Free idempotent generated semigroups: maximal subgroups and the word problem

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Man is condemned to be free.

Jean-Paul Sartre

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Hence:

What can we say about the structure of the free-est idempotent-generated (IG) semigroup with a fixed structure/configuration of idempotents ???

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Alternatively, biordered sets can be (abstractly) described as relational structures $(E(S), \leq^{(l)}, \leq^{(r)})$ with two quasi-orders and several simple rules/axioms (Easdown, Nambooripad, '80s).

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- To every semigroup S with idempotents E associate the free-est semigroup $\mathsf{IG}(E)$ whose idempotents form the same biordered set as in S.
- To every regular semigroup S with idempotents E associate the free-est regular semigroup $\mathsf{RIG}(E)$ in whose idempotents form the same biordered set as in S.

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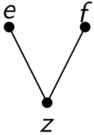
Let $S =$ 

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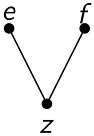
Let $S =$ 

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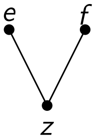
Let $S =$ 

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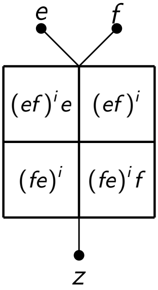

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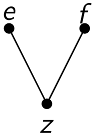
Let $S =$ 

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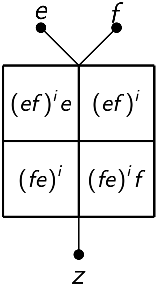

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Example 2: 2×2 rectangular band

$$S = \langle e_{ij} \mid e_{ij} e_{kl} = e_{il} \ (i, j, k, l \in \{1, 2\}) \rangle:$$

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$\mathrm{RIG}(S) = \mathrm{IG}(S)$.

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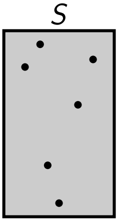
Question

Which groups arise as maximal subgroups of $\mathsf{IG}(E)$ (and thus of $\mathsf{RIG}(E)$)?

The big picture



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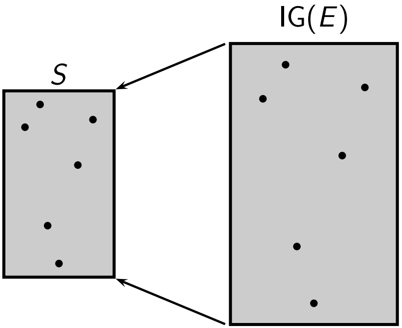

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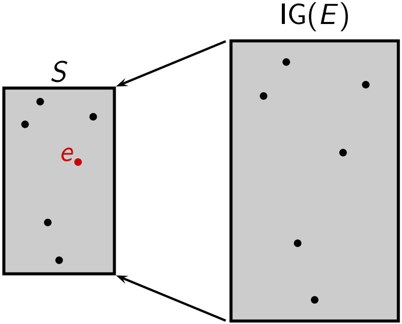

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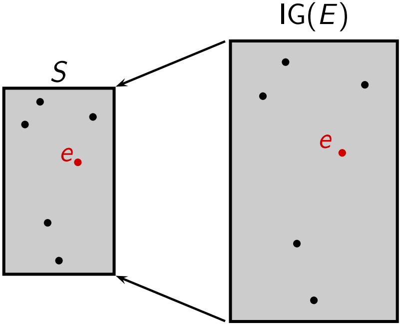

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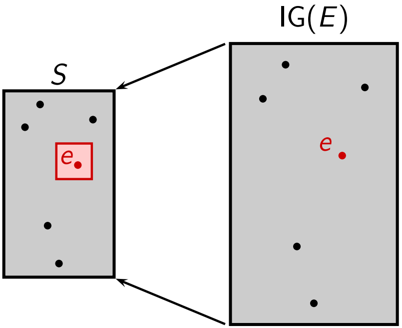

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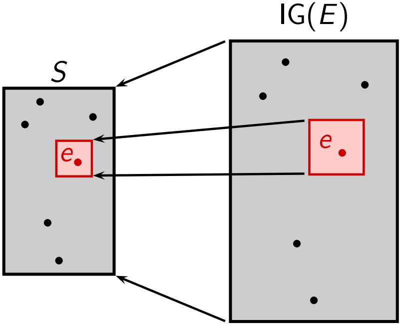

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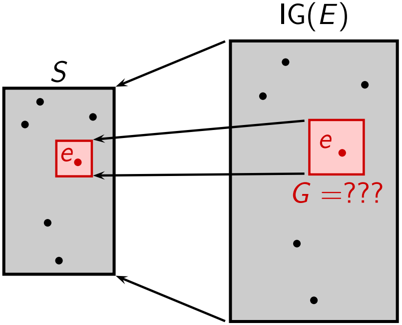
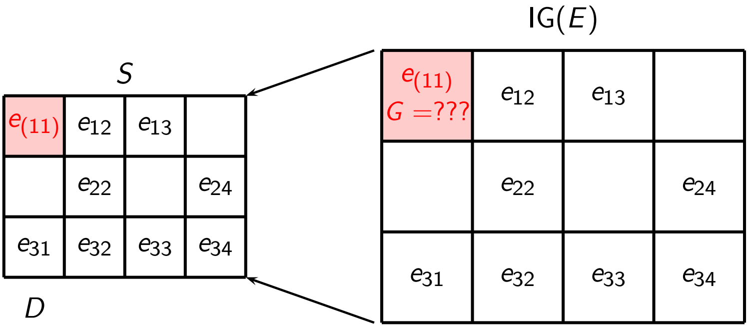


