Variants of semigroups - the case study of finite full transformation monoids

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Prime suspects

Mr. Shady Corleone

Violet Moon
(special undercover agent)
Now seriously... co-authors

I.D. James East
(U. of Western Sydney)
Variants of semigroups

Let \((S, \cdot)\) be a semigroup and \(a \in S\). Given these, one can easily define an alternative product \(\star_a\) on \(S\), namely

\[ x \star_a y = xay. \]

This is the variant \(S^a = (S, \star_a)\) of \(S\) with respect to \(a\).

First mention of variants (as far as we know): Lyapin’s book from 1960 (in Russian).

Magill (1967): Semigroups of functions \(X \to Y\) under an operation defined by

\[ f \cdot g = f \circ \theta \circ g, \]

where \(\theta\) is a fixed function \(Y \to X\). For \(Y = X\), this is exactly a variant of \(\mathcal{T}_X\).
History of variants – continued


G. Y. Tsyaputa (2004/5): variants of finite full transformation semigroups $T_n$

- classification of non-isomorphic variants
- idempotents, Green’s relations
- analogous questions for $P T_n$

A more accessible account of her results may be found in the monograph of Ganyushkin & Mazorchuk Classical Finite Transformation Semigroups (Springer, 2009).
Several examples

For a group $G$ and $a \in G$, we always have $G^a \cong G$ via $x \mapsto xa$. The identity element in $G^a$ is $a^{-1}$.

On the other hand, if $S$ the bicyclic monoid, then $a, b \in S$, $a \neq b$ implies $S^a \not\cong S^b$.

If $S$ is a monoid, $a, u, v \in S$, and $u, v$ are units, then $S^{uav} \cong S^a$ via $x \mapsto vxa$.

Thus, for any $a \in T_X$ there exists $e \in E(T_X)$ such that $T_X^a \cong T_X^e$.

A WORD OF CAUTION: If $S$ is a regular semigroup, $S^a$ is not regular in general! However, for regular $S$ and arbitrary $a \in S$, $\text{Reg}(S^a)$ is always a subsemigroup of $S^a$ (Khan & Lawson).
Egg-box picture of $T_4^a$ for $a = [1, 2, 3, 3]$
A word of caution, you said…?

Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 1, 3, 3]$

Egg-box picture of $\mathcal{T}_4^a$ for $a = [1, 1, 1, 4]$
Three important sets

\[ P_1 = \{ x \in S : xa \mathcal{R} x \}, \quad P_2 = \{ x \in S : ax \mathcal{L} x \}, \]

\[ P = P_1 \cap P_2 \]

Easy facts:

- \( y \in P_1 \Leftrightarrow L_y \subseteq P_1 \),
- \( y \in P_2 \Leftrightarrow R_y \subseteq P_2 \),
- \( \text{Reg}(S^a) \subseteq P \)
Green’s relations: $R^a$, $L^a$, $H^a$, $D^a$

$R^a_x = \begin{cases} 
R_x \cap P_1 & \text{if } x \in P_1 \\
\{x\} & \text{if } x \in S \setminus P_1, 
\end{cases}$

$L^a_x = \begin{cases} 
L_x \cap P_2 & \text{if } x \in P_2 \\
\{x\} & \text{if } x \in S \setminus P_2, 
\end{cases}$

$H^a_x = \begin{cases} 
H_x & \text{if } x \in P \\
\{x\} & \text{if } x \in S \setminus P, 
\end{cases}$

$D^a_x = \begin{cases} 
D_x \cap P & \text{if } x \in P \\
L^a_x & \text{if } x \in P_2 \setminus P_1 \\
R^a_x & \text{if } x \in P_1 \setminus P_2 \\
\{x\} & \text{if } x \in S \setminus (P_1 \cup P_2). 
\end{cases}$

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Group $\mathcal{H}$-classes vs group $\mathcal{H}^a$-classes (in $P$)

Let $S = T_4$ and $a = [1, 2, 3, 3]$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>Is $H_x$ a group $\mathcal{H}$-class of $T_4$?</th>
<th>Is $H_x$ a group $\mathcal{H}^a$-class of $T_4^a$?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[1, 1, 3, 3]$</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$[4, 2, 2, 4]$</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$[2, 4, 2, 4]$</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>$[1, 3, 1, 3]$</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>
Our goal for today...

...is to conduct a thorough algebraic and combinatorial analysis of \( T_X^a \) where \( |X| = n \) and \( a \) is a fixed transformation on \( X \).

As we noted, we may assume that \( a \) is idempotent with \( r = \text{rank}(a) < n \),

\[
a = \begin{pmatrix} A_1 & \cdots & A_r \\ a_1 & \cdots & a_r \end{pmatrix},
\]

so that \( a_i \in A_i \) for all \( i \in [1, r] \).

Here \( A = \text{im}(a) = \{a_1, \ldots, a_r\} \) and \( \alpha = \ker(a) = (A_1| \cdots |A_r) \), with \( \lambda_i = |A_i| \). Furthermore, for \( I = \{i_1, \ldots, i_m\} \subseteq [1, r] \) we write \( \Lambda_I = \lambda_{i_1} \cdots \lambda_{i_m} \) and \( \Lambda = \lambda_1 \cdots \lambda_r \).
Let $B \subseteq X$ and let $\beta$ be an equivalence relation on $X$. We say that $B$ saturates $\beta$ if each $\beta$-class contains at least one element of $B$. Also, we say that $\beta$ separates $B$ if each $\beta$-class contains at most one element of $B$.

$P_1 = \{ f \in \mathcal{T}_X : \text{rank}(fa) = \text{rank}(f) \}$
$= \{ f \in \mathcal{T}_X : \alpha \text{ separates } \text{im}(f) \}$

$P_2 = \{ f \in \mathcal{T}_X : \text{rank}(af) = \text{rank}(f) \}$
$= \{ f \in \mathcal{T}_X : A \text{ saturates } \ker(f) \}$

$P = \{ f \in \mathcal{T}_X : \text{rank}(afa) = \text{rank}(f) \}$ \(= \text{Reg}(\mathcal{T}_X^a) \leq \mathcal{T}_X^a\)
Green’s relations in $\mathcal{T}_X^a$ (Tsyaputa, 2004)

$$R_f^a = \begin{cases} R_f \cap P_1 & \text{if } f \in P_1 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P_1, \end{cases}$$

$$L_f^a = \begin{cases} L_f \cap P_2 & \text{if } f \in P_2 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P_2, \end{cases}$$

$$H_f^a = \begin{cases} H_f & \text{if } f \in P \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus P, \end{cases}$$

$$D_f^a = \begin{cases} D_f \cap P & \text{if } f \in P \\ L_f^a & \text{if } f \in P_2 \setminus P_1 \\ R_f^a & \text{if } f \in P_1 \setminus P_2 \\ \{f\} & \text{if } f \in \mathcal{T}_X \setminus (P_1 \cup P_2). \end{cases}$$
Recall that in $\mathcal{T}_X$, the $\mathcal{D}$-classes form a chain:

$$D_n > D_{n-1} > \cdots > D_2 > D_1.$$ 

Each of the $\mathcal{D}$-classes $D_{r+1}, \ldots, D_n$ is completely ‘shattered’ into singleton ‘shrapnels’ / $\mathcal{D}^a$-classes in $\mathcal{T}_X^a$.

Since all constant maps trivially belong to $P$, $D_1$ is preserved, and remains a right zero band.

For $2 \leq m \leq r$, the class $D_r$ separates into a single regular chunk $D_r \cap P$ and a number of non-regular pieces, as seen on the following picture...
Theorem 4.2 yields an intuitive picture of the Green's structure of $T_x$. Recall that the $D$-classes of $T_x$ are precisely the sets $\{a\}$. For $i \in m, l$, let $g_i \in G_i$. Then we have $f = uag$, where $10$.

Proof.

The maximal

Proposition 4.4.

if and only if $\text{rank}(f)$. We now give some information about the order on the

in

Figure 4: A schematic diagram of the way a

form

non-regular

$D$-class of $T_x$ and

those of the form $\{a\}$. This is all pictured (schematically) in Figure 4; see also Figures 2 and 3.

