Variants of semigroups - the case study of finite full transformation monoids

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The 19th NBSAN Meeting
York, UK, January 14, 2015

Prime suspects

Mr. Shady Corleone

Violet Moon
(special undercover agent)

Now seriously... co-authors

I.D.

James East

(U. of Western Sydney)

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Magill (1967): Semigroups of functions $X \to Y$ under an operation defined by

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where θ is a fixed function $Y \to X$. For $Y = X$, this is exactly a variant of $\mathcal{T}_X$.

History of variants – continued

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A more accessible account of her results may be found in the monograph of Ganyushkin & Mazorchuk *Classical Finite Transformation Semigroups* (Springer, 2009).

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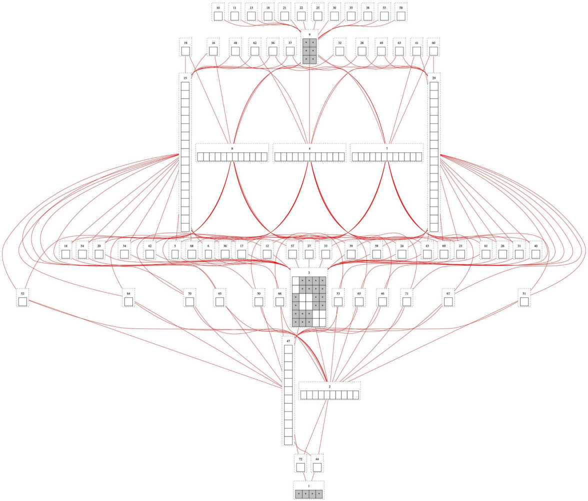
A WORD OF CAUTION: If S is a regular semigroup, S^a is not regular in general! However, for regular S and arbitrary $a \in S$, $\mathrm{Reg}(S^a)$ is always a subsemigroup of S^a (Khan & Lawson).

A word of caution, you said...?



Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 2, 3, 3]$

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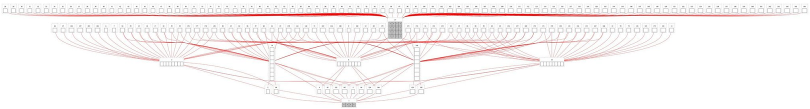

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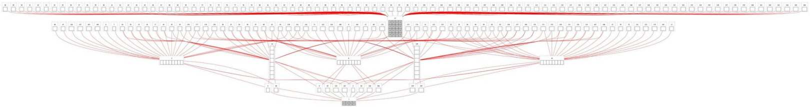

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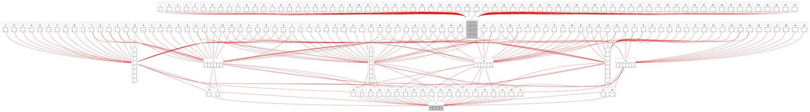

Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 1, 1, 4]$

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Green's relations: $\mathscr{R}^a, \mathscr{L}^a, \mathscr{H}^a, \mathscr{D}^a$

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Group $\mathcal{H}$-classes vs group $\mathcal{H}^a$-classes (in P)

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x	Is H_x a group $\mathscr{H}$-class of $\mathcal{T}_4$?	Is H_x a group $\mathscr{H}^a$-class of $\mathcal{T}_4^a$?
$[1, 1, 3, 3]$	Yes	Yes
$[4, 2, 2, 4]$	Yes	No
$[2, 4, 2, 4]$	No	Yes
$[1, 3, 1, 3]$	No	No

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- Since all constant maps trivially belong to P, D_1 is preserved, and remains a right zero band.
- For $2 \leq m \leq r$, the class D_r separates into a single regular chunk $D_r \cap P$ and a number of non-regular pieces, as seen on the following picture...

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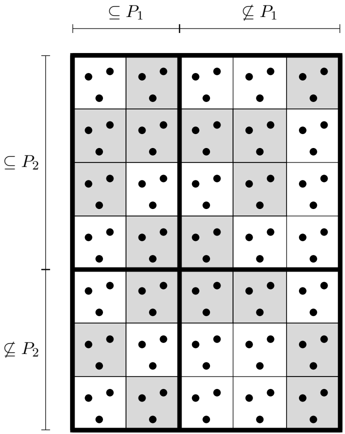

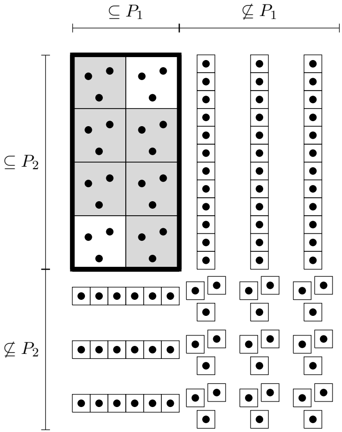

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- $\ker(f) \supseteq \ker(ga)$.

