The word problem for one-relator inverse monoids: new developments

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Algebra szeminárium

Szeged, 2019. december 4.

Starring

Robert D. Gray (UEA, Norwich)

Lt. Col. Frank Slade (US Army)

The word problem

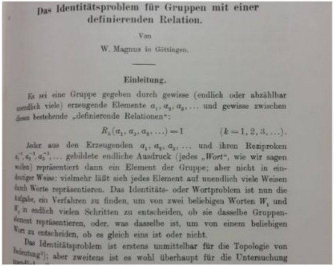
Assume G is a group (finitely) generated by X.

Let $\overline{X} = X \cup X^{-1}$ be a 'doubled' alphabet, and let $\pi : \overline{X}^* \to G$ be the cannonical homomorphism (sending each word w to the element of G represented by w).

The word problem for G is the following algorithmic question.
INPUT: A word $w \in \overline{X}^*$.
QUESTION: Does $w\pi = 1$ hold in G?

Similarly, one defines the word problem for finitely generated monoids / inverse monoids; in that case the input requires two words u, v and the problem asks if $u\pi = v\pi$ holds in the corresponding monoid.

The beginning of the story: back to the Great Depression



Das Identitätsproblem für Gruppen mit einer definierenden Relation.

Von

W. Magnus in Göttingen.

Einleitung.

Es sei eine Gruppe gegeben durch gewisse (endlich oder abzählbar unendlich viele) erzeugende Elemente $a_1, a_2, a_3, \ldots$ und gewisse zwischen diesen bestehende „definierende Relationen":

$$R_k(a_1, a_2, a_3, \ldots) = 1 \qquad (k = 1, 2, 3, \ldots).$$

Jeder aus den Erzeugenden $a_1, a_2, a_3, \ldots$ und ihren Reziproken $a_1^{-1}, a_2^{-1}, a_3^{-1}, \ldots$ gebildete endliche Ausdruck (jedes „*Wort*", wie wir sagen wollen) repräsentiert dann ein Element der Gruppe; aber nicht in eindeutiger Weise; vielmehr läßt sich jedes Element auf unendlich viele Weisen durch Worte repräsentieren. Das Identitäts- oder Wortproblem ist nun die Aufgabe, ein Verfahren zu finden, um von zwei beliebigen Worten W_1 und W_2 in endlich vielen Schritten zu entscheiden, ob sie dasselbe Gruppenelement repräsentieren, oder, was dasselbe ist, um von einem beliebigen Wort zu entscheiden, ob es gleich eins ist oder nicht.

Das Identitätsproblem ist erstens unmittelbar für die Topologie von Bedeutung[1]); aber zweitens ist es wohl überhaupt für die Untersuchung

Gimme the good ol' classics

Theorem (W. Magnus, 1932)

Every one-relator group has decidable word problem.

Theorem (Magnus, 1930, "Der Freiheitssatz")

If $w \in \overline{X}^*$ *is cyclically reduced and* $A \subset X$ *is such that at least one letter from* $X \setminus A$ *appears in* w*, then the subgroup of* $\mathrm{Gp}\langle X \mid w = 1 \rangle$ *generated by* A *is free.*

$\Longrightarrow$ The Magnus method: each one-relator group embeds into an HNN extension of a one-relator group with a shorter relator.

"Da sind Sie also blind gegangen!"

Max Dehn (Magnus' PhD advisor)

Theorem (Shirshov, 1962)

Every one-relator Lie algebra has decidable word problem.

The one-relator monoid Riddle (aka Voldemort)

Open Problem (still! – as of 2019)

Is the word problem decidable for all one-relator monoids $\mathrm{Mon}\langle X \mid u = v\rangle$?

Theorem (Adjan, 1966)

The word problem for $\mathrm{Mon}\langle X \mid u = v\rangle$ *is decidable if either:*

- *one of* u, v *is empty (e.g.* $u = 1$ *– special monoids), or*
- *both* u, v *are non-empty, and have different initial letters and different terminal letters.*

Lallement (1977) and L. Zhang (1992) provided alternative proofs for the result about special monoids. The proof of Zhang is particularly compact and elegant.

The (French) connection

Adyan and Oganessyan (1987): The word problem for one-relator monoids can be reduced to the special case of

$$\mathrm{Mon}\langle X \mid asb = atc \rangle$$

where $a, b, c \in X$, $b \neq c$ and $s, t \in X^*$ (and their duals).

So, where do (one-relator) inverse monoids come into the picture???

Theorem (Ivanov, Margolis & Meakin, 2001)

If the word problem is decidable for all special inverse monoids $\mathrm{Inv}\langle X \mid w = 1 \rangle$ *– where* w *is a reduced word over* $\overline{X}$ *– then the word problem is decidable for every one-relator monoid.*

This holds basically because $M = \mathrm{Mon}\langle X \mid asb = atc \rangle$ embeds into $I = \mathrm{Inv}\langle X \mid asbc^{-1}t^{-1}a^{-1} = 1 \rangle$.

The plot thickens

	$\mathrm{Gp}\langle X \mid w = 1\rangle$	$\mathrm{Mon}\langle X \mid w = 1\rangle$	$\mathrm{Inv}\langle X \mid w = 1\rangle$
decidable WP	✓ (Magnus, 1932)	✓ (Adjan, 1966)	✗ (Gray, 2019)

Conjecture (Margolis, Meakin, Stephen, 1987)

Every inverse monoid of the form $\mathrm{Inv}\langle X \mid w = 1\rangle$ has decidable word problem.

Theorem (RD Gray, 2019)

There exists a one-relator inverse monoid $\mathrm{Inv}\langle X \mid w = 1\rangle$ *with undecidable word problem.*

Inverse monoid basics

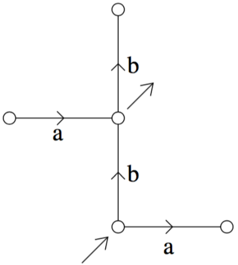
Inverse monoid = a monoid M such that for every $a \in M$ there is a unique $a^{-1} \in M$ such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$.

Inverse monoids form a variety of unary monoids defined by

$$xx^{-1}x = x, \quad (x^{-1})^{-1} = x,$$

$$(xy)^{-1} = y^{-1}x^{-1}, \quad xx^{-1}yy^{-1} = yy^{-1}xx^{-1}.$$

Free inverse monoid $FIM(X)$: Munn, Scheiblich (1973/4)



Elements of $FIM(X)$ are represented as Munn trees = birooted finite subtrees of the Cayley graph of $FG(X)$. The Munn tree on the left illustrates

$$aa^{-1}bb^{-1}ba^{-1}abb^{-1} = bbb^{-1}a^{-1}ab^{-1}aa^{-1}b.$$

A key ingredient: The E-unitary property

An inverse semigroup S is E-unitary if any of the equivalent conditions hold:

- For any $e \in E(S)$ and $x \in S$,
  $e \leq x$ (in the natural inverse semigroup order) $\Rightarrow x \in E(S)$.
- The minimum group congruence σ on S is idempotent-pure, which means that $E(S)$ constitutes a single σ-class. (In particular, if $\theta : M = \mathrm{Inv}\langle X \mid w = 1 \rangle \to \mathrm{Gp}\langle X \mid w = 1 \rangle$ is the natural homomorhpism then we require $1\theta^{-1} = E(M)$.)
- $\sigma = \sim$, where $\sim$ is the compatibility relation (defined by $a \sim b \Leftrightarrow a^{-1}b, ab^{-1} \in E(S)$).
- ...

Theorem (Ivanov, Margolis & Meakin, 2001)

If w is cyclically reduced, then $M = \mathrm{Inv}\langle X \mid w = 1 \rangle$ is E-unitary.

The role of the prefix monoid

For $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ consider its maximum group image $G = \mathrm{Gp}\langle X \mid w = 1\rangle$.

Let P_w denote the submonoid of G generated by its elements represented by all the prefixes of w. This is the prefix monoid of G (relative to w, as P_w depends on the presentation for G). The prefix membership problem for G (defined by a certain one-relator presentation) is the membership problem for P_w within G.

Theorem (Ivanov, Margolis & Meakin, 2001)

Assume that the prefix membership problem is decidable for $G = \mathrm{Gp}\langle X \mid w = 1\rangle$. If, in addition, $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ is E-unitary then the word problem for M is decidable.

This allows to solve the word problem of M for an array of various types of words $w \in \overline{X}^+$.

Gray (2019), Inventiones Mathematicae (to appear)

Theorem A

There is a one-relator inverse monoid $\mathrm{Inv}\langle X \mid w = 1\rangle$ *with undecidable word problem.*

Theorem B

There exist one-relator groups with undecidable submonoid membership problem.

The spice of life: right-angled Artin groups (RAAGs)

(Finite) graph $\Gamma \longrightarrow$ right-angled Artin group $A(\Gamma)$:

$$\mathrm{Gp}\langle V(\Gamma) \mid uv = vu \text{ for all } \{u, v\} \in E(\Gamma)\rangle$$

Properties:

- If Δ is an induced subgraph of Γ then $A(\Delta) \hookrightarrow A(\Gamma)$;
- An isomorphism $\psi : \Delta_1 \to \Delta_2$ of induced subgraphs of Γ gives rise to an isomorphism of subgroups $A(\Delta_1) \to A(\Delta_2)$ of $A(\Gamma)$;
- This, in turn, defines an HNN extension $A(\Gamma, \psi)$ which naturally embeds $A(\Gamma)$.

The importance of choosing the right path

Consider the path graph P_4: 

The isomorphism ψ of subpaths $a \mapsto b$, $b \mapsto c$, $c \mapsto d$ defines an HNN extension $A(P_4, \psi)$ of $A(P_4)$ over $A(P_3)$.

After several rounds of Tietze transformations, it turns out that $A(P_4, \psi)$ is a one-relator group (!), namely

$$\mathrm{Gp}\langle a, t \mid atat^{-1}a^{-1}ta^{-1}t^{-1} = 1\rangle.$$

So, the RAAG $A(P_4)$ embeds into this one-relator group.

Theorem (Lohrey & Steinberg, 2008)

There exists a (fixed) finitely generated submonoid T of $A(P_4)$ with undecidable membership problem.

... $\Longrightarrow$ Theorem B.

A general construction

Let $r, w_1, \dots, w_k \in \overline{X}^*$ be arbitrary, where $X = \{a_1, \dots, a_n\}$.
Set e to be the word

$$a_1 a_1^{-1} \dots a_n a_n^{-1} (tw_1 t^{-1})(tw_1^{-1} t^{-1}) \dots (tw_k t^{-1})(tw_k^{-1} t^{-1}) a_n^{-1} a_n \dots a_1^{-1} a_1.$$

This is a Dyck word (reduces in the free group to 1) $\Leftrightarrow$ represents an idempotent of $FIM(X)$.

Let

$$M = \mathrm{Inv}\langle X, t \mid er = 1\rangle = \mathrm{Inv}\langle X, t \mid e = 1, r = 1\rangle$$

Proposition (!!!)

Let $T = \mathrm{Mon}\langle w_1, \dots, w_k\rangle \leq G = \mathrm{Gp}\langle X \mid r = 1\rangle$. *Then for all* $u \in \overline{X}^*$ *we have*

$$u \in T \leq G \iff (tut^{-1})(tu^{-1}t^{-1}) = 1 \text{ in } M.$$

Touche!

Theorem

If $M = \mathrm{Inv}\langle X \mid er = 1\rangle$ *has decidable word problem then the membership problem for* T *in* $G = \mathrm{Gp}\langle X \mid r = 1\rangle$ *is decidable.*

Proof of Theorem A:

- Take $G = \mathrm{Gp}\langle a, z \mid azaz^{-1}a^{-1}za^{-1}z^{-1} = 1\rangle$;
- Take the Lohrey-Steinberg finitely generated, non-recursive submonoid T of $A(P_4)$;
- Assume that its image T' in G is generated by $w_1, \dots, w_k \in \{a, z, a^{-1}, z^{-1}\}^*$;
- Let
  $$e = aa^{-1}zz^{-1}(tw_1t^{-1})(tw_1^{-1}t^{-1})\dots(tw_kt^{-1})(tw_k^{-1}t^{-1})z^{-1}za^{-1}a;$$
- Then by the above theorem
  $$M = \mathrm{Inv}\langle a, z, t \mid eazaz^{-1}a^{-1}za^{-1}z^{-1} = 1\rangle$$
  has undecidable WP. Q.E.D.

Now, let's keep it positive!

Remember: in the E-unitary case, solving the WP for $\mathrm{Inv}\langle X \mid w = 1\rangle$ is the same as solving the prefix membership problem for $\mathrm{Gp}\langle X \mid w = 1\rangle$.

A sampler of positive results

Inv$\langle X \mid w = 1\rangle$ is proved to have decidable WP when...

- w is a Dyck word = an idempotent of $FIM(X)$ (Birget, Margolis, Meakin, 1993);
- w-strictly positive case (Ivanov, Margolis, Meakin, 2001);
- some Adjan and Baumslag-Solitar types (Margolis, Meakin, Šunić, 2005);
- w is a sparse word (Hermiller, Lindblad, Meakin, 2010);
- some small cancellation conditions (A. Juhász, 2012, 2014).

And now...

- IgD & RD Gray, 2019+

A glimpse into the toolbox (1)

Theorem (Benois, 1969)

Every finitely generated free group has decidable rational subset membership problem. Furthermore, rational subsets of (finitely generated) free groups are closed for intersection and complement.

Let $M = \mathrm{Inv}\langle X \mid w = 1\rangle$. A factorisation

$$w \equiv w_1 \dots w_m$$

is unital (w.r.t. M) if each piece w_i represents an invertible element (a unit) of M.

Lemma

$P_w \leq G = \mathrm{Gp}\langle X \mid w = 1\rangle$ *is generated by* $\bigcup_{i=1}^{m} \mathrm{pref}(w_i)$.

A glimpse into the toolbox (2)

In fact, for any factorisation $w \equiv w_1 \cdots w_m$ we can consider the submonoid of G

$$M(w_1, \ldots, w_m) = \left\langle \bigcup_{i=1}^{m} \mathrm{pref}(w_i) \right\rangle \supseteq P_w.$$

If $=$ holds, we say that the considered factorisation is conservative.

Theorem

(i) *Any unital factorisation is conservative.*

(ii) *If* $\mathrm{Inv}\langle X \mid w = 1\rangle$ *is* E*-unitary then every conservative factorisation if unital.*

Theorem A

Let $G = B *_A C$, where A, B, C are finitely generated groups such that both B, C have decidable word problems, and the membership problem for A in both B and C is decidable. Let M be a submonoid of G such that the following conditions hold:

(i) $A \subseteq M$;

(ii) both $M \cap B$ and $M \cap C$ are finitely generated and
$$M = \mathrm{Mon}\langle (M \cap B) \cup (M \cap C) \rangle;$$

(iii) the membership problem for $M \cap B$ in B is decidable;

(iv) the membership problem for $M \cap C$ in C is decidable.

Then the membership problem for M in G is decidable.

Theorem B

$H \leq G$ closed for rational intersections: $R \in \mathrm{Rat}(G) \Longrightarrow R \cap H \in \mathrm{Rat}(G)$

Effectively closed for rational intersections: algorithmic aspects

NB. Herbst (1991): $R \subseteq H,\ R \in Rat(G) \Longleftrightarrow R \in \mathrm{Rat}(H)$

IgD + RDG: an effective version of this result

Theorem

Let $G = B *_A C$, where A, B, C are finitely generated groups. Let M be a submonoid of G such that both $M \cap B$ and $M \cap C$ are finitely generated and $M = \mathrm{Mon}\langle (M \cap B) \cup (M \cap C)\rangle$. Assume further that the following conditions hold:

(i) *B and C have decidable rational subset membership problems;*

(ii) *$A \leq B$ is effectively closed for rational intersections;*

(iii) *$A \leq C$ is effectively closed for rational intersections.*

Then the membership problem for M in G is decidable.

Applications (1): Unique marker letter theorem

Theorem

Let $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ and assume that $w \equiv u(w_1, \ldots, w_k)$ determines a conservative factorisation of w. Furthermore, suppose that for all $1 \leq i \leq k$ there is a letter $x_i \in X$ which appears exactly once in w_i and does not appear in any w_j, $j \neq i$. Then G has decidable prefix membership problem.

So, if w satisfies these conditions and $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ is E-unitary then M has decidable word problem.

Example

The group

$$G = \mathrm{Gp}\langle a, b, x, y \mid (axb)(ayb)(ayb)(axb)(ayb)(axb) = 1\rangle$$

has decidable prefix membership problem $\Longrightarrow$ the inverse monoid

$$M = \mathrm{Inv}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle$$

has decidable WP.

Applications (2): The (in)famous O'Hare monoid

Constructed by M&M while waiting for a connecting flight at the O'Hare Int. Airport, Chicago, sometime in the 1980s:

$$\mathrm{Inv}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$$

It was set up as a particularly hard test-example for the E-unitary result, and it was unknown until now whether its WP is decidable.

Proposition

Let $M = \mathrm{Inv}\langle Y, a, d \mid (au_{i_1}d)\dots(au_{i_m}d) = 1\rangle$, with a, d not appearing in u_{i_j}'s. Assume further that:

- *some of the u_{i_j}'s is the empty word;*
- *for each $x \in Y$ we have $x \equiv \mathrm{red}(u_{i_r}u_{i_s}^{-1})$ for some r, s;*
- *each $au_{i_j}d$ represents a unit of M.*

Then $G = \mathrm{Gp}\langle Y, a, d \mid (au_{i_1}d)\dots(au_{i_m}d) = 1\rangle$ has decidable prefix membership problem, and so M as decidable WP.

Applications (3): Disjoint alphabets theorem

Theorem

Let $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ where w is cyclically reduced. Again, assume that $w \equiv u(w_1, \ldots, w_k)$ determines a conservative factorisation of w. If $i \neq j \Rightarrow$ (w_i and w_j have no letters in common) then G has decidable prefix membership problem and thus $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ has decidable WP.

Example

The group

$$G = \mathrm{Gp}\langle a, b, c, d \mid (abab)(cdcd)(abab)(cdcd)(cdcd)(abab) = 1\rangle$$

has decidable prefix membership problem $\Longrightarrow$ the inverse monoid

$$M = \mathrm{Inv}\langle a, b, x, y \mid ababcdcdababcdcdcdcdabab = 1\rangle$$

has decidable WP.

Applications (4): Cyclically pinched presentations

Theorem

The prefix membership problem is decidable for one-relator groups defined by cyclically pinched presentations:

$$G = \mathrm{Gp}\langle X \cup Y \mid uv^{-1} = 1 \rangle$$

where u, v are reduced words over disjoint X, Y, respectively.

Example

This implies decidability of the prefix membership problem for surface groups:

- orientable (known)

$$\mathrm{Gp}\langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid [a_1, b_1] \ldots [a_n, b_n] = 1 \rangle,$$

- non-orientable (new)

$$\mathrm{Gp}\langle a_1, \ldots, a_n \mid a_1^2 \ldots a_n^2 = 1 \rangle.$$

Theorem C

Let $G^* = G*_{t,\phi:A\to B}$ be an HNN extension of a finitely generated group G such that A, B are also finitely generated. Assume that G has decidable word problem and that the membership problems of A and B in G are decidable. Let M be a submonoid of G^* such that the following conditions hold:

(i) $A \cup B \subseteq M$;

(ii) $M \cap G$ is finitely generated, and
$$M = \mathrm{Mon}\langle (M \cap G) \cup \{t, t^{-1}\}\rangle;$$

(iii) the membership problem for $M \cap G$ in G is decidable.

Then the membership problem for M in G^* is decidable.

Theorem D

Let $G^* = G*_{t,\phi:A\to B}$ be an HNN extension of a finitely generated group G such that A, B are also finitely generated. Assume that the following conditions hold:

(i) the rational subset membership problem is decidable in G;

(ii) $A \leq G$ is effectively closed for rational intersections.

Then for any finite $W_0, W_1, \dots, W_d, W_1', \dots, W_d' \subseteq G$, $d \geq 0$, the membership problem for

$$M = \mathrm{Mon}\langle W_0 \cup W_1 t \cup W_2 t^2 \cup \cdots \cup W_d t^d \cup tW_1' \cup \cdots \cup t^d W_d' \rangle$$

in G^* is decidable.

Applications (5): Exponent sum zero theorem

Let $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ and assume some $t \in X$ has exponent sum zero in w.

- By general theory (Lyndon, McCool, Moldavanskiĭ,...) $\Rightarrow$ G is an HNN extension of a one-relator group H:
  - take w, replace each letter x by x_{-i} where i is the exponent sum of t in the corresponding prefix, get $\rho_t(w)$;
  - let Ξ_w be the alphabet of this new word;
  - let A be the subgroup of $H = \mathrm{Gp}\langle \Xi_w \mid \rho_t(w) = 1\rangle$ generated by the subset of Ξ_w obtained by removing each letter's "maximum index version";
- Assume further that:
  - $\rho_t(w)$ is cyclically reduced;
  - the rational subset membership problem is decidable in H;
  - $A \leq H$ is effectively closed for rational intersections.

Then G has decidable prefix membership problem.

Applications (5): Exponent sum zero theorem

Examples

- $\mathrm{Gp}\langle a, b, c, t \mid t^{-1}atcbt^{-2}at^2cbt^{-3}at^3c = 1\rangle, \ldots;$
- large classes of Adjan-type presentations not covered by previous results;
- conjugacy pinched presentations $\mathrm{Gp}\langle X, t \mid t^{-1}utv^{-1} = 1\rangle$ where u, v are reduced words over X;
  - in particular, Baumslag-Solitar groups
  $$B(m, n) = \mathrm{Gp}\langle a, b \mid b^{-1}a^m ba^{-n} = 1\rangle;$$
- $\ldots$

The grand finale and an open problem

By modifying slightly the ideas from [Gray, 2019], we obtain

Theorem

There exists a reduced word w over a 3-letter alphabet X such that $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has undecidable prefix membership problem.

Open Problem

Characterise the words $w \in \overline{X}^*$ such that the prefix membership problem for $\mathrm{Gp}\langle X \mid w = 1\rangle$ is decidable. In particular, what about cyclically reduced words?

KÖSZÖNÖM A FIGYELMET!

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