Solving the word problem of the O'Hare monoid far, far away from sweet home Chicago

Igor Dolinka

dockie@dmi.uns.ac.rs

Department of Mathematics and Informatics, University of Novi Sad

Winter One-Relator Workshop (WOW 2018)
Norwich, UK, 10 January 2018

Joint work in progress with Robert D. Gray

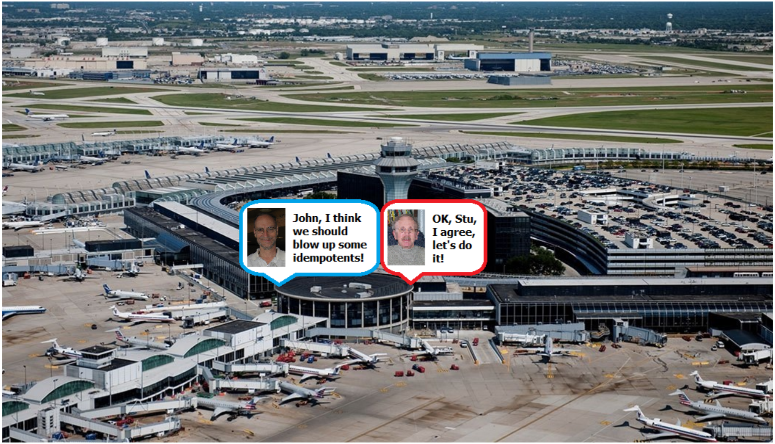
Oh, sorry, wrong pic...!

(Or maybe not that terribly wrong... ☺)

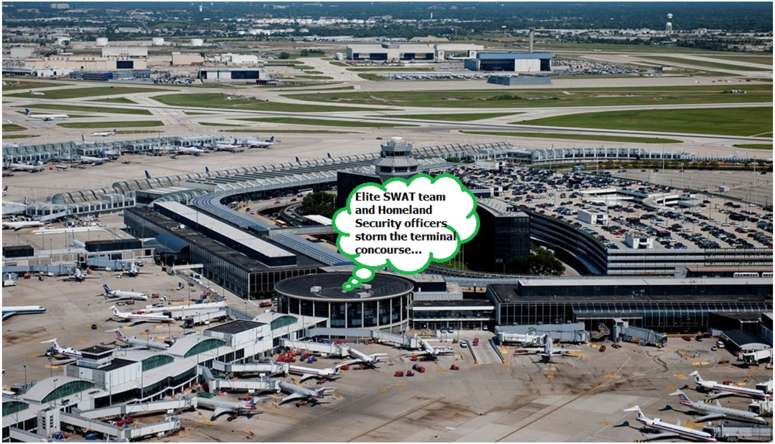
This is the Chicago O'Hare International Airport (IATA code: ORD)

It is the second busiest airport on the planet (after Atlanta).

Two mathematicians engage in a most lovely conversation



Some words have (dire) consequences



Escape

Fortunately, Elwood and Jake show up with their Bluesmobile just in time to save Stu and John from an awkward situation...

She caught the Katy (and left me a mule to ride)

The O'Hare inverse monoid is defined by the presentation

$$\mathrm{Inv}\langle\, a, b, c, d \mid abcdacdadabbcdacd = 1 \,\rangle.$$

Or, as we shall prefer it few minutes later,

$$\mathrm{Inv}\langle\, a, b, c, d \mid abcd \cdot acd \cdot ad \cdot abbcd \cdot acd = 1 \,\rangle$$

It was specifically designed by Margolis and Meakin (while waiting for a connecting flight at ORD) as an example of a special inverse one-relator monoid which eluded thus far the solution of the WP, exhibited interesting/strange geometric properties, and even threatened at some point a positive solution of the E-unitary conjecture...

But: what's the such big fuss about special inverse monoids in the first place?

The old landmark

Theorem (W. Magnus, 1932)

Every one-relator group has a solvable word problem.