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Order of the $\mathcal{D}^a$-classes

Let $f, g \in \mathcal{T}_X$. Then $D^a_f \leq D^a_g$ in $\mathcal{T}_X^a$ if and only if one of the following holds:

- $f = g$,
- $\text{rank}(f) \leq \text{rank}(aga)$,
- $\text{im}(f) \subseteq \text{im}(ag)$,
- $\text{ker}(f) \supseteq \text{ker}(ga)$.

The maximal $\mathcal{D}^a$-classes are those of the form $D^a_f = \{f\}$ where $\text{rank}(f) > r$. 

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Order of the $D^a$-classes
The rank of $\mathcal{T}_X^a$

Let $M = \{ f \in \mathcal{T}_X : \text{rank}(f) > r \}$.

Then $\mathcal{T}_X^a = \langle M \rangle$; furthermore, any generating set for $\mathcal{T}_X^a$ contains $M$.

Consequently, $M$ is the unique minimal (with respect to containment or size) generating set of $\mathcal{T}_X^a$, and

$$\text{rank}(\mathcal{T}_X^a) = |M| = \sum_{m=r+1}^{n} S(n, m) \binom{n}{m} m!,$$

where $S(n, m)$ denotes the Stirling number of the second kind.
‘Positioning’ with respect to the regular classes

- If $f \in P$, then $D^a_f \leq D^a_g$ if and only if $\text{rank}(f) \leq \text{rank}(aga)$.
- If $g \in P$, then $D^a_f \leq D^a_g$ if and only if $\text{rank}(f) \leq \text{rank}(g)$.

Consequences:

- The regular $D^a$-classes of $T^a_X$ form a chain: $D^a_1 < \cdots < D^a_r$ (where $D^a_m = \{ f \in P : \text{rank}(f) = m \}$ for $m \in [1, r]$).
- ‘Co-ordinatisation’ of the non-regular, ‘fragmented’ $D^a$-classes: if $\text{rank}(f) = m \leq r$ and $\text{rank}(afa) = p < m$, then $D^a_f$ sits below $D^a_m$ and above $D^a_p$.
- The ‘crown’: A maximal $D^a$-class $D^a_f = \{ f \}$ sits above $D^a_r$ if and only if $\text{rank}(afa) = r$. The number of such $D^a$-classes is equal to $(n^{n-r} - r^{n-r})r! \Lambda$. 
\( \text{Reg}(\mathcal{T}_X^a) \) – examples

Egg-box diagrams of the regular subsemigroups \( P = \text{Reg}(\mathcal{T}_5^a) \) in the cases (from left to right): \( a = [1, 1, 1, 1, 1] \), \( a = [1, 2, 2, 2, 2] \), \( a = [1, 1, 2, 2, 2] \), \( a = [1, 2, 3, 3, 3] \), \( a = [1, 2, 2, 3, 3] \), \( a = [1, 2, 3, 4, 4] \).
Do you see what I am seeing???

Egg-box diagrams of $\mathcal{T}_3$ (left) and $\text{Reg}(\mathcal{T}_5^a)$ for $a = [1, 2, 2, 3, 3]$ (right).
No, this is not just a coincidence...!

\[ T(X, A) = \{ f \in T_X : \text{im}(f) \subseteq A \} \]

\[ T(X, \alpha) = \{ f \in T_X : \text{ker}(f) \supseteq \alpha \} \]

– transformation semigroups with restricted range (Sanwong & Sommanee, 2008), and restricted kernel (Mendes-Gonçalves & Sullivan, 2010).

Fact:

\[ \text{Reg}(T(X, A)) = T(X, A) \cap P_2 \]

\[ \text{Reg}(T(X, \alpha)) = T(X, \alpha) \cap P_1 \]
Structure Theorem – Part 1

\[ \psi : f \mapsto (fa, af) \]

is a well-defined embedding of \( \text{Reg}(\mathcal{T}_X^a) \) into the direct product \( \text{Reg}(\mathcal{T}(X, A)) \times \text{Reg}(\mathcal{T}(X, \alpha)) \). Its image consists of all pairs \((g, h)\) such that

\[ \text{rank}(g) = \text{rank}(h) \quad \text{and} \quad g|_A = (ha)|_A. \]

