completely regular semigroups can be also considered as unary semigroups, with x^{-1} being the inverse of x in the maximal subgroup to which x belongs. By accepting such an approach, completely regular semigroups form a variety of unary semigroups, which are totally different from regular $*$ -semigroups, since $^{-1}$ is not a semigroup antiautomorphism.

Nevertheless, Petrich discovered that for any variety \mathcal{V} of completely regular semigroups, the class of all completely regular $*$ -semigroups whose semigroup reducts (together with the induced inversion operation) belong to \mathcal{V} is a subvariety of \mathcal{S}^{reg} . This follows from

Lemma 1.4.1. *For any completely regular $*$ -semigroup S and $x \in S$ we have*

$$x^{-1} = xx^*x^*x^*x.$$

In other words, the inversion is in completely regular $*$ -semigroups always expressible by $*$. Therefore, it is possible to consider varieties of completely regular $*$ -semigroups.

Theorem 1.4.2. (Petrich, [87]) *The variety of completely regular $*$ -semigroups is determined within \mathcal{S}^{reg} by the identity*

$$xx^* = xxx^*x^*,$$

while the variety of completely simple regular $$ -semigroups is defined by any of the identities*

$$\begin{aligned} xy^*x^* &= xx^*, \\ yxx^*y^*x^* &= xx^*. \end{aligned}$$

Further, Petrich gives equational descriptions for a number of varieties of completely regular $*$ -semigroups, with various restrictions on the involved groups, or structural relationships of subgroups. Note that an involution on a group does not necessarily coincide with the group inverse; in general, one can prove that any group involution $*$ can be expressed as $x^* = \varphi(x^{-1})$, where φ is some automorphism of the considered group. Still, if we work with regular involutions only, the inverse is easily seen to be the only such involution on a group.

By the following theorem we summarize some of results from [87].

Theorem 1.4.3. *The list below gives equational axiomatizations for several subvarieties of the variety of completely regular $*$ -semigroups:*

- *groups: $xx^* = yy^*$,*
- *Abelian groups: $x = yxy^*$,*
- *Boolean groups: $x = xy^2$,*
- *rectangular groups: $xx^* = xy^*y^*yx^*$,*
- *rectangular Abelian groups: $x = yxy^*y^*x$,*
- *rectangular Boolean groups: $x = xy^2x^2$,*
- *completely simple $*$ -semigroups having only Abelian subgroups: $xx^* = x^2yxx^*x^*y^*x^*$,*
- *completely simple $*$ -semigroups having only Boolean subgroups: $x = xyxyx$,*
- *semilattices of groups: $xx^* = x^*x$,*
- *semilattices of Abelian groups: $xy = yx$,*
- *semilattices of Boolean groups: $x = x^*$,*
- *normal bands: $xyx = xyy^*x$,*
- *normal bands of groups: $xyy^*x^* = xy^*yx^*$,*
- *normal bands of Abelian groups: $x^2x^*x^*xyxzx = xzxyx$,*
- *normal bands of Boolean groups: $x^3yxzx = xzxyx$,*
- *orthodox normal bands of groups: $xyy^*x^* = x^2x^*y^*yx^*$,*
- *orthodox normal bands of Abelian groups: $xyzx = xzyx$,*
- *orthodox normal bands of Boolean groups: $xyx = xy^*x$.*

All these varieties form the ‘skeleton’ of the lower layers of the lattice of completely regular $*$ -semigroup varieties. Of course, by Theorem 1.2.1, the atoms in this lattice are \mathcal{SL}^{id} , \mathcal{RB}^* , and group varieties \mathcal{A}_p (for all prime numbers p).

1.5. Varieties Generated by 0-Direct Unions. Can every semigroup be turned into an involution semigroup? Fortunately, the answer is no. Namely, there are semigroups which do not have antiautomorphisms at all, and the example of left (right) zero bands is probably the first that comes to mind. However, it is true that every semigroup can be embedded into (the semigroup reduct of) an involution semigroup.

The most convenient device to show this is the following. Start with a given semigroup S , and let \tilde{S} denote an anti-isomorphic copy of S , the *dual* of S , $\tilde{S} = \{\tilde{a} : a \in S\}$. Let 0 be a new symbol, and define an associative product on $S \cup \tilde{S} \cup \{0\}$ which works within S and \tilde{S} just as in the original semigroup, all the other products being equal to 0 . The involution $*$ is defined such that we have $0^* = 0$ and for all $a \in S$, $a^* = \tilde{a}$ and $\tilde{a}^* = a$. It is a routine matter to prove that $*$ is an antiautomorphism. The involution semigroup just obtained we denote by $I_0^*(S)$ and it is clear that S embeds into the semigroup reduct of $I_0^*(S)$. In fact, $I_0^*(S)$ is a special case of the construction known in semigroup theory as the *0-direct union* (or *orthogonal sum*) of semigroups. Here we are concerned with a 0-direct union of a semigroup with its dual, enriched by an involution. The involution semilattice Σ_3 , encountered earlier, is just $I_0^*(T)$, where T denotes the trivial semigroup.

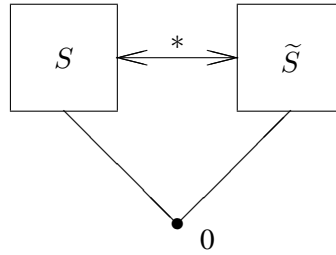


Figure 2. The construction of $I_0^*(S)$

If Θ denotes a set of semigroup identities, by Θ_{hom} we denote the set of all homotypical members of Θ (in case that Θ is the equational theory of a semigroup S , the Θ_{hom} is the equational theory of $S \times \Sigma_2$, where Σ_2 denotes the two-element semilattice). Now we are able to give an explicit axiomatization of the equational theory of an involution semigroup of the form $I_0^*(S)$, provided that the equational theory of S is known.

Theorem 1.5.1. (Crvenković, Dolinka and Vinčić, [15]) *Let Θ be the equational theory of S . Then the equational theory of $I_0^*(S)$ is defined by $\Theta_{hom} \cap \Theta_{hom}^R$ and the following two identities:*

$$\begin{aligned} xx^*y &= xx^*, \\ xyx^* &= xx^*. \end{aligned}$$

In other words, if we assume that the semigroup reduct of $I_0^*(S)$ generates the variety \mathcal{V} , and if we denote by \mathcal{V}^0 the variety generated by $I_0^*(S)$, then \mathcal{V}^0 is defined within \mathcal{V}^* by the above two identities.

From the above theorem and some general results on finite bases of identities for varieties of involution algebras, obtained by Crvenković, Dolinka and Ćsik in [13], one can derive the second main result of [15].

Theorem 1.5.2. *Let S be a semigroup whose identities are closed for reversal and let S^0 denote the semigroup obtained by adjoining a zero to S (even if S already has one). Then S^0 is finitely based if and only if $I_0^*(S)$ is such. If, in addition, S satisfies only homotypical identities, then S is finitely based if and only if $I_0^*(S)$ is finitely based.*

Recall that the 5-element Brandt semigroup B_2 is the one generated by a, b , subject to the following relations:

$$a^2 = b^2 = 0, aba = a, bab = b.$$

By an adjunction of an identity element, we obtain the six-element monoid B_2^1 . It is easy to check that B_2^1 is isomorphic to the semigroup of matrices

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

with the usual matrix multiplication. By a result of Perkins [84], B_2^1 is not finitely based. Moreover, it is *inherently* nonfinitely based (Sapir [98]), which means that there is no finitely based locally finite variety containing B_2^1 . However, it is interesting to note that B_2 is finitely based (see Tiščenko [111] and Trahtman [112]).

Since the equational theory of B_2^1 is homotypical (because B_2^1 contains a zero) and closed for reversal (since B_2^1 has an antiautomorphism, e.g. the matrix transposition), the above considerations give rise to an example of a nonfinitely based involution semigroup.

Corollary 1.5.3. *The 13-element involution semigroup $I_0^*(B_2^1)$ is not finitely based.*

We recall that Sapir [99] proved that no inverse semigroup (considered as an involution semigroup) is inherently nonfinitely based. It would be interesting to see if the above nonfinitely based involution semigroup $I_0^*(B_2^1)$ (which is clearly not inverse) is actually inherently nonfinitely based.

1.6. Lower Floors of the Lattice of Involution Semigroup Varieties. If one is concerned with the investigation of the lattice of involution semigroup varieties, it is reasonable to start with lattices generated by (some of the) atoms, as they are important landmarks of the bottom of the considered ‘big’ lattice. As we have seen, the minimal varieties of involution semigroups, the full list of which was obtained by Fajtlowicz, naturally split into two families: one of them contains four nongroup atoms \mathcal{SL}^{id} , \mathcal{SL}^0 , \mathcal{RB}^* and \mathcal{N}^{id} , while the other consists of two countable sequences of involution group varieties $\mathcal{A}_p^{\text{id}}$ and \mathcal{A}_p (in the latter, the involution is just the group inverse), where p is a prime number. Therefore, one can consider lattices of varieties generated by these two families of atoms, respectively. These lattices were determined by the second author of this survey in [24].

The way to handle the first family is more or less straightforward: it suffices to start with the atoms, and successively calculate joins and meets until no new variety is obtained. The result of such calculations is as follows.

Theorem 1.6.1. (Dolinka, [24]) *The lattice of varieties of involution semigroups generated by the four nongroup atoms is the one given in the following figure.*

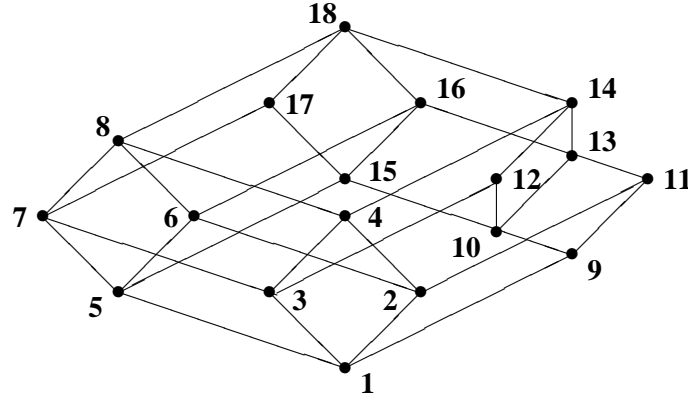


Figure 3. The lattice of involution semigroup varieties generated by nongroup atoms
Here, the numbers 1–18 on the above diagram represent the following varieties:

Label	Variety (given by the list of defining identities)
1	$x = y$
2	$x^2 = x, xy = yx, x^* = x$
3	$x^2 = x, xy = yx, xx^*y = xa^*$
4	$x^2 = x, xy = yx, xx^*y = xa^*y^*$
5	$x^2 = x, xyz = xz$
6	$x^2 = x, xyzt = xzyt, xyz = xy^*z$
7	$x^2 = x, xyzt = xzyt, xyy^*ut = xzz^*vt$
8	$x^2 = x, xyzt = xzyt, xyy^*zt = xyzz^*t$
9	$xy = zt, x^* = x$
10	$xy = zt$
11	$x^2y = xy = yx, x^* = x$
12	$x^2y = xy = yx, xa^*y = xa^*$
13	$x^2y = xy = yx, xy = xy^*$
14	$x^2y = xy = yx, xa^*y = xa^*y^*$
15	$xyz = xz$
16	$x^2y = xy^2 = xy, xyzt = xzyt, xyz = xy^*z$
17	$x^2y = xy^2 = xy, xyzt = xzyt, xyy^*ut = xzz^*vt$
18	$x^2y = xy^2 = xy, xyzt = xzyt, xyy^*zt = xyzz^*t$

The determination of the lattice of varieties generated by involution group atoms is, however, much more involved, and it requires quite a portion of elementary number theory (in particular, Chinese Remainder Theorem, a number of ‘nested’ Euclidean algorithms and a lot of g.c.d.-l.c.c. calculus). The first step is to define \mathcal{A}_n^* to be the variety of all Abelian groups of exponent n with an arbitrary involution, as well as $\mathcal{A}_n^{r,s}$, the subvariety of \mathcal{A}_n^* determined by $(x^*)^r = x^s$. In view of this new notation we have $\mathcal{A}_n^{\text{id}} = \mathcal{A}_n^{1,1}$ and $\mathcal{A}_n = \mathcal{A}_n^{1,n-1}$.

The next step is fairly simple: it consists of the calculation of the meets of the form $\mathcal{A}_m \wedge \mathcal{A}_n^{\text{id}}$.

Proposition 1.6.2.

$$\mathcal{A}_m \wedge \mathcal{A}_n^{\text{id}} = \begin{cases} \mathcal{A}_2, & \text{if } m, n \text{ are even,} \\ \mathcal{T}^*, & \text{otherwise.} \end{cases}$$

Here, of course, \mathcal{T}^* denotes the trivial involution semigroup variety.

For the joins, the situation is summarized by

Proposition 1.6.3. *Let m, n be positive integers, $d = (m, n)$, $m_1 = \frac{m}{d}$ and $n_1 = \frac{n}{d}$. Then we have:*

- (a) *if at least one of m, n is odd, then $\mathcal{A}_m \vee \mathcal{A}_n^{\text{id}} = \mathcal{A}_{[m,n]}^{d, 2m\alpha - d}$,*
- (b) *if m, n are both even, then $\mathcal{A}_m \vee \mathcal{A}_n^{\text{id}} = \mathcal{A}_{[m,n]}^{\frac{d}{2}, m\alpha - \frac{d}{2}}$,*

where in both cases α is a positive integer such that $m_1\alpha \equiv 1 \pmod{n_1}$. In particular, for all $k \geq 1$ we have $\mathcal{A}_{2k-1} \vee \mathcal{A}_{2k-1}^{\text{id}} = \mathcal{A}_{2k-1}^*$ and $\mathcal{A}_{2k} \vee \mathcal{A}_{2k}^{\text{id}} = \mathcal{A}_{2k}^{k,k}$.

Now, it turns out that the set of varieties of the form $\mathcal{A}_m \vee \mathcal{A}_n^{\text{id}}$ (where $\mathcal{A}_1^{\text{id}} = \mathcal{A}_1 = \mathcal{T}^*$) is closed for the lattice operations (i.e. for meets). This immediately yields the required lattice generated by involution group atoms, but on the other hand, this is the part which is the most difficult to prove.

Proposition 1.6.4. $(\mathcal{A}_k \vee \mathcal{A}_\ell^{\text{id}}) \wedge (\mathcal{A}_m \vee \mathcal{A}_n^{\text{id}}) = \mathcal{A}_{(k,m)} \vee \mathcal{A}_{(\ell,n)}^{\text{id}}$.

All these results yield the following

Theorem 1.6.5. *The sublattice of the lattice of involution semigroup varieties generated by its involution group atoms \mathcal{A}_p and $\mathcal{A}_p^{\text{id}}$ for all primes p consists precisely of the following varieties: \mathcal{T}^* , \mathcal{A}_m , $\mathcal{A}_n^{\text{id}}$ and $\mathcal{A}_m \vee \mathcal{A}_n^{\text{id}}$, where $m, n \geq 2$ are arbitrary square-free numbers. Moreover, it is isomorphic to the lattice of finite subsets of a countably infinite set.*

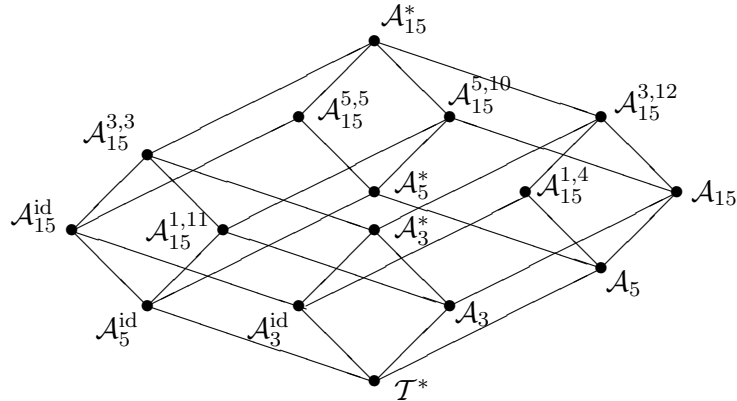


Figure 4. The lattice of involution semigroup varieties generated by \mathcal{A}_3 , $\mathcal{A}_3^{\text{id}}$, \mathcal{A}_5 and $\mathcal{A}_5^{\text{id}}$

1.7. Some Varieties of Involution Bands. After the results described in the previous subsection were obtained, the focus of the attention turned to some special parts of the lattice of involution semigroup varieties. There are several motives for this, and we emphasize two of them. First, the determination of the lattice of all varieties of regular $*$ -bands, achieved by Adair, gives boost to investigations of involution band varieties in general. Secondly, it remains, for the moment, an open question whether Figure 3 represents the lattice of all subvarieties of \mathcal{J}_4 , the join of four nongroup atoms, or there are other subvarieties \mathcal{J}_4 not present in that figure. This question will be answered in the present subsection.

We say that a semigroup variety \mathcal{U} is *central* if its equational theory is closed for reversal, that is, if for each identity $u = v$ holding in \mathcal{U} the identity $u^R = v^R$ is also true in \mathcal{U} . The *type* of an involution semigroup variety \mathcal{V} we define to be the semigroup variety \mathcal{U} determined by the semigroup ($*$ -free) identities of \mathcal{V} . The following result strongly highlights the importance of the varieties of the form \mathcal{V}^0 and \mathcal{V}^{reg} .

Lemma 1.7.1. (Dolinka, [19]) *Let \mathcal{V} be a central homotypical variety of bands. Then \mathcal{V}^0 is contained in every involution band variety of type \mathcal{V} except \mathcal{V}^{reg} .*

This lemma yields the description of all subvarieties of \mathcal{B}^0 , where \mathcal{B} denotes the variety of all bands.

Corollary 1.7.2. *All subvarieties of \mathcal{B}^0 are exhausted by the varieties of the form \mathcal{V}^0 , where \mathcal{V} is a homotypical central variety of bands.*

In the course of studying involution band varieties, the first step is certainly to determine all varieties of involution semilattices. Aside from the trivial variety, we already met two such varieties: these are $\mathcal{S}\mathcal{L}^{\text{id}} = \mathcal{S}\mathcal{L}^{\text{reg}}$ and $\mathcal{S}\mathcal{L}^0$. Now we define $\mathcal{S}\mathcal{L}' = \mathcal{S}\mathcal{L}^{\text{id}} \vee \mathcal{S}\mathcal{L}^0$ (it is not too difficult to see that $\mathcal{S}\mathcal{L}'$ is characterized by the identity $xx^*y = xx^*y^*$), while $\mathcal{S}\mathcal{L}^*$ denotes the variety of all involution semilattices.

Theorem 1.7.3. (Dolinka, [19]) *Every variety of involution semilattices is equal to one of the following: \mathcal{T}^* , $\mathcal{S}\mathcal{L}^{\text{id}}$, $\mathcal{S}\mathcal{L}^0$, $\mathcal{S}\mathcal{L}'$, $\mathcal{S}\mathcal{L}^*$. Thus, the lattice of subvarieties of $\mathcal{S}\mathcal{L}^*$ is the one given in Figure 5.*

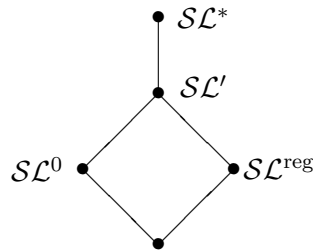


Figure 5. All varieties of involution semilattices

Since $\mathcal{R}\mathcal{B}^*$ is a minimal variety, it follows that it is the only involution band variety of type $\mathcal{R}\mathcal{B}$. A little later we shall give a full account on all varieties of normal bands with involution.

A nice feature of bands with involution is that they exhibit similar structural properties as ordinary bands. Namely, it is well-known that every band is a semilattice of rectangular bands. Analogously, every involution band can be represented as an involution semilattice of rectangular bands, which follows from the fact that the greatest semilattice congruence of an involution band B is actually its $*$ -congruence. Therefore, in a semilattice decomposition of B , the involution $*$ maps the classes in a bijective manner, thus inducing an involution on the structure semilattice Σ of B . Hence, one may wish to characterize involution bands whose structure involution semilattices belong to a specific variety from the above theorem. In the sequel, we list these characterizations (both equational and structural ones). All of them are from [19].

Proposition 1.7.4. *The following conditions are equivalent for an arbitrary involution band B :*

- (1) *the structure involution semilattice of B belongs to \mathcal{SL}^{id} ,*
- (2) *B is $*$ -regular (i.e. it satisfies the identity $x = xx^*x$),*
- (3) *B generated by its projections.*

Proposition 1.7.5. *The following conditions are equivalent for an arbitrary involution band B :*

- (1) *the structure involution semilattice of B belongs to \mathcal{SL}^0 ,*
- (2) *B satisfies the identity $xx^* = xx^*yy^*xx^*$,*
- (3) *the involution subsemigroup of B generated by projections is a rectangular band, which is a $*$ -ideal of B .*

Proposition 1.7.6. *The following conditions are equivalent for an arbitrary involution band B :*

- (1) *the structure involution semilattice of B belongs to \mathcal{SL}' ,*
- (2) *B satisfies the identity $yx x^* y = y x x^* y^* x x^* y$,*
- (3) *B satisfies the identity $xx^*y = xx^*yxx^*y^*xx^*y$,*
- (4) *the involution subsemigroup of B generated by the projections is a $*$ -ideal of B .*

Our next goal here is to determine the lattice of subvarieties of the join $\mathcal{B}^{\text{reg}} \vee \mathcal{B}^0$, the importance of Adair's variety \mathcal{B}^{reg} and of \mathcal{B}^0 already being underlined. By that, we describe the lowest layer in the lattice of involution band varieties and give some hints just how complex this lattice might be. The determination of varieties of involution bands of any particular type is therefore 'based' on the lattice below. But first we give an important result of a structural nature.

Theorem 1.7.7. (Dolinka, [19]) *Let \mathcal{U} and \mathcal{V} be central varieties of bands such that $\mathcal{SL} \subseteq \mathcal{V}$ and $\mathcal{U} \subseteq \mathcal{V}$. The following conditions are equivalent for an arbitrary involution band B :*

- (1) *$B \in \mathcal{U}^{\text{reg}} \vee \mathcal{V}^0$,*
- (2) *B is a retractive ideal extension of a member of \mathcal{U}^{reg} by a member of \mathcal{V}^0 ,*
- (3) *B is a subdirect product of a member of \mathcal{U}^{reg} and a member of \mathcal{V}^0 .*

This theorem admits us to go straight to our aim.

Theorem 1.7.8. (Dolinka, [19]) *The variety $\mathcal{B}^{\text{reg}} \vee \mathcal{B}^0$ is determined within \mathcal{B}^* by the identities $xx^*xyy^*y = xy(xy)^*xy = xyy^*y = xx^*xy$, and the lattice of its subvarieties is given in Figure 6.*

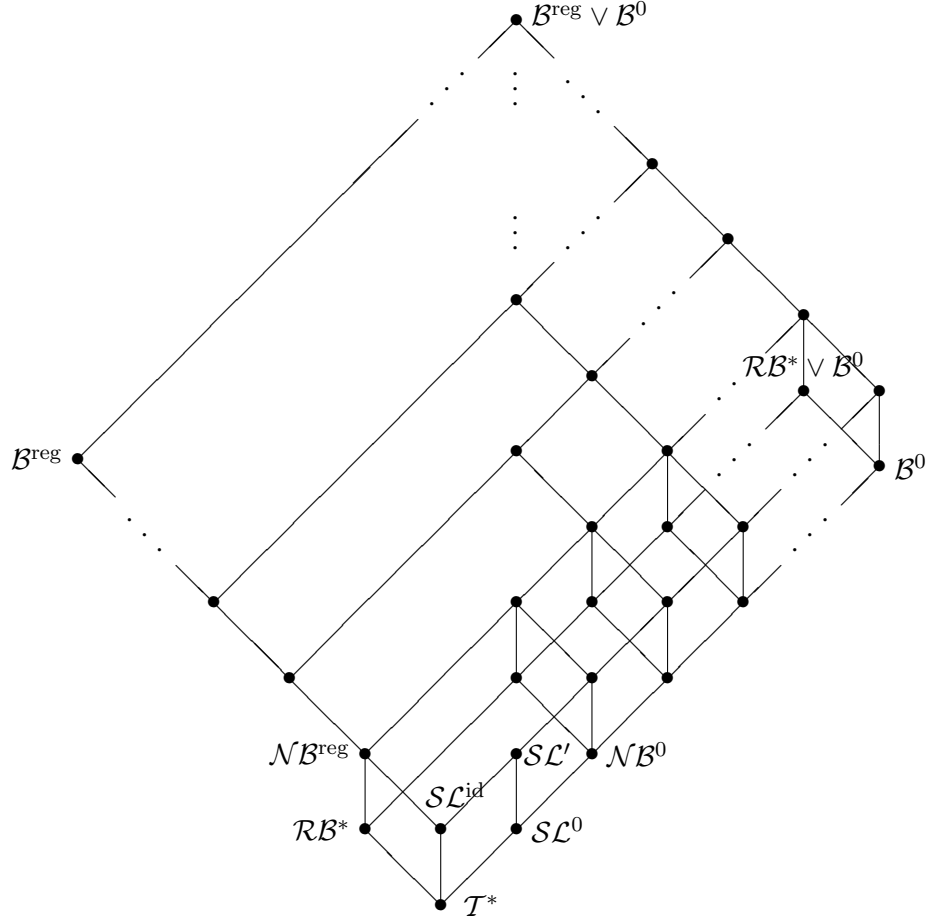


Figure 6. All subvarieties of $\mathcal{B}^{\text{reg}} \vee \mathcal{B}^0$

From the above theorem it is obvious that the lattice of all subvarieties of \mathcal{B}^* is considerably more complicated than the lattice of all band varieties (determined independently by Biryukov [7], Fennemore [38] and Gerhard [40]). For example, the lattice of all subvarieties of \mathcal{B}^* has no finite width, unlike the lattice of all subvarieties of \mathcal{B} . Also, the former lattice is not even modular (which follows from the above theorem), while the latter is known to be distributive.

In [20], the second author of the present survey described the lattice of all subvarieties of \mathcal{NB}^* , the variety of normal involution bands. This was done by classifying all possible involution semigroup identities within the class \mathcal{NB}^* .

Theorem 1.7.9. (Dolinka, [20]) *For each involution semigroup identity $p = q$, one of the following conditions is true in normal bands with involution:*

- (1) $p = q$ is trivial (in the sense that it follows from the defining identities of normal bands with involution),

- (2) $p = q \Rightarrow xy = yx$,
(3) $p = q \Rightarrow xyy^* = xx^*yy^*$,
(4) $p = q \Leftrightarrow xyy^* = yy^*x$,
(5) $p = q \Leftrightarrow xx^*yy^* = yy^*xx^*$.

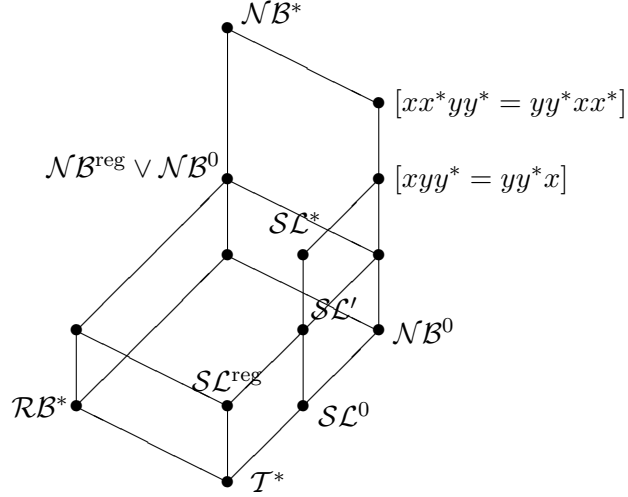


Figure 7. All subvarieties of \mathcal{NB}^*

So, no identity of type (1) defines a proper subvariety of \mathcal{NB}^* , and hence, such identities are of no importance. If a variety satisfies an identity of type (2), it must be a subvariety of \mathcal{SL}^* , when Theorem 1.7.3 applies. On the other hand, from the results of [24] it follows that the identity $xyy^* = xx^*yy^*$ defines the variety $\mathcal{NB}^* \vee \mathcal{NB}^0$ within \mathcal{NB}^* ; thus, if the equational theory of the considered variety contains an identity of type (3), it is a subvariety of $\mathcal{NB}^* \vee \mathcal{NB}^0$, which is a case taken care of by Theorem 1.7.8. Hence, outside \mathcal{SL}^* and $\mathcal{NB}^* \vee \mathcal{NB}^0$, there are at most two proper subvarieties of \mathcal{NB}^* : those defined by $xyy^* = yy^*x$ and $xx^*yy^* = yy^*xx^*$, respectively. It is effectively shown in [20] that these two varieties are different, and so we obtain

Theorem 1.7.10. *The lattice of all varieties of normal bands with involution has the inclusion diagram given in Figure 7.*

Finally, we are going to determine all subvarieties of \mathcal{J}_4 , thereby answering a question from the beginning of this subsection. To do that, we must employ some more notation and define further notions. The material presented below is published for the first time.

An (involution) semigroup S with zero 0 is said to be *null* (or *constant*) if for all $a, b \in S$ we have $ab = 0$. The variety \mathcal{N}^{id} of all null semigroups with trivial involution is a minimal one, i.e. it is on the Fajtlowicz's list. It is easy to prove that it is generated by N_2 , the two element null involution semigroup with a trivial involution. Further, let \mathcal{N}^* denote the variety of all null involution semigroups, while N_3 is the three-element null involution semigroup in which the involution fixes one of its elements and permutes the other two.

The notion of an *inflation* is familiar in semigroup theory for a long time. Namely, a semigroup V is an inflation of its subsemigroup S if there is a homomorphism $\varphi : V \rightarrow S$ such that $\varphi|_S$ is the identity mapping on S and for all $v_1, v_2 \in V$ we have

$$v_1 v_2 = \varphi(v_1) \varphi(v_2).$$

In particular, this means that every product of elements from V lies in S . An inflation of a semigroup S is just a retractive ideal extension of S by the null semigroup $Q \cong V/S$ (see Petrich [85]).

The function φ is often referred to as the *inflation function*.

Now we say that an *involution* semigroup V is a **-inflation* of its involution subsemigroup S if the semigroup reduct of V is an inflation of the semigroup reduct of S , and the corresponding inflation function φ agrees with the star: $\varphi(a^*) = \varphi(a)^*$ for all $a \in S$. Just as in the implication (iii) \Rightarrow (iv) of Theorem 1 from Pastijn [79], it is not difficult to prove

Lemma 1.7.11. *Any *-inflation of an involution semigroup S is a subdirect product of S and a null involution semigroup N .*

Now we describe the structure of involution semigroups belonging to the join $\mathcal{V} \vee \mathcal{N}^*$, where \mathcal{V} is an arbitrary involution semigroup variety.

Lemma 1.7.12. *Let \mathcal{V} be an involution semigroup variety. Then $\mathcal{V} \vee \mathcal{N}^*$ consists precisely of all *-inflations of members of \mathcal{V} .*

Proof. Clearly, both \mathcal{V} and \mathcal{N}^* are contained in the class of all *-inflations of members of \mathcal{V} . On the other hand, by the previous lemma, all involution semigroups from the latter class are contained in $\mathcal{V} \vee \mathcal{N}^*$. Therefore, the proposition will be proved as soon as we show that *-inflations of members of \mathcal{V} constitute a variety.

First of all, for each $i \in I$ (where I is an index set) let V_i be a *-inflation of S_i , with φ_i being the corresponding *-inflation function. Then the *-free reduct of $\prod_{i \in I} V_i$ is an inflation of $\prod_{i \in I} S_i$, the inflation function φ being the target tupling of φ_i 's, that is, $\varphi(\langle v_i : i \in I \rangle) = \langle \varphi_i(v_i) : i \in I \rangle$. But

$$\begin{aligned} \varphi(\langle v_i : i \in I \rangle^*) &= \varphi(\langle v_i^* : i \in I \rangle) = \langle \varphi_i(v_i^*) : i \in I \rangle = \\ &= \langle \varphi_i(v_i)^* : i \in I \rangle = \langle \varphi_i(v_i) : i \in I \rangle^*, \end{aligned}$$

thus *-inflations are closed for direct products.

Further, let V be a *-inflation of S (with φ as the *-inflation function), and let T be an involution subsemigroup of V . Then $T \cap S$ is an involution subsemigroup of S (it is easy to see that it cannot be empty), and, moreover, the *-free reduct of T is an inflation of the *-free reduct of $T \cap S$ respect to $\varphi|_T$. Yet, φ agrees with *, and so does $\varphi|_T$. So, T is a *-inflation of $T \cap S \in \mathcal{V}$.

Finally, with the same setting as above, let P be a homomorphic image of V under homomorphism α . Then the *-free reduct of $P = \alpha(V)$ is an inflation of the *-free reduct of $T = \alpha(S)$, and the corresponding inflation function is φ' defined by $\varphi'(p) = t$ if and only if there are $s \in S$, $v \in V$, such that $\alpha(s) = t$, $\alpha(v) = p$ and $\varphi(v) = s$ (one easily shows that such a definition is logically correct). However, α is a *-homomorphism, so $t^* = \alpha(s^*)$ and $p^* = \alpha(v^*)$. Since

$\varphi(v^*) = \varphi(v)^* = s^*$, we have $\varphi'(p^*) = t^* = \varphi'(p)^*$, whence we conclude that φ' agrees with $*$. \square

Since it is easy to calculate that N_2 and N_3 are the only subdirectly irreducibles in \mathcal{N}^* (thus N_3 , or any other null involution semigroup with a nonidentical involution, generates \mathcal{N}^*), it follows that the list of subdirectly irreducibles of a variety of the form $\mathcal{V} \vee \mathcal{N}^*$ exhausts with the subdirectly irreducibles of \mathcal{V} , N_2 and N_3 . Therefore, any subvariety of $\mathcal{V} \vee \mathcal{N}^*$ is either of the form $\mathcal{W} \vee \mathcal{N}^{\text{id}}$, or of the form $\mathcal{W} \vee \mathcal{N}^*$, where $\mathcal{W} \subseteq \mathcal{V}$. So, to determine the structure of the lattice of subvarieties of $\mathcal{V} \vee \mathcal{N}^*$, it remains to establish which of the above joins are mutually different. To this end the following auxiliary result will be helpful.

Lemma 1.7.13. *If \mathcal{W} is an involution semigroup variety which does not satisfy $x = x^*$, then $\mathcal{W} \vee \mathcal{N}^{\text{id}} = \mathcal{W} \vee \mathcal{N}^*$.*

Proof. Let $S \in \mathcal{W}$ be an involution semigroup in which $x = x^*$ fails. Denote the elements of N_2 by 0 and 1, and consider the direct product $T = S \times N_2$. Let $P = S \times \{0\}$ and consider the equivalence $\theta = \Delta_{T \setminus P} \cup (P \times P)$ of T (it collapses all pairs whose second coordinate is 0). Obviously, θ is a $*$ -congruence of T , and $N = T/\theta$ is null. As S has elements which are not fixed by $*$, so has N (because if $a \neq a^*$ for some $a \in S$, then $(a, 1)^* = (a^*, 1) \neq (a, 1)$). Thus, N generates \mathcal{N}^* , implying that $\mathcal{N}^* \subseteq \mathcal{W} \vee \mathcal{N}^{\text{id}}$. The lemma now easily follows. \square

Our general result (which is related to the main results of Graczyńska [46] and Mel'nik [75]) is now as follows.