Order of the $\mathscr{D}^a$-classes

Let $f, g \in \mathcal{T}_X$. Then $D^a_f \leq D^a_g$ in $\mathcal{T}^a_X$ if and only if one of the following holds:

- $f = g$,
- $\mathrm{rank}(f) \leq \mathrm{rank}(aga)$,
- $\mathrm{im}(f) \subseteq \mathrm{im}(ag)$,
- $\ker(f) \supseteq \ker(ga)$.

The maximal $\mathscr{D}^a$-classes are those of the form $D^a_f = \{f\}$ where $\mathrm{rank}(f) > r$.

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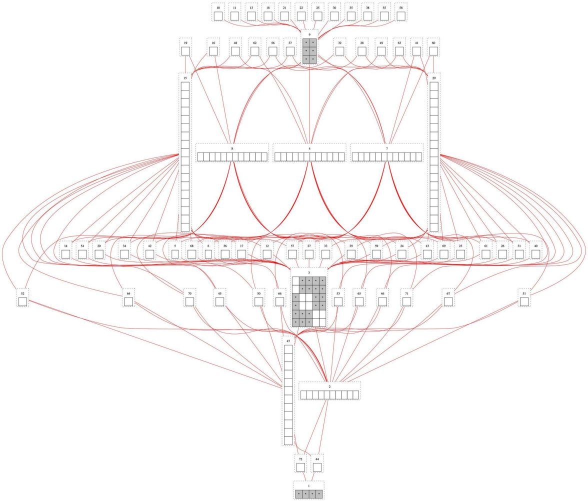

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Then $\mathcal{T}_X^a = \langle M \rangle$; furthermore, any generating set for $\mathcal{T}_X^a$ contains M.

Consequently, M is the unique minimal (with respect to containment or size) generating set of $\mathcal{T}_X^a$, and

$$\mathrm{rank}(\mathcal{T}_X^a) = |M| = \sum_{m=r+1}^{n} S(n,m) \binom{n}{m} m!,$$

where $S(n,m)$ denotes the Stirling number of the second kind.

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- The regular $\mathscr{D}^a$-classes of $\mathcal{T}^a_X$ form a chain: $D^a_1 < \cdots < D^a_r$ (where $D^a_m = \{f \in P : \mathrm{rank}(f) = m\}$ for $m \in [1, r]$).

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- ‘Co-ordinatisation’ of the non-regular, ‘fragmented’ $\mathscr{D}^a$-classes: if $\mathrm{rank}(f) = m \leq r$ and $\mathrm{rank}(afa) = p < m$, then D_f^a sits below D_m^a and above D_p^a.

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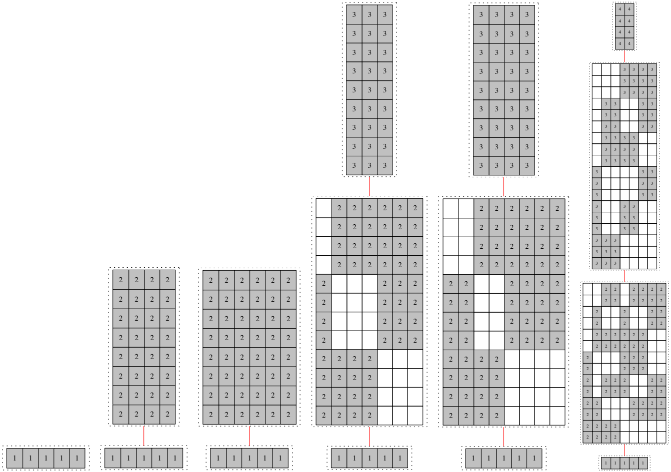
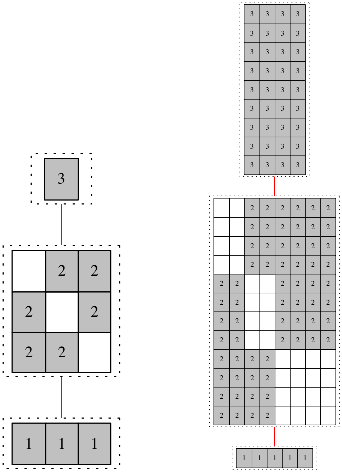
- The ‘crown’: A maximal $\mathscr{D}^a$-class $D^a_f = \{f\}$ sits above D^a_r if and only if $\mathrm{rank}(afa) = r$. The number of such $\mathscr{D}^a$-classes is equal to $(n^{n-r} - r^{n-r})r!\Lambda$.

$\mathrm{Reg}(\mathcal{T}_X^a)$ – examples



Egg-box diagrams of the regular subsemigroups $P = \mathrm{Reg}(\mathcal{T}_5^a)$ in the cases (from left to right): $a = [1, 1, 1, 1, 1]$, $a = [1, 2, 2, 2, 2]$, $a = [1, 1, 2, 2, 2]$, $a = [1, 2, 3, 3, 3]$, $a = [1, 2, 2, 3, 3]$, $a = [1, 2, 3, 4, 4]$.

Do you see what I am seeing???



Egg-box diagrams of $\mathcal{T}_3$ (left) and $\mathrm{Reg}(\mathcal{T}_5^a)$ for $a = [1, 2, 2, 3, 3]$ (right).

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Fact:

$$\mathrm{Reg}(\mathcal{T}(X, A)) = \mathcal{T}(X, A) \cap P_2$$

$$\mathrm{Reg}(\mathcal{T}(X, \alpha)) = \mathcal{T}(X, \alpha) \cap P_1$$

Structure Theorem – Part 1

$$\psi : f \mapsto (fa, af)$$

is a well-defined embedding of $\mathrm{Reg}(\mathcal{T}_X^a)$ into the direct product $\mathrm{Reg}(\mathcal{T}(X, A)) \times \mathrm{Reg}(\mathcal{T}(X, \alpha))$.

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Thus $\mathrm{Reg}(\mathcal{T}_X^a)$ is a subdirect product of $\mathrm{Reg}(\mathcal{T}_X^a)$ and $\mathrm{Reg}(\mathcal{T}(X,\alpha))$.

Structure Theorem – Part 2

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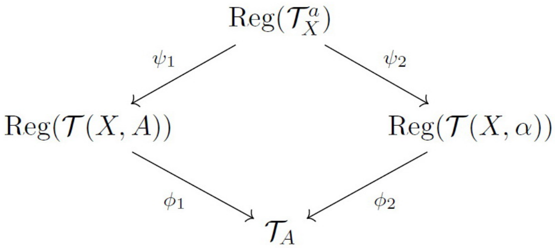

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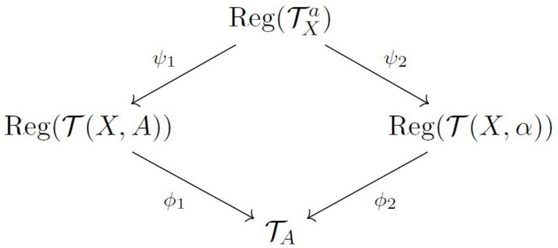

Further, the induced map $\phi = \psi_1\phi_1 = \psi_2\phi_2 = \mathrm{Reg}(\mathcal{T}_X^a) \to \mathcal{T}_A$ is an epimorphism that is 'group / non-group preserving'.

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$$\mathrm{rank}(P) = \mathrm{rank}(D) + \mathrm{rank}(P:D) = r^{n-r} + 1$$

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- We obtain a pleasing generalisation of celebrated Howie's Theorem:

$$\mathcal{E}_X^a = \langle E_a(\mathcal{T}_X^a)\rangle_a = E_a(D) \cup (P \setminus D).$$

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  where $\rho_2 = 2$ and $\rho_r = \binom{r}{2}$ if $r \geq 3$.
- The number of idempotent generating sets of $\mathcal{E}_X^a$ of the minimal possible size is
  $$\left[(r-1)^{n-r}\Lambda\right]^{\rho_r}\Lambda! S(r^{n-r},\Lambda) \sum_{\Gamma\in\mathbb{T}_r} \frac{1}{\lambda_1^{d_\Gamma^+(1)}\cdots\lambda_r^{d_\Gamma^+(r)}}.$$
  where $\mathbb{T}_r$ is the set of all strongly connected tournaments on r vertices.

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- $$\mathrm{rank}(I_m^a) = \mathrm{idrank}(I_m^a) = \begin{cases} m^{n-r} S(r, m) & \text{if } 1 < m < r \\ n & \text{if } m = 1. \end{cases}$$

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  These sandwich semigroups generalise the variants.
  - applicable to functions, matrices, diagrams,. . .

THANK YOU!

Questions and comments to:

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http://people.dmi.uns.ac.rs/∼dockie