Theorem (Adjan, 1966)

The word problem for $\mathrm{Mon}\langle\, A \mid u = v \,\rangle$ *is decidable if either:*

- *one of* u, v *is empty, or*
- *both* u, v *are non-empty, and have different initial letters and different terminal letters.*

Lallement (1977) and L. Zhang (1992) provided alternative proofs for the first case (of special monoids $\mathrm{Mon}\langle\, A \mid u = 1 \,\rangle$). The proof of Zhang is particularly compact and elegant.

(You better) Think

Adjan and Oganessian (1987): The word problem for one-relator monoids can be reduced to the special case of

$$\mathrm{Mon}\langle\, A \mid asb = atc \,\rangle$$

where $a, b, c \in A$, $b \neq c$ and $s, t \in A^*$ (and their duals). It is known that all such monoids are right (resp. left) cancellative.

Theorem (Ivanov, Margolis & Meakin, 2001)

If the word problem is decidable for all special inverse monoids $\mathrm{Inv}\langle\, A \mid w = 1 \,\rangle$ *– where w is a reduced word over $A \cup A^{-1}$ – then the word problem is decidable for every one-relator monoid.*

This holds basically because $M = \mathrm{Mon}\langle\, A \mid asb = atc \,\rangle$ embeds into $I = \mathrm{Inv}\langle\, A \mid asbc^{-1}t^{-1}a^{-1} = 1 \,\rangle$.

Changing the perspective

Note that the word $asbc^{-1}t^{-1}a^{-1}$ is always reduced, but not cyclically reduced.

Hence, studying the word problem for $\mathrm{Inv}\langle A \mid w = 1 \rangle$ where w is cyclically reduced might be more manageable.

Even though this case seems to have zero intersection with the one-relator monoid problem, it is still important to study in order to gain some understanding how the WP works for special one-relator inverse monoids.

The prefix monoid

For $M = \mathrm{Inv}\langle\, A \,|\, w = 1 \,\rangle$ consider its greatest group image $G = \mathrm{Gp}\langle\, A \,|\, w = 1 \,\rangle$.

Let P_w denote the submonoid of G generated by its elements represented by all the prefixes of w. This is the prefix monoid of G relative to w.

Theorem (Ivanov, Margolis & Meakin, 2001)

Let w be cyclically reduced. Then $\mathrm{Inv}\langle\, A \,|\, w = 1 \,\rangle$ has a soluble word problem provided that the membership problem for P_w in G is decidable.

This allows to solve the word problem of M for an array of various types of words $w \in (A \cup A^{-1})^+$.

A key ingredient: The E-unitary property

An inverse semigroup S is E-unitary if any of the equivalent conditions hold:

- For any $e \in E(S)$ and $x \in S$,
  $e \leq x$ (in the natural inverse semigroup order) $\Rightarrow x \in E(S)$.
- The minimum group congruence σ on S is idempotent-pure, which means that $E(S)$ constitutes a single σ-class.
- $\sigma = \sim$, where $\sim$ is the compatibility relation (defined by $a \sim b \Leftrightarrow a^{-1}b, ab^{-1} \in E(S)$).
- ...

A key ingredient: The E-unitary property

Theorem (Ivanov, Margolis & Meakin, 2001)

If w is cyclically reduced, then $M = \mathrm{Inv}\langle A \mid w = 1 \rangle$ is E-unitary.

This confirmed a conjecture by M, M & Stephen published way back in 1987.

In particular, this implies that U_M, the group of units of M, embeds into $G = \mathrm{Gp}\langle A \mid w = 1 \rangle$. In fact, its image is already contained in P_w (as the group of *its* units).

E-unitary non-examples:

- $\mathrm{Inv}\langle a, b, c, d \mid abc = 1,\ adc = 1 \rangle$.
- $\mathrm{Inv}\langle A \mid uvu^{-1} = 1 \rangle$ provided $u, v \in A^+$ have different terminal letters (so that uvu^{-1} is reduced as written).

Searching for simpler generators of P_w

A factorisation

$$w \equiv \beta_1 \cdots \beta_k$$

is called unital if all β_i represent elements of U_M, where $M = \mathrm{Inv}\langle\, A \,|\, w = 1 \,\rangle$. Then it is not difficult to show

Lemma

P_w is generated by $\bigcup_{i=1}^{k} \mathrm{pref}(\beta_i)$, i.e. by the elements of $G = \mathrm{Gp}\langle\, A \,|\, w = 1 \,\rangle$ represented by prefixes of individual 'invertible factors' β_i.

In fact, for any factorisation $w \equiv \beta_1 \cdots \beta_k$ we can consider the submonoid of G

$$M(\beta_1, \ldots, \beta_k) = \left\langle \bigcup_{i=1}^{k} \mathrm{pref}(\beta_i) \right\rangle \supseteq P_w.$$

If $=$ holds, we say that the considered factorisation is conservative.

Searching for simpler generators of P_w

So, the previous lemma reads as:

Lemma

Every unital factorisation of w is conservative.

However,

Lemma (ID & RDG, 2017)

If $\mathrm{Inv}\langle A \mid w = 1 \rangle$ is E-unitary (e.g. if w is cyclically reduced), then every conservative factorisation of w is unital.

Theorem (ID & RDG, 2017)

There is a (unique) finest conservative factorisation $w \equiv \beta_1 \cdots \beta_k$ of w. In the E-unitary case,

$$U_M = \langle \beta_1, \ldots, \beta_k \rangle$$

Gimme some lovin'

Back to the O'Hare inverse monoid. Recall, this is given by

$$\mathsf{Inv}\langle\, a, b, c, d \mid abcdacdadabbcdacd = 1 \,\rangle.$$

I'd like to convince you that

$$w = \underbrace{abcd}_{\alpha} \cdot \underbrace{acd}_{\beta} \cdot \underbrace{ad}_{\gamma} \cdot \underbrace{abbcd}_{\delta} \cdot \underbrace{acd}_{\beta}$$

is the finest conservative/unital factorisation of the O'Hare word w.

First I am going to show that it is a) unital, and then that it is b) finest. For each of these statements I am going to show you two proofs: one 'geometric', and one 'combinatorial'.

Stephen's procedure

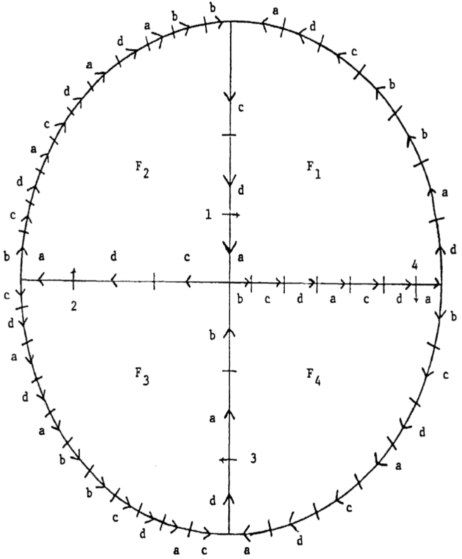
J. B. Stephen ('Presentations of inverse monoids', JPAA, 1990) gives an effective procedure which results (at ∞) in the Schützenberger graph of an inverse monoid presentation = the Cayley graph of the monoid restricted to right invertible elements (aka the $\mathscr{R}$-class of 1).

Roughly, in the case of $\mathrm{Inv}\langle\, A \mid w = 1 \,\rangle$ it consists of two operations:

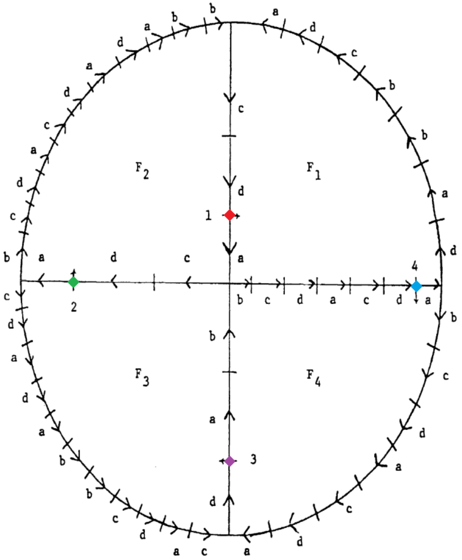
- add ('sew') cycle labelled by w at any vertex constructed so far;
- 'fold' – identify outgoing/incoming edges from/to a vertex labelled by the same letter.

Any graph obtained after a finite number of sewings+foldings is called a finite approximation of the Schützenberger graph in question, and it represents a particular piece of that graph.

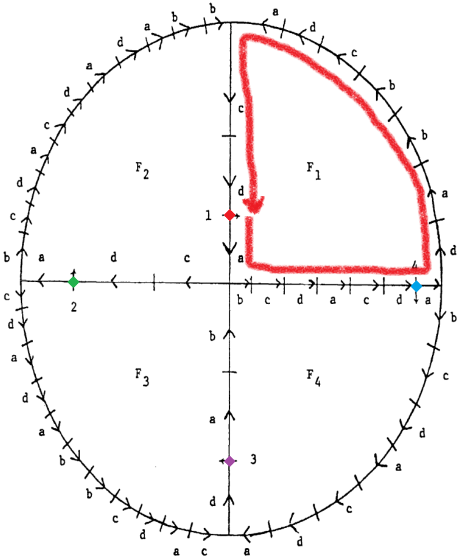
Shake your tail feather



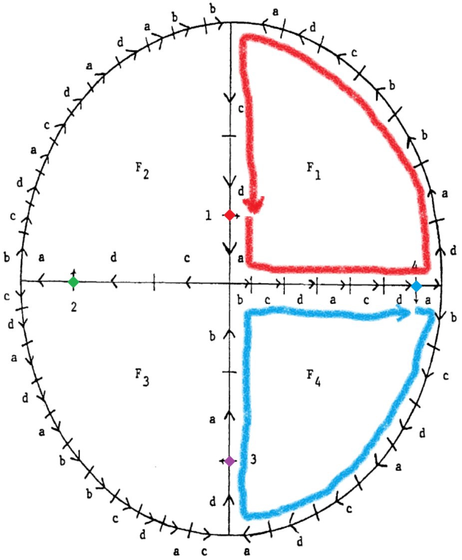
Shake your tail feather



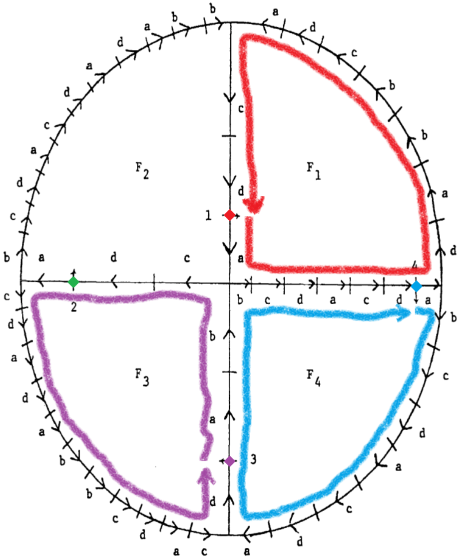
Shake your tail feather



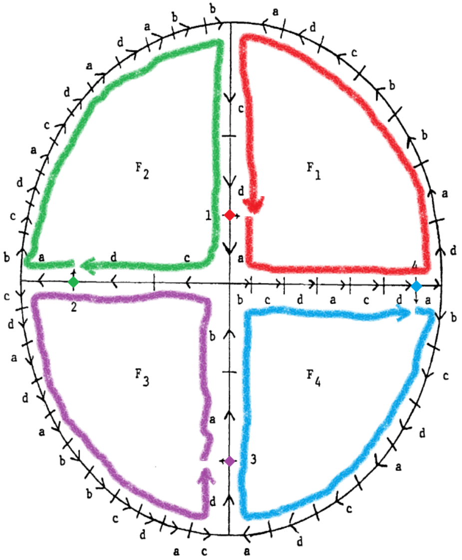
Shake your tail feather



Shake your tail feather



Shake your tail feather



Check, please!

- The original relation:
$$abcd \cdot acd \cdot ad \cdot abbcd \cdot acd = 1$$
- The red cycle from the blue initial vertex:
$$ad \cdot abbcd \cdot acd \cdot abcd \cdot acd = 1$$
- The blue cycle from the violet initial vertex:
$$abbcd \cdot acd \cdot abcd \cdot acd \cdot ad = 1$$
- The violet cycle from the green initial vertex:
$$acd \cdot ad \cdot abbcd \cdot acd \cdot abcd = 1$$
- The green cycle from the red initial vertex:
$$acd \cdot abcd \cdot acd \cdot ad \cdot abbcd = 1$$

So, each of $abcd$, acd, ad, $abbcd$ is both right and left invertible.

Invertible pieces of w reloaded

Lemma

Let $u \in (A \cup A^{-1})^*$ be any word representing a right invertible element of $M = \mathrm{Inv}\langle\, A \,|\, w = 1 \,\rangle$, and let $\overline{u}$ be the (free-group-)reduced form of u. Then $u = \overline{u}$ holds in M.

So, since

$$\beta = \alpha\delta^{-1}\alpha = (\alpha\beta)(\delta\beta)^{-1}\alpha$$

holds in $FG(A)$ $\Rightarrow$ it also holds in M $\Rightarrow$ β is (right) invertible. Similarly, $(\alpha\beta\gamma\delta)^{-1} = \beta(\alpha\beta\gamma\delta\beta)^{-1}$ holding in $FG(A)$ $\Rightarrow$ $\alpha\beta\gamma\delta$ is (left) invertible.

In a similar fashion we obtain that $\alpha\beta\gamma$, $\alpha\beta$ and α are invertible, and so are γ and δ.

Finest unital factorisation – Take 1

An easy (inductive) analysis of the Stephen procedure for the O'Hare monoid shows that the initial vertex (corresponding to $1 \in M$) is incident with precisely two edges: an outgoing edge labelled a and an incoming edge labelled d.

Hence, any word representing a right invertible element of M must begin with either a or d^{-1}. Analogously, any word representing a left invertible element of M must end with either a^{-1} or d.

It follows immediately that there can be no unital factorisation of the O'Hare word finer than

$$abcd \cdot acd \cdot ad \cdot abbcd \cdot acd.$$

Finest unital factorisation – Take 2

Deductions of the type:

$$ab \text{ invertible} \Rightarrow bcd \text{ invertible (because of } abbcd\text{)}$$
$$\Rightarrow a \text{ invertible (because of } abcd\text{)} \Rightarrow d \text{ invertible (because of } ad\text{)}$$
$$\Rightarrow c \text{ invertible (becuase of } acd\text{)} \Rightarrow b \text{ invertible (because of } abcd\text{)}$$

All possible cases lead to the same conclusion: if there would be a finer unital factorisation $\Rightarrow$ all of a, b, c, d would be invertible and M would be a group.

However, this is not the case (thank you, Nik!) as M admits a homomorphism onto the bicyclic monoid $B = \text{Inv}\langle\, x, y \mid xy = 1\,\rangle$ via $a \mapsto x$, $b, c \mapsto 1$, $d \mapsto y$ (taking the O'Hare word to $xyxyxyxy$, a relator in B).

Corollary

$U_M = \langle abcd, acd, ad, abbcd\rangle = \langle aba^{-1}, aca^{-1}, ad\rangle$

(even as a monoid).

(Dancin' to the) Jailhouse rock

$$G = \mathrm{Gp}\langle\, a, b, c, d \mid abcdacdadabbcdacd = 1 \,\rangle$$
$$= \mathrm{Gp}\langle\, a, b, c, d, x, y, z \mid x = aba^{-1},\ y = aca^{-1},\ z = ad,\ xyzyzzxxyzyz = 1 \,\rangle$$
$$= \mathrm{Gp}\langle\, a, b, c, d, x, y, z \mid b = a^{-1}xa,\ c = a^{-1}za,\ d = a^{-1}z,\ xyzyzzxxyzyz = 1 \,\rangle$$
$$= \mathrm{Gp}\langle\, a, x, y, z \mid xyzyzzxxyzyz = 1 \,\rangle$$

$$P_w = \mathrm{Mon}\langle a, ab, abc, abcd, ac, acd, ad, abb, abbc, abbcd\rangle$$
$$= \mathrm{Mon}\langle a, aba^{-1}, aca^{-1}, ad\rangle = \mathrm{Mon}\langle a, x, y, z\rangle$$

So, the prefix monoid P_w of G w.r.t. the O'Hare presentation is in fact the positive part/submonoid of G w.r.t. the new presentation $\langle\, a, x, y, z \mid xyzyzzxxyzyz = 1 \,\rangle$!!!

The band! The band!! I can see the light!!!

Theorem (Blues Brothers, 2017)

Let u be a strictly positive word over A. Then the positive part of $\mathrm{Gp}\langle\, A \mid u = 1\,\rangle$ *has a decidable membership problem.*

Proof sketch.

Let $C \subseteq A$ be the set of all letters that actually appear in u, and let $B = A \setminus C$. Then $G = FG(B) * \mathrm{Gp}\langle\, C \mid u = 1\,\rangle$. As the inverse of any letter from C can be expressed in G by a positive word over C, $\mathrm{Gp}\langle\, C \mid u = 1\,\rangle$ coincides with its postive part. Thus the positive part of G is $B^* * \mathrm{Gp}\langle\, C \mid u = 1\,\rangle$ (here $*$ refers to the monoid free product). So, a word v over $A \cup A^{-1}$ represents an element from the positive part of G if and only if $\overline{v}$ fails to contain any letter from B^{-1}. □

This implies that the prefix monoid P_w of the O'Hare group has a decidable membership problem. By the Ivanov-Margolis-Meakin Theorem, the WP of the O'Hare inverse monoid is soluble.

Everybody needs somebody (or some problem) to love

- Can we at least prove (via the prefix monoid method) that $\mathrm{Inv}\langle A \mid w = 1 \rangle$ has a solvable WP if w is a positive word (i.e. $\in A^+$)? Do clever changes of generators + Tietze transformations suffice? Some weaker generalisations?
- We have seen that for E-unitary $M = \mathrm{Inv}\langle A \mid w = 1 \rangle$ we have

  $$U_M = U_{P_w} \leq P_w \leq G = \mathrm{Gp}\langle A \mid w = 1 \rangle.$$

  It would be worthwhile to study the situation $H \leq S \leq G$ where G, H are groups, G is one-relator, and S is a monoid (then S is a union of some cosets of H). Can we 'decompose' the membership problem of S in G to the membership problem of H in G and an additional condition on the cosets involved?
- This points to the old & famous problem: the generalised WP for one-relator groups. In particular, what about the subgroups generated by $\alpha_1, \ldots, \alpha_k$ for an arbitrary factorisation $\alpha_1 \cdots \alpha_k$ of the (positive) relator w?

THANK YOU!

Questions and comments to:

dockie@dmi.uns.ac.rs

Further information may be found at:

http://people.dmi.uns.ac.rs/$\sim$dockie