Finite homomorphism-homogeneous permutations via edge colourings of chains

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First of all there is Blue. Later there is White, and then there is Black, and before the beginning there is Brown.

Paul Auster: Ghosts (The New York Trilogy)

(Ultra)homogeneity

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Remark

If we restrict to relational structures, 'finitely generated' becomes simply 'finite'.

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Theorem (Fraïssé)

Let $\mathbf{C}$ *be a Fraïssé class. Then there exists a unique countably infinite ultrahomogeneous structure* $\mathcal{F}$ *such that* $\mathrm{Age}(\mathcal{F}) = \mathbf{C}$.

Fraïssé theory (continued)

The structure $\mathcal{F}$ from the previous theorem is called the Fraïssé limit of **C**.

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between its finitely generated substructures is a restriction of an endomorphism ψ of $\mathcal{A}$: $\varphi = \psi|_B$.

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A submonoid M of A^A is the endomorphism monoid of a HH structure on A in a residually finite relational language if and only if it is closed (in the pointwise convergence topology) and oligomorphic.

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Theorem (M & P, 2011)

A structure $\mathcal{A}$ is HH if and only if $\mathrm{End}(\mathcal{A})$ is oligomorphic (i.e. $\mathcal{A}$ is weakly oligomorphic) and $\mathcal{A}$ admits quatifier elimination for positive formulæ.

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- Fraïssé limits (ID, 2014) – the 'one-point homomorphism extension property'

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co-NP-complete classes of finite HH structures:

- finite undirected graphs with loops (Rusinov & Schweitzer, 2010)
- finite algebras of a (fixed) similarity type containing either a symbol of arity ≥ 2, or at least two unary symbols (Mašulović, 2013)
- ...

Few questions

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How, on Earth, is a permutation considered in the role of a structure???

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- To an algebraist: an element of the symmetric group $\mathrm{Sym}(X)$, a bijection $\pi : X \to X$, e.g.

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{pmatrix}$$

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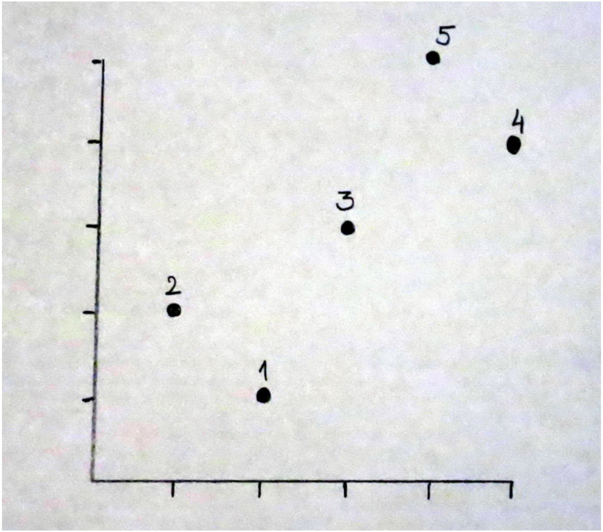

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- To a model theorist: a structure $(X, \leq_1, \leq_2)$, where the set X is equipped by two linear orders, e.g.

$$1 <_1 2 <_1 3 <_1 4 <_1 5 \quad \text{and} \quad 2 <_2 1 <_2 3 <_2 5 <_2 4.$$

  Very suitable for infinite generalisations.

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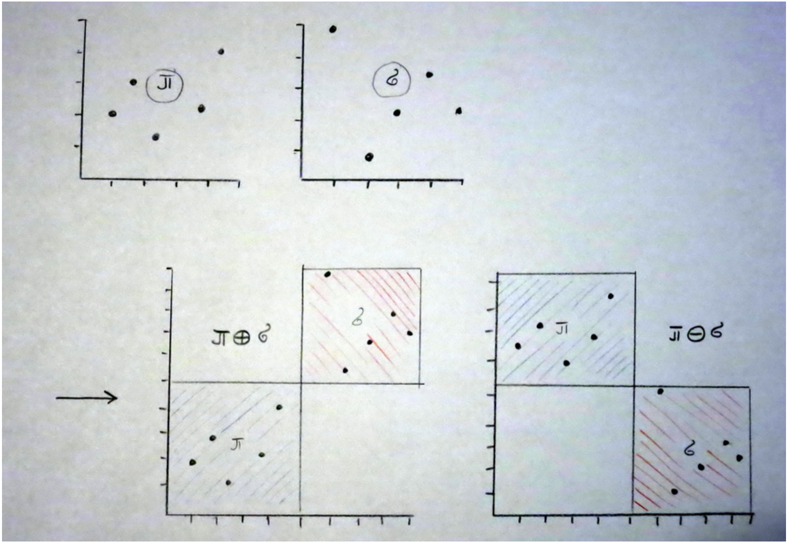

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Easily generalises to infinite permutations.

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Theorem (Cameron, 2002)

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A model for Π: an everywhere dense and independent subset of $\mathbb{Q} \times \mathbb{Q}$.

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Now we have $\sqsubseteq_1 \cup \sqsubseteq_2 = \leq_1$ and $\sqsubseteq_1 \cap \sqsubseteq_2 = \Delta_A$. So, in fact, we have a colouring of the non-loop edges of the graph of $(A, \leq_1)$ into two colours: blue and red, such that each coloured component induces a poset on A.

Changing the view

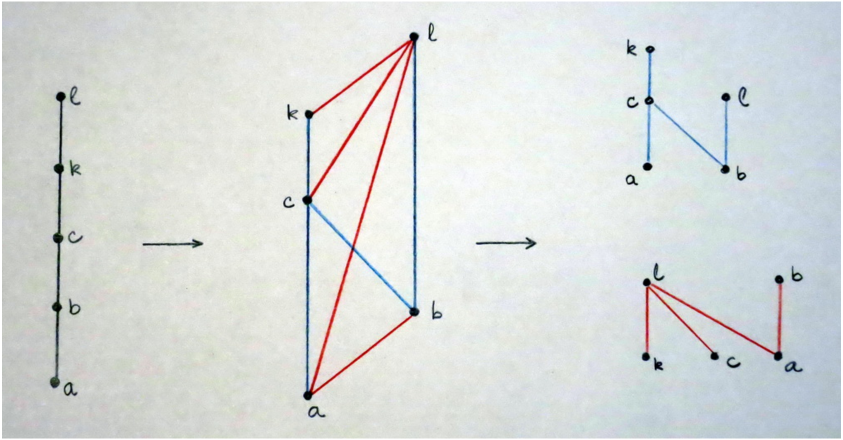
Let us now call a permutation a structure of the form $(A, \leq, \sqsubseteq_1, \sqsubseteq_2)$, where

- $\leq$ is a linear order of A, and
- $(\sqsubseteq_1, \sqsubseteq_2)$ is a partition of $\leq$ into two partial orders on A, in the sense that $\sqsubseteq_1 \cup \sqsubseteq_2 = \leq$ and $\sqsubseteq_1 \cap \sqsubseteq_2 = \Delta_A$ (so all loops are violet).

Changing the view

Example

Permutation *black* (of the set $\{a, b, c, k, l\}$)



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Lemma

A permutation $\pi = (A, \leq_1, \leq_2)$ is (homomorphism-)homogeneous if and only if it adjoined 'permutation' $\mathcal{P}_\pi = (A, \leq_1, \sqsubseteq_1, \sqsubseteq_2)$ is (homomorphism-)homogeneous.

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(i) $m = n$ *and* $r_1 = \cdots = r_n = 1$;

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Let π be a permutation of $[1, n]$. Then π is HH if and only if either $\pi = \overline{\iota_{r_1}} \oplus \cdots \oplus \overline{\iota_{r_m}}$, or $\pi = \iota_{r_1} \ominus \cdots \ominus \iota_{r_m}$, where the sequence $(r_1, \ldots, r_m)$ satisfies one of the following conditions:

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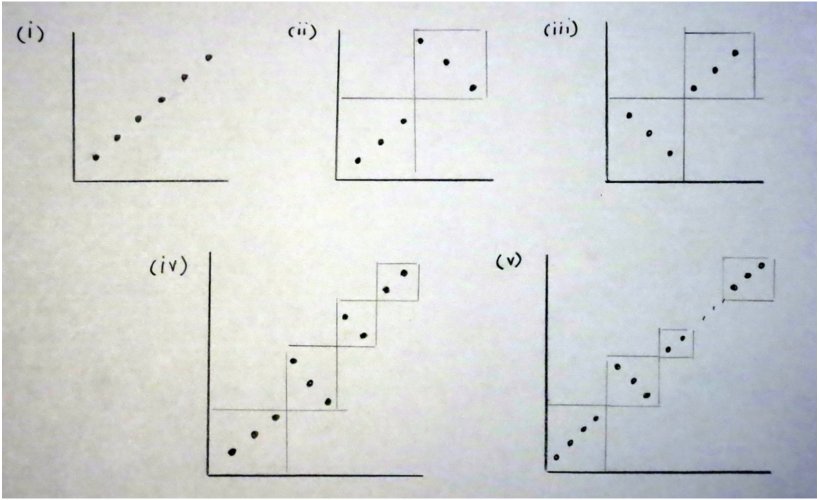

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For a permutation $\mathcal{P}_\pi = (A, \leq, \sqsubseteq_1, \sqsubseteq_2)$ let $\mathsf{B}_\pi = (A, \sqsubseteq_1)$ be the 'blue poset' (agreement), while $\mathsf{R}_\pi = (A, \sqsubseteq_2)$ is the 'red poset' (inversion).

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The key step (continued)

Corollary

If $\mathcal{P}_\pi = (A, \leq, \sqsubseteq_1, \sqsubseteq_2)$ is a finite HH permutation and $|A| > 1$, then at least one of the posets B_π and R_π are disconnected and thus a free sum of at least two chains.

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Hence,

$$\pi = \iota_{r_1} \ominus \cdots \ominus \iota_{r_m}$$

for some positive integers $(r_1, \ldots, r_m)$ such that $r_1 + \cdots + r_m = n$; these are the lengths of maximal blue chains.

The cases

Case 1: R_π is a free sum of chains $\Longrightarrow$ (i)

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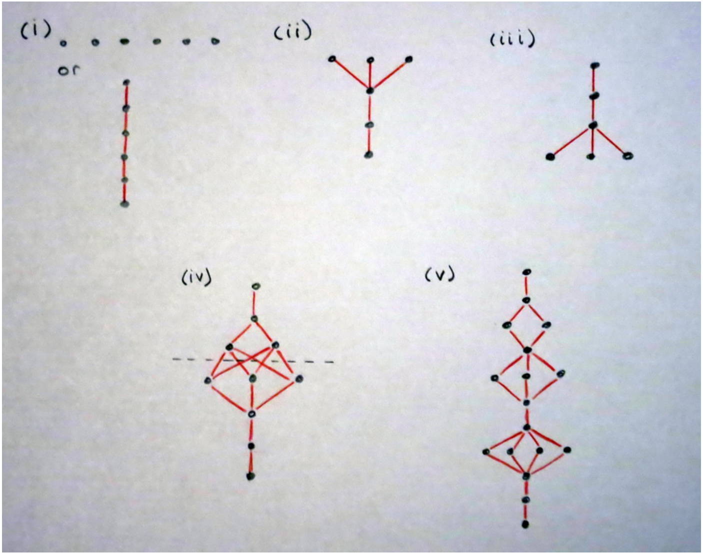
Case 5: R_π is a lattice $\Longrightarrow$ (v)

The cases



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For details, see

I. Dolinka, É. Jungábel, Finite homomorphism-homogeneous permutations via edge colourings of chains, *Electronic Journal of Combinatorics* **19(4)** (2012), #P17, 15 pp.

Problems

Open Problem

Describe countably infinite homomorphism-homogeneous permutations.

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Describe the finite homomorphism-homogeneous structures with n independent linear orders, $n \geq 3$.

On certain nights, when it is clear to Blue that Black will not be going anywhere, he slips out to a bar not far away for a beer or two, enjoying the conversations he sometimes has with the bartender, whose name is Red, and who bears an uncanny resemblance to Green, the bartender from the Gray Case so long ago.

Paul Auster: Ghosts (The New York Trilogy)

THANK YOU!

Questions and comments to:

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http://people.dmi.uns.ac.rs/∼dockie