Thus \( \text{Reg}(\mathcal{T}_X^a) \) is a subdirect product of \( \text{Reg}(\mathcal{T}_X^a) \) and \( \text{Reg}(\mathcal{T}(X, \alpha)) \).
Structure Theorem – Part 2

The maps

\[ \phi_1 : \text{Reg}(\mathcal{T}(X, A)) \to \mathcal{T}_A : g \mapsto g|_A \]

\[ \phi_2 : \text{Reg}(\mathcal{T}(X, \alpha)) \to \mathcal{T}_A : g \mapsto (ga)|_A \]

are epimorphisms, and the following diagram commutes:

Further, the induced map \( \phi = \psi_1 \phi_1 = \psi_2 \phi_2 = \text{Reg}(\mathcal{T}_X^a) \to \mathcal{T}_A \) is an epimorphism that is ‘group / non-group preserving’.

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Size and rank of $P = \text{Reg}(\mathcal{T}_X^a)$

\[ |P| = \sum_{m=1}^{r} m! \cdot m^{n-r} \cdot S(r, m) \sum_{l \in \binom{[1, r]}{m}} \Lambda_l. \]

Let $D$ be the top (rank-$r$) $\mathcal{D}^a$-class of $P$.

\[ \text{rank}(P) = \text{rank}(D) + \text{rank}(P : D) = r^{n-r} + 1 \]
The idempotent generated subsemigroup $\langle E_{a}(T_{X}^{a}) \rangle_{a}$

$E_{a}(T_{X}^{a}) = \{ f \in T_{X} : (af)|_{\text{im}(f)} = \text{id}|_{\text{im}(f)} \}.$

$|E_{a}(T_{X}^{a})| = \sum_{m=1}^{r} m^{n-m} \sum_{I \in \binom{[1, r]}{m}} \land I.$

We obtain a pleasing generalisation of celebrated Howie’s Theorem:

$E_{X}^{a} = \langle E_{a}(T_{X}^{a}) \rangle_{a} = E_{a}(D) \cup (P \setminus D).$
The idempotent generated subsemigroup \( \langle E_a(T^a_X) \rangle \)

\[
\text{rank}(E^a_X) = \text{idrank}(E^a_X) = r^{n-r} + \rho_r,
\]

where \( \rho_2 = 2 \) and \( \rho_r = \binom{r}{2} \) if \( r \geq 3 \).

The number of idempotent generating sets of \( E^a_X \) of the minimal possible size is

\[
\left[ (r - 1)^{n-r} \Lambda \right]^{\rho_r} \Lambda! S(r^{n-r}, \Lambda) \sum_{\Gamma \in \mathbb{T}_r} \frac{1}{\lambda_1^{d_1^+(1)} \cdots \lambda_r^{d_r^+(r)}}.
\]

where \( \mathbb{T}_r \) is the set of all strongly connected tournaments on \( r \) vertices.
The ideals of $P$

- The ideals of $P$ are precisely

$$I^a_m = \{ f \in P : \text{rank}(f) \leq m \}$$

for $m \in [1, r]$.

- They are all idempotent generated (by $E_a(D^a_m)$) except $P = I^a_r$ itself.

- $\text{rank}(I^a_m) = \text{idrank}(I^a_m) = \begin{cases} m^{n-r} S(r, m) & \text{if } 1 < m < r \\ n & \text{if } m = 1. \end{cases}$
Future work

- Conduct an analogous study for variants of:
  - full linear (matrix) monoids
  - symmetric inverse semigroups
  - various diagram semigroups (partition, (partial) Brauer, (partial) Jones, wire, Kaufmann, . . .)
  - . . .

- Consider an ‘Ehresmann-style’ defined small (semi)category (aka partial monoid / semigroup) $S$. One can turn each hom-set $S_{ij}$ ($i$ - domain, $j$ - codomain) into a semigroup by fixing a ‘sandwich’ element $a \in S_{ji}$ and defining

$$x \star y = x \circ a \circ y.$$

These **sandwich semigroups** generalise the variants.

- applicable to functions, matrices, diagrams, . . .
THANK YOU!

Questions and comments to:

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Further information may be found at:
http://people.dmi.uns.ac.rs/~dockie