Theorem 1.7.14. *Let \mathcal{V} be an involution semigroup variety which does not contain nontrivial null involution semigroups. Let \mathcal{U} be the greatest subvariety of \mathcal{V} satisfying $x = x^*$. Then the lattice of subvarieties of $\mathcal{V} \vee \mathcal{N}^*$ has the structure as depicted in Figure 8, where the interval $[\mathcal{N}^{\text{id}}, \mathcal{U} \vee \mathcal{N}^{\text{id}}]$ is isomorphic to the lattice of subvarieties of \mathcal{U} , while the interval $[\mathcal{N}^*, \mathcal{V} \vee \mathcal{N}^*]$ is isomorphic to the lattice of subvarieties of \mathcal{V} .*

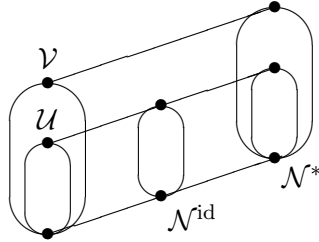


Figure 8. The lattice of subvarieties of $\mathcal{V} \vee \mathcal{N}^*$

Proof. If $\mathcal{W} \subseteq \mathcal{U}$, then $\mathcal{W} \vee \mathcal{N}^{\text{id}}$ satisfies $x = x^*$, and thus it differs from any variety of the form $\mathcal{V}' \vee \mathcal{N}^*$, where $\mathcal{V}' \subseteq \mathcal{V}$. Moreover, from the previous remarks it follows that $\mathcal{W}_1 \vee \mathcal{N}^{\text{id}} = \mathcal{W}_2 \vee \mathcal{N}^{\text{id}}$ implies $\mathcal{W}_1 = \mathcal{W}_2$ for all $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{U}$. On the other hand, if $\mathcal{W} \not\subseteq \mathcal{U}$, then by Lemma 1.7.13 we have $\mathcal{W} \vee \mathcal{N}^{\text{id}} = \mathcal{W} \vee \mathcal{N}^*$.

Now if $\mathcal{W} \subseteq \mathcal{V}$ is arbitrary, then by listing the subdirectly irreducible members of $\mathcal{W} \vee \mathcal{N}^*$, we obtain that the correspondence $\mathcal{W} \mapsto \mathcal{W} \vee \mathcal{N}^*$ (as well as $\mathcal{W}' \mapsto$

$\mathcal{W}' \cup \mathcal{N}^{\text{id}}$ for $\mathcal{W}' \subseteq \mathcal{U}$) is a bijective one. Thus, to prove the theorem, we need to show that these two correspondences are lattice homomorphisms.

It is immediate to see that these mappings agree with \vee , the varietal join operation. For the intersection, i.e. for the equalities

$$(\mathcal{W}_1 \vee \mathcal{N}^{\text{id}}) \cap (\mathcal{W}_2 \vee \mathcal{N}^{\text{id}}) = (\mathcal{W}_1 \cap \mathcal{W}_2) \vee \mathcal{N}^{\text{id}},$$

and

$$(\mathcal{Z}_1 \vee \mathcal{N}^*) \cap (\mathcal{Z}_2 \vee \mathcal{N}^*) = (\mathcal{Z}_1 \cap \mathcal{Z}_2) \vee \mathcal{N}^*,$$

where $\mathcal{W}_1, \mathcal{W}_2 \subseteq \mathcal{U}$ and $\mathcal{Z}_1, \mathcal{Z}_2 \subseteq \mathcal{V}$, it suffices to inspect once more the list of subdirectly irreducibles in corresponding varieties, just as above. The theorem is proved. \square

The variety to which we intend to apply the above theorem is

$$\begin{aligned} \mathcal{J}_4 &= \mathcal{RB}^* \vee \mathcal{SL}^{\text{id}} \vee \mathcal{SL}^0 \vee \mathcal{N}^{\text{id}} = \\ &= \mathcal{NB}^{\text{reg}} \vee \mathcal{SL}^0 \vee \mathcal{N}^{\text{id}} = \\ &= (\mathcal{NB}^{\text{reg}} \vee \mathcal{NB}^0) \vee \mathcal{N}^{\text{id}} = \\ &= (\mathcal{NB}^{\text{reg}} \vee \mathcal{NB}^0) \vee \mathcal{N}^*, \end{aligned}$$

as $\mathcal{NB}^{\text{reg}} \vee \mathcal{NB}^0$ does not satisfy $x = x^*$. On the other hand, \mathcal{SL}^{id} is the *only* non-trivial subvariety of $\mathcal{NB}^{\text{reg}} \vee \mathcal{NB}^0$ equipped with an identical involution, and since $\mathcal{NB}^{\text{reg}} \vee \mathcal{NB}^0$ has 10 subvarieties (as proved by Theorem 1.7.8), it follows from the above theorem that \mathcal{J}_4 has exactly 22 subvarieties. The subvarieties missing from Figure 3 are \mathcal{NB}^0 , $\mathcal{SL}^{\text{id}} \vee \mathcal{NB}^0$ and the joins of these two with \mathcal{N}^* .

1.8. Subdirectly Irreducible Involution Bands. Subdirectly irreducible algebras are very important building blocks of a variety, determining a great deal its structure and relationships to other varieties (just as it was experienced in the previous considerations). As long as semigroups are concerned, probably the first paper dealing with subdirect decompositions was the one of Thierrin [110]. The main contribution to the topic in the sixties was given by Schein [104], while Gerhard [41] described subdirectly irreducible bands. The characterizations presented in the sequel are just in the style of those given in [41], and they are all due to Dolinka [25].

The first task is certainly to describe subdirectly irreducible involution semilattices. We already met two distinguished semilattices with involution: these are Σ_2 (the two-element semilattice with the identical involution) and Σ_3 (the 0-direct union of a trivial semigroup with its copy). By Σ_4 we denote the involution semilattice obtained from Σ_3 by adjoining an identity element (which is, of course, fixed by the involution). Σ_4 is depicted in the following figure.

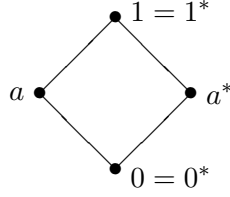


Figure 9. The involution semilattice Σ_4

Theorem 1.8.1. *There are exactly three (nontrivial) subdirectly irreducible involution semilattices: Σ_2 , Σ_3 and Σ_4 .*

As in the case of bands, it is necessary to distinguish between those involution bands which do or do not contain a zero element. Also similarly to bands, it is much easier to obtain the characterization for involution bands without zero. Recall that if B is an involution band and $a, b \in B$, then $\theta(a, b)$ is a customary notation designed for the *principal* congruence generated by (a, b) , that is, for the least congruence containing the indicated pair.

Theorem 1.8.2. *An involution band B without zero is subdirectly irreducible if and only if B is an ideal extension of a rectangular involution band K such that there exist distinct $a, b \in K$ for which $\theta(a, b) \subseteq \theta(c, d)$ holds for all distinct $c, d \in K$, and for all $p, q \in B$, the condition $pk = qk$ and $kp = kq$ for all $k \in K$ implies $p = q$.*

Of course, as one might expect, every subdirectly irreducible involution band has a core — the least non-null $*$ -ideal. So, the above theorem guarantees that in a subdirectly irreducible involution band without zero, its core K is a rectangular band with involution. However, *unlike* ordinary bands, the case when a zero is present splits into two essentially different cases. Namely, it turns out that the core of a subdirectly irreducible can be either a rectangular involution band with zero adjoined (i.e. with structure involution semilattice Σ_2), or of the form $I_0^*(A)$ for some rectangular band A (i.e. with structure involution semilattice Σ_3). The first of these two possibilities is handled easily, while the other is much more involved.

Theorem 1.8.3. *Let B be an involution band with zero. Then it is subdirectly irreducible and has a rectangular involution band with adjoined zero as the core if and only if $B = (B_1)^0$ for some subdirectly irreducible involution band B_1 without zero.*

Theorem 1.8.4. *An involution band B with zero which is not an involution semilattice, whose core has the structure based on Σ_3 , is subdirectly irreducible if and only if B is an ideal extension of an involution band of the form $I_0^*(L)$ for some left zero band L , such that there exist distinct $a, b \in L$ for which $\theta(a, b) \subseteq \theta(c, d)$ holds for all distinct $c, d \in L$, and for all $p, q \in B$, the condition $p\ell = q\ell$ and $\ell^*p = \ell^*q$ for all $\ell \in L$ implies $p = q$.*

An interesting special case of the above theorem describes the subdirectly irreducibles in \mathcal{B}^0 .

Theorem 1.8.5. *An involution band $B \in \mathcal{B}^0$ is subdirectly irreducible if and only if it is of the form $I_0^*(T)$, where T is either a subdirectly irreducible band without zero, or the trivial semigroup.*

In light of Theorem 1.7.8, it follows that all the subdirectly irreducibles of $\mathcal{B}^{\text{reg}} \vee \mathcal{B}^0$ belong either to \mathcal{B}^{reg} , or to \mathcal{B}^0 .

For regular $*$ -normal bands (i.e. for the variety $\mathcal{NB}^{\text{reg}}$) we can explicitly point out the subdirectly irreducibles. Namely, by Theorem 2.2 of Scheiblich [101], every normal $*$ -regular involution band B can be represented as a spined product of a left normal band L and its anti-isomorphic copy (that is, its dual) R , which is a right normal band, while the involution simply reverses pairs. (Recently, this assertion was generalized to arbitrary involution bands in [26]: if ϱ^b denotes the congruence opening of an equivalence ϱ , then every involution band B can be represented as a spined product of the band B/\mathcal{R}^b and its dual over B/\mathcal{D}' , where $\mathcal{D}' = \mathcal{L}^b \circ \mathcal{R}^b$, so that the involution is again the reversal of pairs.) It is not difficult to prove that such a band B is subdirectly irreducible as an involution band if and only if L is subdirectly irreducible as a band. But II.2 of [41] lists all left normal subdirectly irreducible bands: these are the trivial semigroup, the two element semilattice, the two element left zero band, and the latter band with adjoined zero. Thus the nontrivial subdirectly irreducible members of $\mathcal{NB}^{\text{reg}}$ are: Σ_2 , the 2×2 rectangular involution band RB_2 (which is the only nontrivial subdirectly irreducible rectangular involution band) and RB_2^0 . On the other hand, the above theorem implies that the only (nontrivial) subdirectly irreducibles in \mathcal{NB}^0 are Σ_3 and $I_0^*(L_2)$, where L_2 denotes the two-element left zero band. In [25], it was proved that the list of all subdirectly irreducible normal bands with involution is completed by Σ_4 and two more normal involution bands, one containing six, and another containing nine elements. This in passing shows that the variety \mathcal{NB}^* is residually < 10 .

Theorem 1.8.6. *Aside from those contained in $\mathcal{NB}^{\text{reg}} \vee \mathcal{NB}^0$ and $S\mathcal{L}^*$, there are exactly two more subdirectly irreducible members of \mathcal{NB}^* , both with core $I_0^*(L_2)$: one extended by Σ_2 (this one having 6 elements), and one extended by RB_2^0 (thus, having 9 elements), denoted by N_6 and N_9 , respectively.*

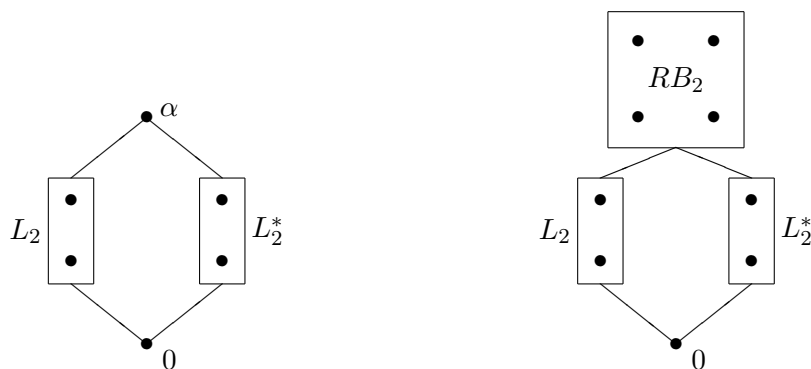


Figure 10. Normal involution bands N_6 and N_9

Since N_9 has noncommuting projections, it must generate the whole \mathcal{NB}^* , bearing in mind Theorem 1.7.10. On the other hand, N_6 belongs to the subvariety of \mathcal{NB}^* determined by $xx^*yy^* = yy^*xx^*$ (since $0\alpha = \alpha 0 = 0$), but does not belong to the subvariety given by $xx^*y = yxx^*$, as $\ell_1\alpha = \ell_1 \neq \ell_2 = \ell_2\ell_1 = \alpha\ell_1$, where $L_2 = \{\ell_1, \ell_2\}$. Hence, N_6 generates the former subvariety, whence all members of the latter one turn out to be subdirect products of involution semilattices and normal involution bands from \mathcal{NB}^0 .

1.9. Varieties of Regular $*$ -Semigroups with the Amalgamation Property. Let $\{A_\alpha : \alpha \in I\}$ be a family of universal algebras, sharing a common subalgebra U such that for each $\alpha, \beta \in I$, $\alpha \neq \beta$, we have $A_\alpha \cap A_\beta = U$. Such a family (which is in fact a partial algebra) is called an *amalgam*. It is said to be *weakly embeddable* into an algebra B if there exist injective homomorphisms $\varphi_\alpha : A_\alpha \rightarrow B$, $\alpha \in I$, agreeing on U ($\varphi_\alpha|_U = \varphi_\beta|_U$ for all $\alpha, \beta \in I$). If, in addition, we have $\varphi_\alpha(A_\alpha) \cap \varphi_\beta(A_\beta) = \varphi_\alpha(U)$ for all different $\alpha, \beta \in I$, then the considered amalgam is *strongly embedded* into B . A variety of algebras \mathcal{V} has the *weak (strong) amalgamation property* if any amalgam of algebras from \mathcal{V} can be weakly (strongly) embedded into an algebra from \mathcal{V} .

As known, semigroup amalgams and amalgamation properties in semigroup varieties constitute a well developed and established part of semigroup theory. Yet, there is a major obstacle in completing a number of characterization results which concern amalgams, namely the group varieties. It is still an open question whether there exists a proper nonabelian variety of groups with the weak (strong) amalgamation property (for the strong variant, this is just Problem 6 from [77]). Therefore, it is quite expectable that in considering amalgamation problems for various involution semigroup varieties, groups, and in fact completely simple $*$ -semigroups will remain out of range, so that we obtain descriptions modulo these classes.

For inverse semigroups (recall that they can be considered as regular $*$ -semigroups with the identity $xx^*x^*x = x^*xx^*$), the following theorem is a result of combined efforts of Hall [47] and Bíró, Kiss and Pálffy [6] (see also [48, 57]).

Theorem 1.9.1. *Aside from the hypothetical proper nonabelian weakly (strongly) amalgamable group varieties, precisely the following inverse semigroup varieties have the weak (strong) amalgamation property:*

- (1) *the variety of all inverse semigroups,*
- (2) *the variety of all groups,*
- (3) *all varieties of commutative inverse semigroups (these are the varieties of semilattices of Abelian groups).*

Later, the focus moved onto *generalized inverse semigroups* — orthodox $*$ -semigroups in which idempotents form a regular $*$ -normal band (of course, inverse semigroups are characterized by the condition that idempotents form a semilattice from \mathcal{SL}^{id}). The investigation along this line was initiated by Imaoka in [58], and the contribution of Hall and Imaoka [50] should be also singled out.

The second author of this survey noted that the results from the last section of [50], when put together with some techniques applied earlier to existence varieties of regular semigroups [49], give a sufficient basis for describing regular $*$ -semigroup varieties with the weak (strong) amalgamation property. In that sense, the paper [28] (where the following result appears) is a continuation of [50]. Note that all varieties listed below are either generalized inverse, or completely simple.

Theorem 1.9.2. *A regular $*$ -semigroup variety \mathcal{V} has the weak (strong) amalgamation property if and only if one of the following conditions is satisfied:*

- (1) \mathcal{V} is an inverse semigroup variety with the weak (strong) amalgamation property,
- (2) $\mathcal{V} = \mathcal{U} \vee \mathcal{RB}^*$, where \mathcal{U} is an inverse semigroup variety with the weak (strong) amalgamation property,
- (3) \mathcal{V} is a completely simple $*$ -semigroup variety with the weak (strong) amalgamation property.

It is worth mentioning one more ingredient used in obtaining the above result. First of all, note that the Brandt semigroup B_2 can be considered as an inverse semigroup (then it is generated as a regular $*$ -semigroup by a single generator a , subject to the relation $a^2 = 0$). It was proved by Schein [105] (and reproved in [47]) that an inverse semigroup variety consists entirely of semilattices of groups if and only if it omits B_2 . This was extended to regular $*$ -semigroup varieties in [28], so that for such a variety \mathcal{V} , $B_2 \notin \mathcal{V}$ is equivalent to the fact that \mathcal{V} consists entirely of completely regular $*$ -semigroups, and further, to the fact that \mathcal{V} satisfies an identity of the form $x = ux^2$, where $u = u(x)$ is an involution semigroup word.

However, quite recently it turned out that even the above indicator characterization is just a part of a more general setting. We finish by quoting the main result of [29].

Theorem 1.9.3. *Let \mathcal{V} be an involution semigroup variety. Then the following conditions are equivalent:*

- (1) any member of \mathcal{V} can be decomposed into an involution semilattice of Archimedean semigroups,
- (2) \mathcal{V} does not contain B_2 and $I_0^*(B_2)$.

Analogous descriptions for varieties consisting of semilattices of Archimedean semigroups (without involution) were obtained earlier by Sapir and Sukhanov [100] for periodic case, and for the general case by Ćirić and Bogdanović [9].

2. VARIETIES OF INVOLUTION SEMIRINGS

2.1. The Role of Involution Semirings in Theoretical Computer Science. First of all, we recall that by our definition, a semiring is an algebra with two binary operations, $(S, +, \cdot)$, the first of which is commutative. On the other hand, there are several authors which, while referring to semirings, do not assume the commutativity of $+$, see e.g. [80, 81, 82]. Also, one may often encounter definitions in which $(S, +)$ is required to be a monoid, and its neutral element 0 is then considered as a fundamental constant. However, the latter difference will not cause

any major problems: we shall use the semirings with a zero in the present subsection (conforming to the practice in theoretical computer science), and then pass in the subsequent two subsections to the (more general) approach in which the zero is dropped, but the results are always easily transformed from the one variant to another.

Also, in this subsection we shall use another symbol for semiring involutions, namely \vee instead of the star. There are fairly good reasons for the change of notation. Namely, if Σ is an alphabet, then it is a quite wide-spread notational convention to denote the free monoid on Σ by Σ^* , which consist of all words (finite sequences) over Σ , and free monoids will be important for us in the sequel. Indeed, we may define a semiring with unit

$$L_\Sigma = (\mathcal{P}(\Sigma^*), +, \cdot, \emptyset, \{\lambda\}),$$

where $+$ (for traditional reasons) denotes the set-theoretical union, λ is the empty word, and for $A, B \subseteq \Sigma^*$ we have

$$A \cdot B = \{uv : u \in A, v \in B\}.$$

The subsets of Σ^* are usually called *languages* (over Σ), and AB is called the *concatenation* of languages A and B . Therefore, we obtain the *language semiring* over Σ . Actually, it is not difficult to see that we can obtain a semiring (with unit) form an arbitrary semigroup (monoid) S , by defining analogous operations of the power set of S ,

$$P_S = (\mathcal{P}(S), +, \cdot, \emptyset)$$

(in case S is a monoid, the unit $\{1\}$ is added to the above system). According to the above notation, L_Σ is in fact the same thing as P_{Σ^*} .

Now, one can define a unary operation $A \mapsto A^*$ in P_S (provided S is a monoid) by

$$A^* = \sum_{n \geq 0} A^n$$

(the sum operator denoting the union), where $A^{n+1} = A \cdot A^n$ and by convention, $A^0 = \{1\}$. If we consider the language semiring L_Σ , the above definition introduces the *Kleene star* operation, which is well-known in theoretical computer science, especially in automata theory. By equipping L_Σ with $*$, we obtain the *language algebra* L_Σ^* . Note that $*$ is here by no means an involution; actually, it satisfies the fixed-point identity $x^{**} = x^*$.

On the other hand, there is an obvious way to define an involution on L_Σ . Namely, if w^R denotes the reverse of the word w , just as in the previous section, for $L \subseteq \Sigma^*$ we may define

$$L^\vee = \{w^R : w \in L\}.$$

It is pretty easy to see that \vee gives L_Σ the structure of a involution semiring with unit, which we denote by L_Σ^\vee . If both \vee and $*$ are considered, we obtain the *involution language algebra* $L_\Sigma^{*\vee}$.

Another important examples of involution semirings come from binary relations. If A is an arbitrary set, we define the algebra

$$Rel(A) = (\mathcal{P}(A \times A), \cup, \circ, \emptyset, \Delta_A),$$

where \circ is the relational composition and Δ_A is the diagonal (identity) relation. $Rel(A)$ also turns out to be a semiring with unit, and it can be made into an involution semiring $Rel^\vee(A)$ by considering the operation of the *converse* of relations:

$$\varrho^\vee = \{(b, a) : (a, b) \in \varrho\}.$$

Similarly as above, we can iterate the relational composition, thus obtaining a unary operation

$$\varrho^* = \bigcup_{n \geq 0} \varrho^n,$$

where $\varrho^{n+1} = \varrho \circ \varrho^n$ and $\varrho^0 = \Delta_A$. The relation ϱ^* is actually the reflexive-transitive closure of ϱ . By adding $*$ to (involution) semirings of relations $Rel(A)$ and $Rel^\vee(A)$, we obtain *Kleene relation algebras (with involution)* $Rel^*(A)$ and $Rel^{*\vee}(A)$, respectively, cf. [59, 70, 22, 23].

Language and relation semirings are just special cases of *complete semirings*, which are of at most importance in the mathematical foundations of computer science, cf. [4, 8, 34, 63, 65, 67]. These are semirings in which an infinite summation operator $\sum_{i \in I}$ is defined, such that if $\{a_i : i \in I\}$ is any family of elements of the considered semiring, we have:

$$\begin{aligned} \sum_{1 \leq i \leq n} a_i &= a_1 + \cdots + a_n, \\ \sum_{(i,j) \in I \times J} a_i b_j &= \left(\sum_{i \in I} a_i \right) \left(\sum_{j \in J} b_j \right), \\ \sum_{i \in I} a_i &= \sum_{j \in J} \sum_{i \in I_j} a_i, \end{aligned}$$

where I is the disjoint union of the sets I_j , $j \in J$. Of course, the summation is commutative, associative and completely distributive. Further, a complete semiring is *completely additively idempotent* if $\sum_{i \in I} a = a$ holds for any index set I (clearly, each completely additively idempotent semiring is additively idempotent). Note that all the above examples are such. Finally, in any complete semiring one can define the *iteration operation* $*$ by

$$a^* = \sum_{n=0}^{\infty} a^n.$$

Now we have the following observation.

Lemma 2.1.1. *Every language algebra can be embedded into a Kleene relation algebra. Consequently, every language semiring is isomorphic to a semiring of binary relations.*

Proof. (sketch) Consider the mapping $\xi : \mathcal{P}(\Sigma^*) \rightarrow \mathcal{P}(\Sigma^* \times \Sigma^*)$ defined for every $A \subseteq \Sigma^*$ by

$$\xi(A) = \{(w, wx) : w \in \Sigma^*, x \in A\}.$$

It is a routine matter to show that ξ is, in fact, an embedding of the algebra L_Σ^* into $Rel^*(\Sigma^*)$. \square

Hence, if we denote by \mathcal{L} the variety generated by all language algebras, while \mathcal{KA} denotes the variety of *Kleene algebras*, generated by all Kleene relation algebras, we have $\mathcal{L} \subseteq \mathcal{KA}$, and in particular, all language algebras are Kleene algebras. However, the above inclusion is in fact an equality, $\mathcal{L} = \mathcal{KA}$, because by the Kozen-Németi Theorem (cf. [64, 70]), the free Kleene algebra on Σ is just the subalgebra of L_Σ^* formed by the regular subsets of the free monoid Σ^* . Using this, and knowing the explicit equational axiomatization of Kleene algebras (which is necessarily infinite, cf. [10, 66, 13]), one can easily derive the following result.

Theorem 2.1.2. *Both language semirings and relation semirings generate the variety of idempotent semirings with unit.*

But what is the situation if the involution is present? The above Lemma 2.1.1 is no longer true for the involution case: in fact no involution semiring of the form L_Σ^\vee can be embedded in an involution semiring of relations. In other words, if \mathcal{L}^\vee denotes the variety generated by involution language algebras, while \mathcal{KA}^\vee is the variety of *Kleene algebras with involution* generated by all algebras $Rel^{*\vee}(A)$, one can prove that $\mathcal{KA}^\vee \subseteq \mathcal{L}^\vee$, but this inclusion is *proper*. It is just the involution that distinguishes between them, even if we drop the iteration operations and work with involution semirings only. Consider the following identity:

$$x + xx^\vee x = xx^\vee x.$$

It is a routine to see that the above identity is true in binary relations. However, it suffices to consider the one-element alphabet $\Sigma = \{a\}$ and substitute the language $\{a\}$ for x to see that the above identity fails in all involution semirings of languages. In fact, we have a more accurate information concerning this matter.

Theorem 2.1.3. (Bloom, Ésik and Stefanescu, [8]) *The variety \mathcal{L}^\vee is defined by the identities of Kleene algebras, axioms of semiring involution (including $0^\vee = 0$) and*

$$(x^*)^\vee = (x^\vee)^*.$$

Theorem 2.1.4. (Ésik and Bernátsky, [35]) *The variety \mathcal{KA}^\vee is defined as a subvariety of \mathcal{L}^\vee by the identity*

$$x + xx^\vee x = xx^\vee x.$$

From these results it is not difficult to obtain

Corollary 2.1.5. *The involution semirings of languages generate the variety of idempotent involution semirings with unit, while the relational involution semirings generate its subvariety determined by $x + xx^\vee x = xx^\vee x$.*

Let us also mention some related results obtained by the authors of this survey and Z.Ésik.

Theorem 2.1.6. (Crvenković, Dolinka and Ésik, [13, 14]) *Varieties \mathcal{L}^\vee and \mathcal{KA}^\vee are both not finitely based. Also, if we drop the union (addition) operation from Kleene relation algebras with involution (resp. involution language algebras), the equational theories of the so obtained varieties consist precisely of those identities of \mathcal{KA}^\vee (resp. \mathcal{L}^\vee) which do not contain occurrences of $+$, and these theories are too nonfinitely based.*

The simplest explanation for the second part of the above result is that the interaction between the concatenation and $*$ is from the equational point of view ‘too complicated’, and exactly this interaction is the origin of all nonfinite axiomatizability results of the above type which concern algebras of formal languages.

It is interesting to remark that there is a ‘technical’ connection between the first part of the above theorem and Theorem 1.5.2. Namely, there are two ways to prove that \mathcal{L}^\vee and \mathcal{KA}^\vee are not finitely based, knowing that the same holds for \mathcal{L} and \mathcal{KA} , respectively, and knowing, of course, Theorems 2.1.3 and 2.1.4. One of these proofs — more syntactical in nature — relies on the same proposition on involutorial identities (proved in [13]), which allowed us to obtain in [15] the result of Theorem 1.5.2. Probably there are some further links between the identities of general algebraic systems with involution and of their involution-free reducts respectively, which are yet to be discovered and explored.

2.2. Minimal Varieties of Involution Semirings. Minimal varieties of involution semirings were described by the second author of this survey in [21]. Towards that goal, an important help was the already known list of minimal varieties of ordinary semirings, determined by Polin [92], cf. also [109]. To recall Polin’s result and to formulate the main result of [21], we define some binary and unary operations on finite sets $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$ and $4 = \{0, 1, 2, 3\}$.

\vee 0 1	\wedge 0 1	\circ 0 1	$*_\ell$ 0 1	$*_r$ 0 1
0 0 1	0 0 0	0 0 0	0 0 0	0 0 1
1 1 1	1 0 1	1 0 0	1 1 1	1 0 1
\wedge_3 0 1 2	\circ_3 0 1 2	\diamond 0 1 2 3	\square 0 1 2 3	
0 0 0 0	0 0 0 0	0 0 1 2 3	0 0 1 0 1	
1 0 1 0	1 0 0 0	1 1 1 3 3	1 0 1 0 1	
2 0 0 2	2 0 0 0	2 2 3 2 3	2 2 3 2 3	
		3 3 3 3 3	3 2 3 2 3	
	a 0 1 2	a 0 1 2 3		
	\bar{a} 0 2 1	\tilde{a} 0 2 1 3		

Theorem 2.2.1. (Polin, [92]) *A variety of semirings is minimal if and only if it is generated by one of the following semirings:*

- (1) $(2, \circ, \wedge)$, $(2, \circ, \circ)$, $(2, \vee, \vee)$, $(2, \vee, \wedge)$, $(2, \vee, \circ)$, $(2, \wedge, \circ)$,
- (2) $(2, \vee, *_\ell)$, $(2, \vee, *_r)$,

- (3) $Z_p = (\{0, 1, \dots, p-1\}, +_p, \cdot_p)$, where p is a prime number, and $+_p$ and \cdot_p are respectively the addition and the multiplication modulo p (i.e. Z_p is the finite field with p elements),
- (4) $N_p = (\{0, 1, \dots, p-1\}, +_p, \circ_p)$, where p is a prime number, and \circ_p is the zero multiplication of the set $\{0, 1, \dots, p-1\}$.

Note that all varieties of involution semirings having a trivial involution ($x^* = x$) are exhausted by varieties of commutative semirings augmented with the identity mapping, and this conclusion applies to minimal varieties as well. Clearly, (1), (3) and (4) of the above theorem provide all such varieties.

Theorem 2.2.2. (Dolinka, [21]) *A variety of semirings with nontrivial involution is minimal if and only if it is generated by one of:*

- (1) $(3, \wedge_3, \wedge_3, \bar{})$, $(3, \wedge_3, \circ_3, \bar{})$, $(3, \circ_3, \wedge_3, \bar{})$,
- (2) $(4, \diamond, \square, \bar{})$.
- (3) $(\{0, 1, \dots, p-1\}, +_p, \circ_p, -_p)$, where $-_p$ is the operation of additive inverse modulo a prime number $p \geq 3$.

It is more or less in the universal algebraic folklore that all of the algebras above generate minimal (equationally complete) varieties. The proof of the other implication, on the other hand, resembles somewhat to the way in which Fajtlowicz obtained the minimal varieties of involution semigroups, because it consists of considering cases according to the properties of Hermitian elements (involution fixed points).

Firstly, one can prove that if an involution semiring which generates a minimal variety \mathcal{V} contains a Hermitian element a which is either not additively idempotent ($a + a \neq a$), or not multiplicatively idempotent ($a^2 \neq a$), then \mathcal{V} consists of commutative involution semirings with a trivial involution, and in that case Theorem 2.2.1 settles the problem. Otherwise, it can be assumed that all Hermitian elements a under consideration satisfy $a + a = a^2 = a$. Now, in any involution semiring S which is not additively idempotent and which belongs to a minimal variety, there is a unique Hermitian element which is:

- (1) the multiplicative zero of S ,
- (2) either the additive zero, or the additive unit of S .

In the latter of the two cases given in (2) above, S must be a ring, $a^\vee = -a$, and, moreover, there is a monogenic subring S' of S and a prime p such that N_p , augmented by the additive inverse modulo p , is a homomorphic image of S' . On the other hand, in the former of the two described cases, S generates the same variety as $(3, \circ_3, \wedge_3, \bar{})$ does.

So, it remains to consider minimal varieties generated by additively idempotent involution semirings. If such a variety contains a nontrivial involution semiring with a unique Hermitian element, then it has to contain one of $(3, \wedge_3, \wedge_3, \bar{})$, $(3, \wedge_3, \circ_3, \bar{})$. Finally, if an involution semiring contains at least two Hermitian elements and generates a minimal variety (even without the condition of the additive idempotency), then it contains an involution subsemiring isomorphic to $(4, \diamond, \square, \bar{})$, whence our theorem is established.

2.3. Idempotent Distributive Involution Semirings. A semiring is *distributive* if it satisfies the dual distributive identity

$$x + yz = (x + y)(x + z)$$

(of course, the above identity and commutativity of $+$ together imply $xy + z = (x + z)(y + z)$). A semiring is *idempotent* if both of its operations are such (we already referred to additive and multiplicative idempotency in semirings). Note that if a semiring S is (additively) idempotent, then $(S, +)$ is a semilattice. If both of the binary reducts $(S, +)$ and (S, \cdot) of S are idempotent and commutative, then S is called a *bisemilattice*. A bisemilattice in which the two operations coincide (i.e. which satisfy $x + y = xy$) is called a *mono-bisemilattice*. Of course, it causes no confusion if we identify (in the notational sense) semilattices and mono-bisemilattices.

Idempotent and distributive semirings are called *ID-semirings* for short. The study of ID-semirings started in the late sixties and continued in the seventies, see e.g. [61, 73, 88], with investigations on distributive bisemilattices. However, the topic gained attention in the early eighties, mainly with contributions of Pastijn and Romanowska [80, 82, 95, 96]. In particular, the lattice of all varieties of ID-semirings (with $+$ commutative) is given in [96]: it is the four-dimensional cube. Recently, Kuřil and Polák [68] found a way to determine all varieties of idempotent semirings (without the requirement of distributivity of $+$ over \cdot). On the other hand, Pastijn and Guo [81] described the lattice of all ID-semirings without $+$ being commutative. It is a countably infinite distributive lattice.

Motivated by the result of Romanowska [96], the second author of this survey obtained the lattice of all varieties of ID-semirings with involution. The corresponding result is as follows.

Theorem 2.3.1. (Dolinka, [27]) *There are exactly 64 varieties of ID-semirings with involution, and their lattice coincides with the one depicted in Figure 11.*

As semilattices and mono-bisemilattices can be identified, so can involution semilattices and mono-bisemilattices with involution. Therefore, Σ_2 , Σ_3 and Σ_4 will also denote involution semirings in which both operations define the corresponding semilattice with involution. It is easy to see that all of the above algebras are in fact ID-semirings.

It was proved in Theorem 2.1 of [82] that the multiplicative reduct of an ID-semiring must be a normal band. Further, by Theorem 1.6 of the same paper, each ID-semiring is a Płonka sum of a semilattice ordered system of ID-semirings satisfying

$$x + xyx = x.$$

The latter semirings are, in turn, obtained by a special kind of a composition of a distributive lattice ordered system of ID-semirings in which the multiplicative reduct is a rectangular band.

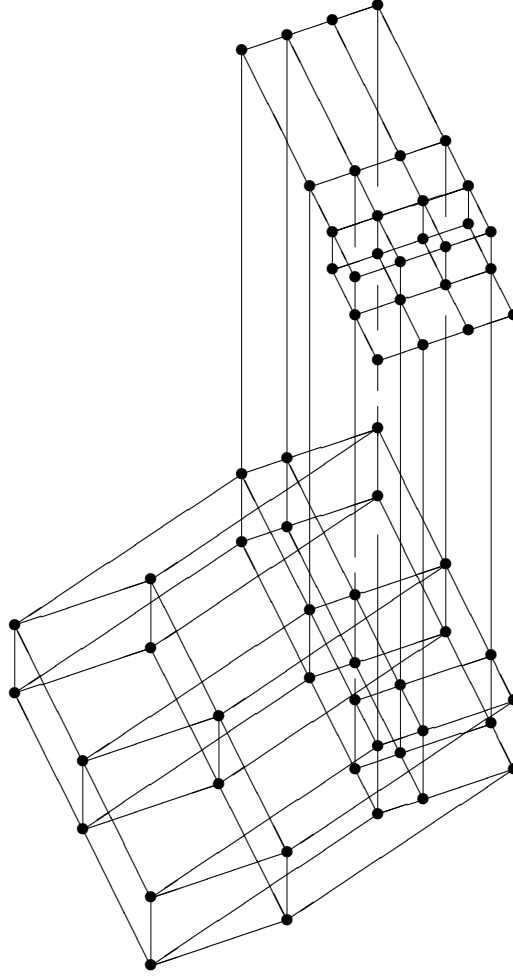


Figure 11. The lattice of all varieties of ID-semirings with involution

All these results can be extended for ID-semirings with involution as well. First of all, one must replace the well-known general algebraic construction of a Płonka sum by the *involutional Płonka sum of algebras*, introduced in [31]. Here we give the basic definition, restricted to the case of semirings.

Let Y be an involution semilattice. A family of semirings $\{S_i : i \in Y\}$, together with a system of homomorphisms $\{\phi_{i,j} : i, j \in Y, i \geq j\}$ and a bijection $*$ on $\bigcup_{i \in Y} S_i$, is called an Y -ordered system of semirings if the following conditions are satisfied:

- (1) for each $i \in Y$, $\phi_{i,i}$ is the identity mapping on S_i ,
- (2) for each $i, j, k \in Y$ such that $i \geq j \geq k$ we have

$$\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k},$$

- (3) for each $i \in Y$, $*$: $S_i \rightarrow S_{i^*}$ is a semiring anti-isomorphism,
- (4) $\phi_{i^*,j^*}(x) = (\phi_{i,j}(x^*))^*$, for all $i, j \in Y$ such that $i \geq j$ and all $x \in S_{i^*}$.

The *involutional Płonka sum* of such a system is a semiring with involution S , where $S = \bigcup_{i \in Y} S_i$, with the operations given by

$$\begin{aligned} a + b &= \phi_{i,ij}(a) + \phi_{j,ij}(b), \\ ab &= \phi_{i,ij}(a)\phi_{j,ij}(b), \end{aligned}$$

where $a \in S_i$ and $b \in S_j$.

Theorem 2.3.2. (Dolinka and Vinčić, [31]) *Each ID-semiring with involution can be represented as an involutional Płonka sum of an involution semilattice-ordered system of ID-semirings satisfying the identity $x + xyx = x$. Conversely, the involutional Płonka sum of every such system is an ID-semiring with involution.*

As we mentioned above, in [95] Romanowska proved that each ID-semiring satisfying $x + xyx = x$ is the sum of a distributive lattice-ordered m -system of rectangular ID-semirings (i.e. with rectangular multiplicative reduct). This means that we have given a system of disjoint semirings S_i indexed by a distributive lattice (D, \vee, \wedge) (so that $i \in D$), and for each $i, j \in D$ such that $i \geq j$ an embedding $\psi_{i,j} : S_i \rightarrow S_j$ such that

- (i) $\psi_{i,i}$ is the identity map on S_i for all $i \in D$,
- (ii) $\psi_{i,j} \circ \psi_{j,k} = \psi_{i,k}$ for all $i, j, k \in D$ such that $i \geq j \geq k$,
- (iii) $\psi_{i,i \wedge j}(S_i) + \psi_{j,i \wedge j}(S_j) \subseteq \psi_{i \vee j, i \wedge j}(S_{i \vee j})$ for all $i, j \in D$.

The sum of this system is defined in such a way that the operations in the resulting semiring $(S, +, \cdot)$ (where $S = \bigcup_{i \in D} S_i$) are given by

$$\begin{aligned} a_i b_j &= \psi_{i,i \wedge j}(a_i)\psi_{j,i \wedge j}(b_j), \\ a_i + b_j &= \psi_{i \vee j, i \wedge j}^{-1}(\psi_{i,i \wedge j}(a_i) + \psi_{j,i \wedge j}(b_j)), \end{aligned}$$

where $a_i \in S_i$ and $b_j \in S_j$.

Now, we are going to call an m^* -system of semirings a family of semirings S_i indexed by a distributive lattice with involution $(D, \vee, \wedge, *)$, endowed with semiring embeddings $\psi_{i,j}$ for each pair $i \geq j$ and a bijection $*$ on $\bigcup_{i \in D} S_i$ such that the conditions (i)-(iii) above are satisfied, as well as the following conditions:

- (iv) $*$: $S_i \rightarrow S_{i^*}$ is a semiring anti-isomorphism for all $i \in D$,
- (v) $\psi_{i^*,j^*}(x) = (\psi_{i,j}(x^*))^*$, for all $i, j \in D$ such that $i \geq j$ and all $x \in S_i$,

which express the compatibility of $*$ with the m -system structure and, respectively, the ‘symmetry’ of the m -system with respect to the involution.

Theorem 2.3.3. (Dolinka, [27]) *An algebra $(S, +, \cdot, *)$ is an ID-semiring with involution satisfying $x + xyx = x$ if and only if it is the sum of an m^* -system of rectangular ID-semirings.*

Finally, it remains to provide some information about rectangular ID-semirings with involution. We recall here a construction which is well-known in universal algebra, called the *matrix power*. Namely, for a universal algebra (A, \mathcal{F}) (where \mathcal{F} is a family of finitary operations on A) and $n \in \mathbb{N}$, the n -th matrix power is defined on the set $A^n = A \times \cdots \times A$ such that all fundamental operations from the original

algebra are inherited by applying them coordinatewise in A^n , while two operations are added: the n -ary *diagonal operation* d given by

$$d(\mathbf{x}_1, \dots, \mathbf{x}_n) = \langle x_{11}, x_{22}, \dots, x_{nn} \rangle,$$

where $\mathbf{x}_i = \langle x_{i1}, \dots, x_{in} \rangle$ for all $1 \leq i \leq n$, and a unary operation p determined by

$$p(\langle x_1, x_2, \dots, x_n \rangle) = \langle x_2, \dots, x_n, x_1 \rangle.$$

It is known that for a variety of algebras \mathcal{V} and a given positive integer n , all isomorphic copies of all n -th matrix powers of members of \mathcal{V} also form a variety, denoted by $\mathcal{V}^{[n]}$. Also, it is known that this construction preserves equational completeness, see [71]. For more information about matrix powers and their application in universal algebra, we refer to [53] and [71]. Now we obtain the following theorem, which does not have its non-involutorial analogue.

Theorem 2.3.4. (Dolinka, [27]) *Every rectangular ID-semiring with involution is the matrix square of some semilattice and conversely, every matrix square of a semilattice is a rectangular ID-semiring with involution. In other words, the variety of rectangular ID-semirings is just $\mathcal{SL}^{[2]}$ and thus it has no proper subvarieties (cf. [21]).*

Another nice and in this setting important feature of the paper [31] is that it admits a direct calculation of those involutorial Płonka sums which are subdirectly irreducible, provided that the subdirectly irreducibles are known in the class of (involution) algebras from which the components of the sum are taken. So, the results in [31] generalize the corresponding results on subdirectly irreducible Płonka sums, given in [69]. With a little amount of technical work, one can find the explicit list of subdirectly irreducible ID-semirings, and thereby show that the variety of ID-semirings is — similarly to the variety of normal bands with involution — residually < 10 (in fact, the results presented in the last subsection of the section on involution semigroups can be also derived from the general theorems of [31]). In particular, if an involutorial Płonka sum is subdirectly irreducible, then its structure involution semilattice must be trivial, or it is subdirectly irreducible itself, that is, one of Σ_2 , Σ_3 and Σ_4 (by our Theorem 1.8.1).

But first, let L_2 denote the (unique) two-element ID-semirings whose multiplicative reduct is a left zero band. Dually, we have the semiring R_2 . These semirings, as well as their direct product $L_2 \times R_2$, are examples of a rectangular ID-semirings. By defining the exchange involution (the reversing of pairs) on the latter one, one obtain a four-element involution semiring, which is isomorphic to the matrix square of the two-element semilattice. This one we denote by RS_2^* .

The two-element and the four-element distributive lattice we denote by D_2 and D_4 , respectively. Of course, we can equip the first one by the identity mapping as the involution, thus obtaining the involution lattice D_2^* . In turn, D_4 can be enriched to the involution lattice D_4^* by defining an involution which fixes the top and the bottom element, and exchanges the other two.

Similarly to semigroups, one can adjoin an absorbing element to a semiring (with involution). This is the same as to compose into an involutorial Płonka sum

a Σ_2 -ordered system consisting of a trivial involution semiring and an arbitrary involution semiring S , and such a construction yields an algebra denoted by S^0 . Also, one can perform 0-direct unions by taking a semiring S (without involution), its anti-isomorphic copy \tilde{S} , and a trivial involution semiring (which, considered together, form a Σ_3 -ordered system of semirings) and constructing their involutorial Płonka sum. Such a sum is denoted by $I_0^*(S)$.

Finally, assume we are concerned with a Σ_4 -ordered system of semirings, where the (involution) semiring assigned to the index 0 is trivial. Further, we have the anti-isomorphic semirings $S = S_a$ and $\tilde{S} = S_{a^*}$, and the involution semiring S_1 , with the structure semiring homomorphism $\phi = \phi_{1,a}$ satisfying the required conditions. The resulting sum we denote by $\diamond_0^*(S, S_1, \phi)$. We omit ϕ if it is (up to an isomorphism of the resulting sum) uniquely determined by the components. Moreover, if S_1 is trivial, then it will be omitted too. The desired key result on subdirectly irreducible ID-semirings is now the following.

Theorem 2.3.5. (Dolinka, [27]) *A nontrivial ID-semiring with involution is subdirectly irreducible if and only if it is isomorphic to one of the following 17 semirings with involution:*

- (1) RS_2^*, D_2^*, D_4^* ,
- (2) $\Sigma_2, (RS_2^*)^0, (D_2^*)^0, (D_4^*)^0$,
- (3) $\Sigma_3, I_0^*(L_2), I_0^*(D_2)$,
- (4) $\Sigma_4, \diamond_0^*(L_2), \diamond_0^*(D_2, \phi_0), \diamond_0^*(D_2, \phi_1), \diamond_0^*(L_2, RS_2^*), \diamond_0^*(D_2, D_2^*),$ and $\diamond_0^*(D_2, D_4^*),$

where ϕ_0 maps the only element of the trivial semiring into the lower element of D_2 , while ϕ_1 maps into the upper element of D_2 .

The above theorem, together with the other structural results presented in this section, are the main ingredients in a lengthy and involved argument, with a number of subtle details, which leads to the result of Theorem 2.3.1. In Figure 11, there are three clearly distinguished intervals of the lattice. The lattice is, of course, intentionally drawn in such a way, because the corresponding proof splits into three separate parts, each producing one of those intervals, starting from the bottom and proceeding to the top.

2.4. Some Varieties of Involution Rings. Let $(R, +, \cdot, -, 0)$ be a ring and assume that $*$ is its semiring involution. Then it is very easy to deduce from the ring axioms that for all $r \in R$ we have $(-r)^* = -r^*$ and $0^* = 0$, so that $*$ agrees with the whole ring structure of R . In the way just described, we obtain a *ring with involution* (or a $*$ -ring).

Involution rings are probably the most important and best studied algebraic structures with involution in mathematics in general. It would take too much space to attempt to give even a shortest account on the results concerning involution rings and their applications. This topic originates back to von Neumann, who considered the adjoint (as an involution) in the algebra of bounded linear operators on a Hilbert space (such an involution algebra is widely used in theoretical physics, especially

in quantum mechanics). Classical books on involution rings are e.g. Berberian [3] and Herstein [52].

However, the point of view of considering an involutorial antiautomorphism of a ring as a fundamental operation (and thus, of considering related universal algebraic questions) is somewhat more recent and much easier to review. Such an approach has been taken, for example, in Rowen [97] and in the survey article of Wiegandt [115].

One of the (historically) most important classes of involution rings is the one of *regular *-rings*. Originally, it were *regular rings* which were considered by von Neumann in his fundamental treatise [114] (see also [108]), and which turned out to be the starting point (and the main motivation) for the whole theory of regular semigroups. Regular rings and regular *-rings are in a quite fascinating way strongly related to (orthocomplemented) modular lattices, and thus, in particular, to projective geometries. This link is described by the well-known von Neumann's Full Coordinatization Theorem (which generalizes the classical coordinatization theorems of projective spaces).

Theorem 2.4.1. (von Neumann, [114], Roddy, [94]) *Let M be a (orthocomplemented) modular lattice. Then there is a regular ring (with involution) R whose principal right ideals form a lattice, which is isomorphic to M . Moreover, R can be obtained as a ring of matrices (of a certain finite dimension) over a ring D such that $D \subseteq M$ and the ring operations of D are expressed as polynomials of the lattice M . In the case of ortholattices, the orthocomplementation is uniquely determined by the involution on R .*

It is easy to prove that the condition of a regularity of a *-ring is equivalent to the condition that every principal right ideal is generated by a *projection*, an idempotent fixed by the involution. Therefore, in the orthocomplemented version of the above theorem, one can replace the lattice of principal right ideals of R by the lattice of projections of R with respect to the partial order defined by $e \leq f$ is and only if $ef = e$. Hence, every modular ortholattice can be represented by projections of some regular *-ring.

Further, one can show that the regularity of a *-ring R is equivalent to the implication

$$rr^* = 0 \Rightarrow r = 0,$$

for all $r \in R$. This form of regularity provides an obvious way to equationally define a special ring involution which guarantees the regularity of the underlying ring. Following Yamada [117], we call a *special regular *-ring* an involution ring which satisfies the identity

$$xx^*x = x.$$

Using some results from Nambooripad and Pasijn [76], Yamada first proved that the multiplicative reduct of any special regular *-ring is a semilattice of groups, and moreover, we have $2x = (2x)(2x)^*(2x) = 8xx^*x = 8x$, so $6x = 0$. In light of this, the following result is not so surprising.

Theorem 2.4.2. (Yamada, [117]) *Any special regular $*$ -ring R can be decomposed into a direct sum $R = R_2 \oplus R_3$, such that R_2 and R_3 are the $*$ -ideals (ideals closed for $*$) of R consisting of all the elements of R of order 2 and 3, respectively. Moreover, R_2 satisfies $x^4 = x$, while R_3 satisfies $x^3 = x$, so that R satisfies $x^7 = x$. Consequently (by Jacobson's Theorem), every special regular $*$ -ring is commutative.*

Going in more detail, Yamada in [117] described the subdirectly irreducible special regular $*$ -rings.

Theorem 2.4.3. (Yamada, [117]) *The only subdirectly irreducible special regular $*$ -rings are the finite fields with 2, 3 and 4 elements, with the inverse operation as the involution ($x^* = x^{-1}$ for all $x \neq 0$ and $0^* = 0$).*

Of course, it is well-known that a ring which satisfies the identity $x^{n+1} = x$ for some $n \in \mathbb{N}$ is subdirectly irreducible if and only if it is a field (satisfying the same identity). This fact, and the above theorems of Yamada serve as good inspiration to investigate in general the subdirect decomposition of involution rings obeying an identity of the form $x^{n+1} = x$.

Given a ring R , denote by R^{opp} its opposite ring, i.e. its anti-isomorphic copy. Clearly, the direct sum $R \oplus R^{opp}$ is isomorphic to their direct product, and one can define the exchange involution on this sum. The resulting involution ring we denote by $Ex(R)$. Of course, to each ideal I of R it corresponds a $*$ -ideal of $Ex(R)$ obtained as the direct sum of I and I^* . Also, if R is a ring with involution and I is an ideal of the ring reduct of R such that $R = I \oplus I^*$, then it follows that $R \cong Ex(I)$.

It is not difficult to analyze all the possible involutions on a finite field $GF(p^k)$. The required involution defines an involutorial automorphism of that field, and it is well-known that every automorphism of the specified finite field is of the form

$$x \mapsto x^{p^m}$$

for some integer $0 \leq m \leq k - 1$. Thus, we have

$$x = (x^*)^* = (x^{p^m})^* = x^{p^{2m}}.$$

As the multiplicative group of our field must be cyclic of order $p^k - 1$, we obtain that $(p^k - 1) \mid (p^{2m} - 1)$, that is, $k \mid 2m$. Since $2m < 2k$, this yields two possibilities: $m = 0$, whence the involution is just the identity mapping, and $m = \frac{k}{2}$, provided k is even (otherwise, this case is impossible). The resulting field with involution we denote by $GF(p^k)$ in the former case (abusing slightly the notation), and by $GF^*(p^k)$ in the latter case. Now we have prepared the way for stating our next result.

Theorem 2.4.4. (Crvenković, Dolinka and Vinčić, [16]) *A ring with involution R is subdirectly irreducible and obeys the identity $x^{n+1} = x$ if and only if there is a prime number p and an integer $k \geq 1$ satisfying $(p^k - 1) \mid n$, such that R is isomorphic to one of the following:*

- (1) $GF(p^k)$,

- (2) if k is even, $GF^*(p^k)$,
- (3) $Ex(GF(p^k))$.

The key lemma in the course of proving the above theorem is that if R is a ring with involution satisfying the given conditions, then R has an identity element (which is, clearly, fixed by the involution) and R is actually $*$ -simple (meaning that R has no nontrivial $*$ -ideals). The other main ingredient for the proof comes from the paper of Birkenmeier, Groenewald and Heatherly [5] in which the relationships between the ideal and the $*$ -ideal structure of an involution ring were studied. In particular, the result we need is that if R is $*$ -simple, then it is either simple as a ring, or $R \cong Ex(K)$, where K and K^* are the only nontrivial proper ideals of R and $R^2 \neq 0$. From these facts it is possible to derive the previous theorem.

One of the principal applications of the above result is that it helps a lot in determining the lattice $L^{(n)}$ of all subvarieties of the (involution) ring variety $\mathcal{V}^{(n)}$ defined by $x^{n+1} = x$ for a given value of n . Towards this aim, the following observation is very useful. Let $\mathcal{V}_p^{(n)}$ denote the subvariety of $\mathcal{V}^{(n)}$ determined by $px = 0$ (formed by all members of the latter variety of characteristic p), and let $L_p^{(n)}$ be its lattice of subvarieties. Clearly, $\mathcal{V}_p^{(n)}$ is nontrivial if and only if $(p-1) \mid n$. Now if $\{p_1, \dots, p_k\}$ is the set of all prime numbers with this property, then it can be easily shown that the varieties $\mathcal{V}_{p_i}^{(n)}$, $1 \leq i \leq k$, are *independent*, which means that there is a term $t(x_1, \dots, x_k)$ such that the identity $t(x_1, \dots, x_k) = x_i$ holds in $\mathcal{V}_{p_i}^{(n)}$. If a variety is equal to the join of some of its independent subvarieties, it is usual in universal algebra to say that the variety under consideration decomposes into a *variatal product* of these subvarieties (cf. [74]). In our case, we write $\mathcal{V}^{(n)} = \mathcal{V}_{p_1}^{(n)} \otimes \dots \otimes \mathcal{V}_{p_k}^{(n)}$. It is well-known that variatal product decompositions induce direct decompositions of the lattice of subvarieties, thus we have

$$L^{(n)} \cong L_{p_1}^{(n)} \times \dots \times L_{p_k}^{(n)}.$$

Hence, the task of finding the lattice of varieties of rings (with involution) satisfying $x^{n+1} = x$ reduces to the same task in a fixed prime characteristic p , where $(p-1) \mid n$. This is just where Theorem 2.4.4 can be used, for it supplies the corresponding subdirectly irreducibles. It remains then to study their mutual relationships in order to obtain the exact list of varieties they generate.

This is just what have been done in the recent note [30]. Namely, let \mathcal{F}_p denote the set of all finite fields of characteristic p , while \mathcal{F}_p^* denotes the set of all (subdirectly irreducible) involution rings from the above theorem which are of characteristic p . Furthermore, write $R \hookrightarrow S$ if R embeds into S . This relation turns \mathcal{F}_p and \mathcal{F}_p^* into partially ordered sets. Clearly, $(\mathcal{F}_p, \hookrightarrow)$ is isomorphic to the divisibility order of natural numbers (as $GF(p^k)$ embeds into $GF(p^\ell)$ if and only if $k \mid \ell$), but it was shown in [30] that $(\mathcal{F}_p^*, \hookrightarrow)$ can be effectively described as well.

Now let $\mathcal{F}_p(n)$ ($\mathcal{F}_p^*(n)$) denote the set of those $GF(p^k)$ (and $GF^*(p^k)$ and $Ex(GF(p^k))$ in the involutorial case) for which $(p^k - 1) \mid n$. The main result of [30] is as follows.

Theorem 2.4.5. *Let $n \geq 1$ be an integer and p a prime such that $(p - 1) \mid n$. Then $L_p^{(n)}$ is isomorphic to the lattice of all ideals of the ordered set $(\mathcal{F}_p(n), \hookrightarrow)$ (resp. $(\mathcal{F}_p^*(n), \hookrightarrow)$).*

By the previous remarks, the finite partial orders from the above theorem turn out to be computable, which establishes an effective algorithm for constructing $L_p^{(n)}$, as required.

Bearing in mind Theorem 2.4.2, let us finish this survey by discussing the case $n = 6$ as an example, so that $(p - 1) \mid n$ (where p is a prime) if and only if $p \in \{2, 3, 7\}$.

When we consider ordinary rings, the situation is clear: for $p = 2$ we have two subdirectly irreducibles, $GF(2)$ and $GF(4)$, where $GF(2)$ embeds into $GF(4)$; for $p = 3$ we have $GF(3)$, and for $p = 7$ we have $GF(7)$. Thus, it is easy (using the above theorem) to conclude that there are 12 ring varieties satisfying $x^7 = x$, and that the lattice formed by them is the product of a three-element chain and the square of a two-element chain.

In the case of involution ring varieties, for each of $p = 3, 7$ we have two subdirectly irreducibles, so that $GF(3) \hookrightarrow Ex(GF(3))$ and $GF(7) \hookrightarrow Ex(GF(7))$, and both $L_3^{(6)}$ and $L_7^{(6)}$ are three-element chains. For $p = 2$, a routine calculation shows that $L_2^{(6)}$ is isomorphic to the lattice given in the following figure.

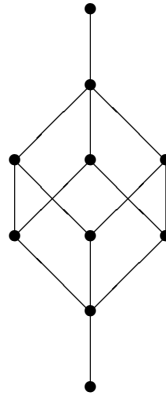


Figure 12. The lattice of all involution ring varieties satisfying $x^7 = x$ and $2x = 0$

Hence, we obtain exactly 90 varieties of involution rings satisfying $x^7 = x$. Only six of them have a special involution, cf. Theorem 2.4.3.

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