Let's zoom in



Presentation for a max. subgroup of IG(E): Generators

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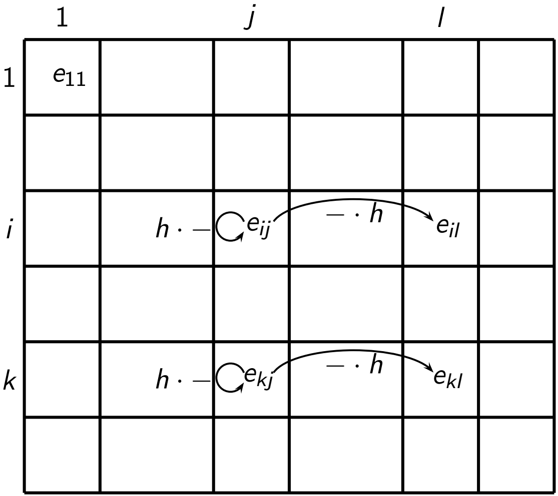
$$G = \langle f_{ij} \ (e_{ij} \in D \cap E) \mid ??? \rangle$$

Typical relations: $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$

- $h = h^2$



Singular square $\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix} \Rightarrow$ relation $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$.

Presentation – Approach #1

Theorem (Nambooripad 1979; Gray, Ruškuc 2012)

The maximal subgroup G of $e \in E$ in $IG(E)$ or $RIG(E)$ is defined by the presentation:

$$\begin{aligned}
\langle f_{ij} \mid f_{i,\pi(i)} &= 1 && (i \in I),\\
f_{ij} &= f_{il} && (\text{if } r_j e_{il} = r_l \text{ is a Schreier rep.}),\\
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Proof: Reidemeister–Schreier rewriting process followed by Tietze transformations.

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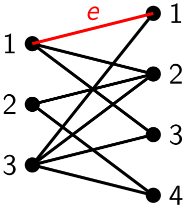

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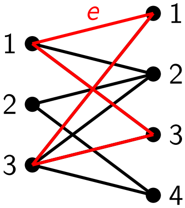

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Theorem (Brittenham, Margolis, Meakin, 2009)

The fundamental group of $GH(S)$ at any point of its connected component C_e containing the edge $e \cong$ the maximal subgroup of $RIG(E(S))$ (and thus of $IG(E(S))$) containing e.

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Obviously, a clever choice of $\mathcal{T}$ may speed up the computation.

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- Finally, Gray and Ruškuc (2012) proved that every group arises as a maximal subgroup of some free idempotent generated semigroup (!!!).

Gray & Ruškuc, 2012

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Every group is a maximal subgroup of some free idempotent generated semigroup (over a regular semigroup).

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Is every finitely presented group a maximal subgroup in some free idempotent generated semigroup over a finite regular semigroup?

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- Endomorphism monoid of a free G-act: Dolinka, Gould, Yang (wreath products of G by symmetric groups).

Bands

Theorem (ID)

For every left- or right seminormal band B, all maximal subgroups of $IG(B)$ are free. For every variety $\mathbf{V}$ *not contained in* $\mathbf{LSNB} \cup \mathbf{RSNB}$ *there exists $B \in \mathbf{V}$ such that $IG(B)$ contains a non-free maximal subgroup.*

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Answer (ID, Ruškuc, 2013): All of them! (Resp. all finitely presented ones.)

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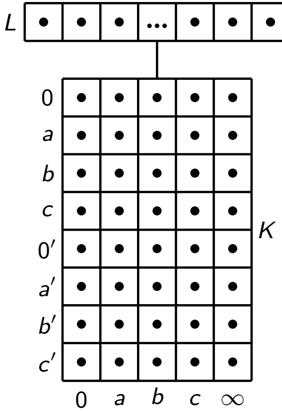
- $J = \{0, a, b, c, \dots, \infty\}$;
- $\mathcal{T} = \mathcal{T}_I^* \times \mathcal{T}_J$;
- the minimal ideal: $K = \{(\sigma, \tau) : \sigma, \tau \text{ constants}\}$;
- K is an $I \times J$ rectangular band.

New construction: set-up

- $B = K \cup L$, where L is a left zero semigroup.



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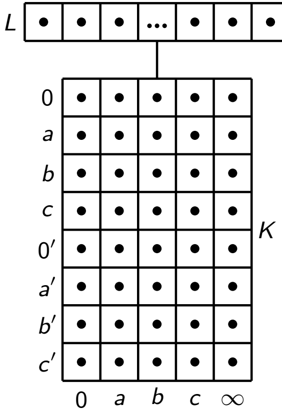

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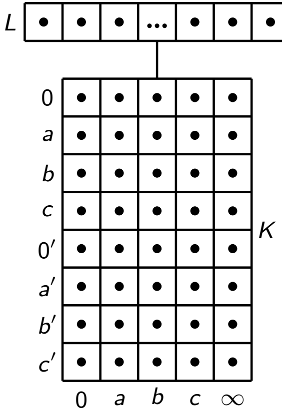

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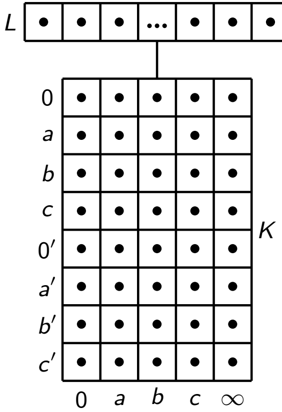

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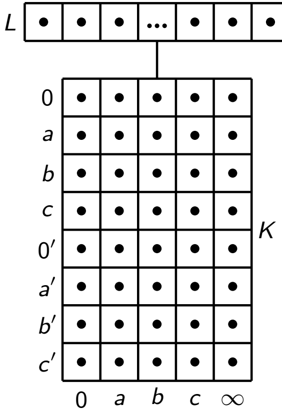

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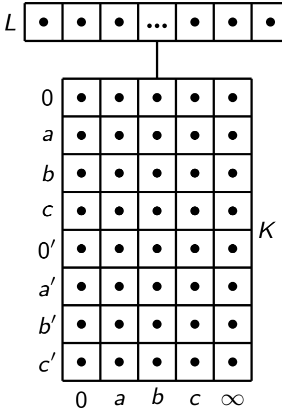

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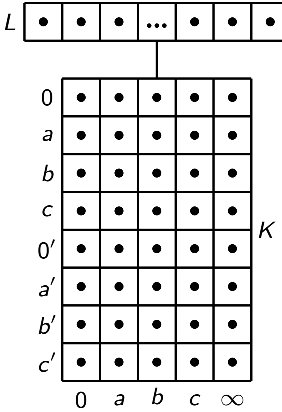

New construction: the action of L on K

Notation	**Indexing**	$\mathrm{im}(\boldsymbol{\sigma})$	$(\boldsymbol{\infty})\boldsymbol{\tau}$
(σ_0, τ_0)	–	$\{0, 0'\}$	0
(σ_a, τ_a)	$a \in A$	$\{0, a'\}$	a
$(\overline{\sigma}_a, \overline{\tau}_a)$	$a \in A$	$\{a, a'\}$	0
$(\sigma_{\mathbf{r}}, \tau_{\mathbf{r}})$	$\mathbf{r} = (ab, c) \in R$	$\{b, c'\}$	a

New construction: the endgame

	0	a	b	c	∞
0	f_{00}	f_{0a}	f_{0b}	f_{0c}	$f_{0\infty}$
a	f_{a0}	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	f_{b0}	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	f_{c0}	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	$f_{0'0}$	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	$f_{a'0}$	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	$f_{b'0}$	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
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c'	$f_{c'0}$	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\xrightarrow{\substack{(\sigma_0, \tau_0) \\ (\sigma_a, \tau_a) \\ (\overline{\sigma}_a, \overline{\tau}_a)}}$$

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
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The word problem (ongoing joint work with R.D.Gray and N.Ruškuc)

Let S be a semigroup with a finite biordered set $E = E(S)$.

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Corollary

The word problem for $RIG(E)$ is solvable iff the word problem for each of its maximal subgroups is solvable.

The word problem(?)

So, the following question naturally arises:

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Is it true that the word problem for $\mathrm{IG}(E)$ (where $E = E(S)$ is finite) is solvable iff the word problem for each of its maximal subgroups is solvable?

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Theorem

There exists a finite band B such that all the maximal subgroups of $IG(E(B))$ are free, but the word problem of $IG(E(B))$ is still undecidable.

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- the action of L on K' and K'' is exactly the same as in the ID-NR construction.

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- as known from the properties of $\mathrm{IG}(E)$, to the rectangular bands K' and K'' correspond two $\mathcal{D}$-classes M' and M'' of $\mathrm{IG}(E)$ which are completely simple subsemigroups, with typical elements

$$(i', g_1, j') \quad \text{and} \quad (i'', g_2, j''),$$

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Proposition

$(1', 1, 1')(1'', 1, 1'') = (1', 1, 1')(1'', g, 1'')$ *if and only if* $g \in H$.

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If $G = F_2 \times F_2$ and W is a f.p. 2-generated group with an undecidable problem, then taking H to be the fibre product w.r.t. the natural homomorphism $\pi : F_2 \to W$ (i.e. $H = \ker \pi$) suffices.

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Open Problem

Is it at least true that the word problem for $\mathrm{IG}(E)$ is solvable when E is the biorder of a finite normal band?

THANK YOU!

Questions and comments to:

dockie@dmi.uns.ac.rs

Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie