Universal locally finite maximally homogeneous semigroups

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The 27th NBSAN Meeting

A conference honouring the 70th birthday of Laci Márki

York, UK, 8 January 2018

Isten éltesen, Laci! Boldog évfordulót!

Joint work with *Monsieur le docteur* Robert D'Gray

Special thanks go to UEA campus bunnies for creating a thoroughly pleasant working environment! ☺

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- Universal: It containeth a copy of every finite group as a subgroup.
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- Homogeneous: Every two isomorphic subgroups A, B were conjugate. In fact, any isomorphism $\phi : A \to B$ was a restriction of some inner automorphism (of the whole beast).

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Naaah, I'm just kidding you folks ☺, it was constructed by Philip Hall in 1959 in his beautiful paper *Some constructions for locally finite groups*.

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Regardless of the initial group G_0, the direct limit of this chain is, up to isomorphism, one and the same countable group $\mathcal{U}$. It is universal (for finite groups), locally finite, and homogeneous; moreover, it is the unique countable group with these properties.

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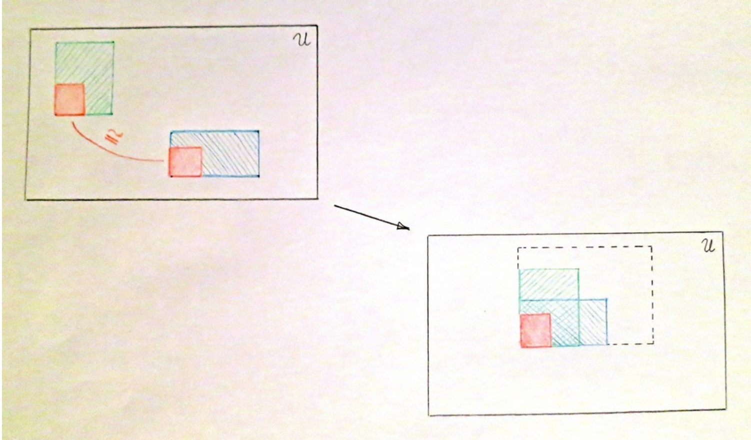
Manfred Droste at AAA83, March 2012, Novi Sad

Is there a countable universal
locally finite homogeneous (inverse) semigroup?

The keyword: Amalgamation



Amalgamation and the Fraïssé Theorem

An amalgam in a class $\mathcal{K}$ of first-order structures is an ensemble (S, A, B, f_1, f_2) consisting of structures $S, A, B \in \mathcal{K}$ along with two embeddings $f_1 : S \to A$ and $f_2 : S \to B$.

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An amalgam (S, A, B, f_1, f_2) in $\mathcal{K}$ is embeddable into a structure C if there are embeddings $g_1 : A \to C$ and $g_2 : B \to C$ such that $f_1 g_1 = f_2 g_2$, that is:

$$\begin{tikzcd} & A \arrow[dr, dashed, "g_1"] & \\ S \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & C \\ & B \arrow[ur, dashed, "g_2"'] & \end{tikzcd}$$

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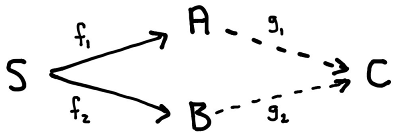

$\mathcal{K}$ has the amalgamation property (AP) if any amalgam in $\mathcal{K}$ can be embedded into some structure $C \in \mathcal{K}$.

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So, in model-theoretical terminology, Hall's universal group $\mathcal{U}$ is the Fraïssé limit of the class of all finite groups.

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Conclusion: There is no countable universal locally finite homogeneous semigroup. There is no such inverse semigroup either.

So...(?)

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A finite (inverse) semigroup S is an amalgamation base for the class of all finite (inverse) semigroups if every amalgam based on S (i.e. an amalgam of the form $(S, \ldots, \ldots, \ldots, \ldots)$) embeds into some finite (inverse) semigroup.

Amalgamation bases

- T.E.Hall (1975): A finite inverse semigroup S belongs to $\mathcal{A}$, the class of all amalgamation bases for finite inverse semigroups, if and only if S is $\mathscr{J}$-linear (i.e. $\mathscr{J}$-classes form a chain).

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- The class $\mathcal{B}$ of all amalgamation bases for finite semigroups has not been characterised so far. We do know that any semigroup in $\mathcal{B}$ must be $\mathscr{J}$-linear, but the converse is not true.
- Known: $\mathcal{B}$ contains all finite groups, all reducts of inverse semigroups from $\mathcal{A}$, and, most importantly, all full transformation semigroups $\mathcal{T}_n$ (K.Shoji, 2016).

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Proposition

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T is maximally homogeneous if $\mathrm{Aut}(T)$ acts homogeneously on copies of S in T for all $S \in \mathcal{B}$ (resp. $S \in \mathcal{A}$).

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Theorem (ID & Gray, 2017)

There is a unique maximally homogeneous full transformation limit semigroup $\mathcal{T}$.

The maximally homogeneous inverse semigroup $\mathcal{I}$

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  However, this will fail:

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  However, this will fail: we can prove that (most of) the maximal subgroups of these limits are not isomorphic to $\mathcal{U}$, whereas all the maximal subgroups of both $\mathcal{T}$ and $\mathcal{I}$ are isomorphic to $\mathcal{U}$.

Structure of $\mathcal{I}$ and $\mathcal{T}$

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In particular, we hope to stumble upon a number of well-known homogeneous objects in the course of studying the structural features of $\mathcal{I}$ and $\mathcal{T}$.

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4. *The semilattice of idempotents $E(\mathcal{I})$ is isomorphic to the countable universal homogeneous semilattice Ω.*

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To establish (4), we used the characterisation of Ω by Droste, Kuske & Truss (1999), stating that a countable $\wedge$-semilattice is $\cong \Omega$ if and only if:

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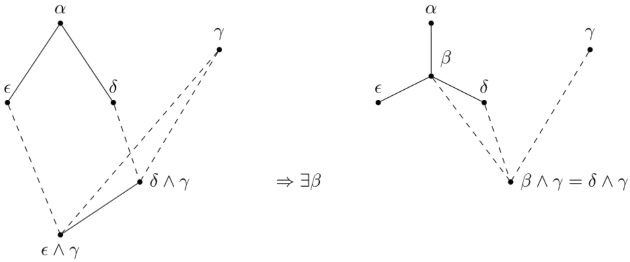
- the following axiom $(*)$ holds:



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- Using the fact that $\mathcal{I}_n \in \mathcal{A}$ is an amalgamation base for finite inverse semigroups (because it is $\mathscr{J}$-linear) and the Extension Property from the Hrushovski construction, we 'tuck in' β back into $\mathcal{I}$ to conclude that $E(\mathcal{I})$ has $(*)$.

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4. *$\mathcal{T}$ is regular, idempotent generated, and self-dual (isomorphic to its opposite semigroup).*
5. *The Graham-Houghton graph of every $\mathscr{D}$-class of $\mathcal{T}$ is isomorphic to the countable random bipartite graph.*

Graham-Houghton graphs: a reminder

Vertices: $\mathscr{R}$-classes (one part) and $\mathscr{L}$-classes (the other part) of a fixed $\mathscr{D}$-class of a semigroup S.

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Example: The rank 2 $\mathscr{D}$-class of $\mathcal{T}_4$.

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123|4			*		*	*
124|3		*		*		*
134|2	*			*	*	
234|1	*	*	*			
12|34		*	*	*	*	
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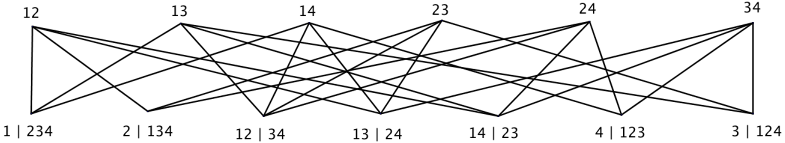

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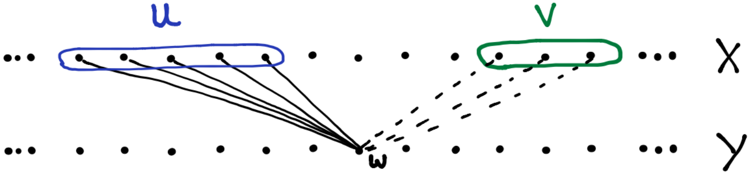
The countable random bipartite graph = the Fraïssé limit of the class of all finite bipartite graphs

It is uniquely characterised among countably infinite bipartite graph by the condition:

> *For any two finite disjoint sets U, V from one part of the bipartition, there is a vertex w in the other part such that $w \sim U$ and $w \not\sim V$.*



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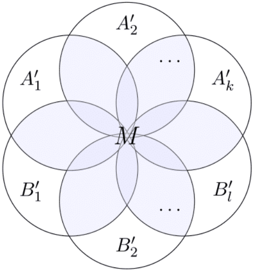

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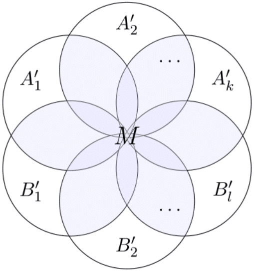

Lemma

Let $A_1, \ldots, A_k, B_1, \ldots, B_\ell$ be t-element subsets of

$$[m] = \{1, \ldots, m\}.$$

If $|M| < t$ then there exists a partition P of $[m]$ with t parts such that $P \perp A_i$ and $P \not\perp B_j$.

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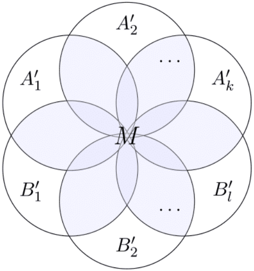

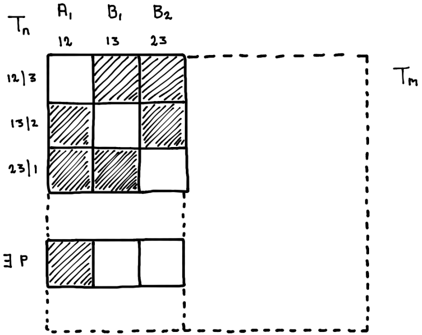

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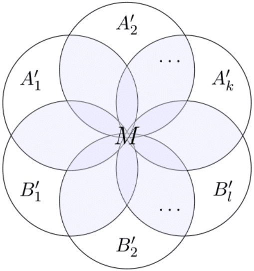

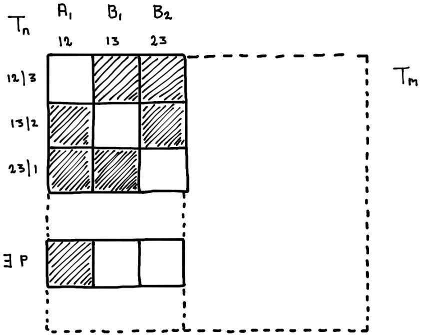

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Proposition

Let $1 < r < n$. Then $\exists \phi : \mathcal{T}_n \to \mathcal{T}_m$ such that $\forall a_1, \dots, a_k, b_1, \dots, b_\ell \in J_r \subseteq \mathcal{T}_n$ from distinct $\mathscr{L}$-classes $\exists c \in \mathcal{T}_m$ such that in $\mathcal{T}_m$:

- *$R_c \cap L_{a_i\phi}$ are groups*
- *$R_c \cap L_{b_j\phi}$ are not groups*

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- It is completely unclear what are the full combinatorial ramifications of this dual result.
- It is related to the following interesting combinatorial question: Given a family of distinct partitions $P_1, \dots, P_k$, $Q_1, \dots, Q_\ell$ of $[m]$, each with exactly t non-empty parts, under what conditions can one guarantee that there is a t-element subset A of $[m]$ which is a transversal of each of P_i, and of none of Q_j?

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Remark

There exist $2^{\aleph_0}$ non-isomorphic countable locally finite groups, and $\mathcal{U}$ embeds all of them.

THANK YOU!

Questions and comments to:

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Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie