Free idempotent generated semigroups: maximal subgroups and the word problem

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Mathematics Colloquium
MI SANU, Belgrade, 16 December 2016

I'm free like a river
Flowin' freely to infinity
I'm free to be sure of what
I am and who I need not be
I'm much freer - like the meaning
Of the word 'free' that crazy man defines
Free - free like the vision that
The mind of only you are ever gonna see

Stevie Wonder: Free

Man is condemned to be free.

Jean-Paul Sartre

Idempotent generated semigroups

Many natural semigroups are idempotent-generated ($S = \langle E(S) \rangle$):

- The semigroup $\mathcal{T}_n \setminus \mathcal{S}_n$ of singular (non-invertible) transformations on a finite set (Howie, 1966);
- The singular part of $\mathcal{M}_n(\mathbb{F})$, the semigroup of all $n \times n$ matrices over a field $\mathbb{F}$ (Erdos (not Paul!), 1967);
- In 2006, Putcha completed the classification of linear algebraic monoids that are idempotent-generated;
- The singular part of $\mathcal{P}_n$, the singular part of the partition monoid on a finite set (East, FitzGerald, 2012);

Hence:

What can we say about the structure of the free-est idempotent-generated (IG) semigroup with a fixed structure/configuration of idempotents ???

Biordered sets of idempotents

'Configuration of idempotents' = biordered sets = relational structures $(E(S), \leq^{(l)}, \leq^{(r)})$ with two quasi-orders such that

$$e \leq^{(l)} f \Leftrightarrow e = ef, \qquad e \leq^{(r)} f \Leftrightarrow e = fe.$$

Biordered sets can be finitely axiomatised by several simple rules (Easdown, Nambooripad, '80s).

Basic pair $\{e, f\}$ of idempotents:

$$\{e, f\} \cap \{ef, fe\} \neq \varnothing$$

that is, $ef = e$ or $ef = f$ or $fe = e$ or $fe = f$.
(Note: if, for example, $ef \in \{e, f\}$, then $(fe)^2 = fe$.)

Alternatively: Biordered set of a semigroup S = the partial algebra on $E(S)$ obtained by retaining the products of basic pairs (in S).

Free IG semigroups: idea

- To every semigroup S with idempotents E associate the free-est semigroup $\mathrm{IG}(E)$ whose idempotents form the same biordered set as in S.
- To every regular semigroup S with idempotents E associate the free-est regular semigroup $\mathrm{RIG}(E)$ in whose idempotents form the same biordered set as in S.

Free IG semigroups: formal definitions

Let E be the biordered set of idempotents of a semigroup S.

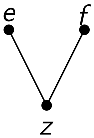
$$\mathrm{IG}(E) := \langle E \mid e \cdot f = ef \text{ where } \{e, f\} \text{ is a basic pair} \rangle.$$

Suppose now S is regular. We define the sandwich sets:

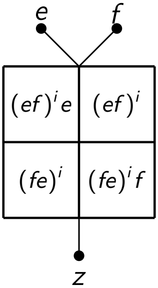
$$S(e, f) = \{h \in E \;:\; ehf = ef,\; fhe = h\} \neq \varnothing$$

$$\mathrm{RIG}(E) := \langle E \mid \mathrm{IG},\; ehf = ef \;(e, f \in E,\; h \in S(e, f))\rangle.$$

Example 1: Three-element meet semilattice

Let $S =$ 

$\mathrm{IG}(S) = \langle e, f, z \mid e^2 = e,\ f^2 = f,\ z^2 = z,\ ez = ze = fz = zf = z \rangle$:



$RIG(S) = \langle e, f, z \mid \mathrm{IG},\ ef = fe = z \rangle = S.$

Example 2: 2×2 rectangular band

$$S = \langle e_{ij} \mid e_{ij}e_{kl} = e_{il} \ (i,j,k,l \in \{1,2\})\rangle:$$

e_{11}	e_{12}
e_{21}	e_{22}

$$\mathrm{IG}(S) = \langle e_{ij} \mid e_{ij}e_{kl} = e_{il} \ (i,j,k,l \in \{1,2\},\ i = k \text{ or } j = l)\rangle:$$

$(e_{11}e_{22})^i e_{11}$ $(e_{12}e_{21})^i$	$(e_{12}e_{21})^i e_{12}$ $(e_{11}e_{22})^i$
$(e_{21}e_{12})^i e_{21}$ $(e_{22}e_{11})^i$	$(e_{22}e_{11})^i e_{22}$ $(e_{21}e_{12})^i$

$$\mathrm{RIG}(S) = \mathrm{IG}(S).$$

Relationships between $S = \langle E \rangle$, $\mathrm{IG}(E)$, and $\mathrm{RIG}(E)$

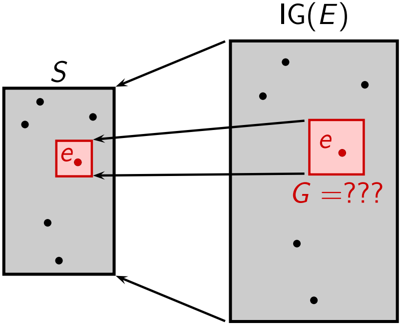
- (Easdown, 1985) The natural (surjective) homomorphism $\phi : \mathrm{IG}(E) \to S$ ($S = \langle E(S) \rangle$) has the following properties:
  - the restriction of ϕ to E is an isomorphism of biordered sets;
  - the maximal subgroup H_e in S is the ϕ-image of its counterpart in $\mathrm{IG}(E)$ (which is in turn isomorphic to its counterpart in $\mathrm{RIG}(E)$).
- The 'eggbox picture' of the $\mathcal{D}$-class of e has the same dimensions in all three.
- $\mathrm{IG}(E)$ may contain other, non-regular $\mathcal{D}$-classes.

So, understanding $\mathrm{IG}(E)$ is essential in understanding the structure of arbitrary IG semigroups.

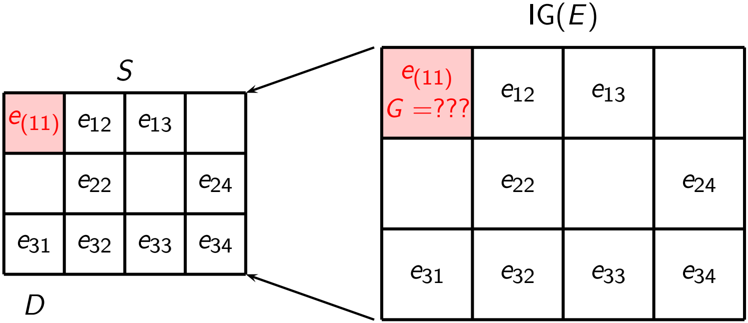
Question

Which groups arise as maximal subgroups of $\mathrm{IG}(E)$ (and thus of $\mathrm{RIG}(E)$)?

The big picture



Let's zoom in



S
$e_{(11)}$ e_{12} e_{13}
e_{22} e_{24}
e_{31} e_{32} e_{33} e_{34}
D
$\mathrm{IG}(E)$
$e_{(11)}$
$G = ???$
e_{12} e_{13}
e_{22} e_{24}
e_{31} e_{32} e_{33} e_{34}

Presentation for a max. subgroup of IG(E): Generators

Fact

G is generated by a set in 1-1 correspondence with $D \cap E(S)$.

D

$e_{(11)}$	e_{12}	e_{13}	
	e_{22}		e_{24}
e_{31}	e_{32}	e_{33}	e_{34}

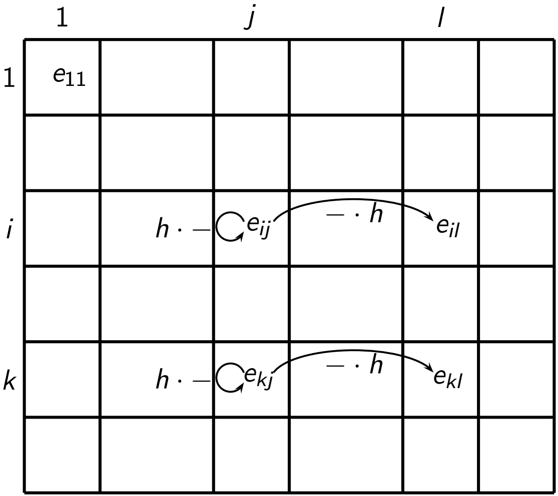
generators of G

f_{11}	f_{12}	f_{13}	
	f_{22}		f_{24}
f_{31}	f_{32}	f_{33}	f_{34}

$$G = \langle f_{ij} \ (e_{ij} \in D \cap E) \mid ??? \rangle$$

Typical relations: $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$

- $h = h^2$



Singular square $\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix} \Rightarrow$ relation $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$.

Presentation – Approach #1

Theorem (Nambooripad 1979; Gray, Ruškuc 2012)

The maximal subgroup G of $e \in E$ in $IG(E)$ or $RIG(E)$ is defined by the presentation:

$$\begin{aligned}\langle f_{ij} \mid f_{i,\pi(i)} &= 1 && (i \in I),\\ f_{ij} &= f_{il} && (\text{if } r_j e_{il} = r_l \text{ is a Schreier rep.}),\\ f_{ij}^{-1} f_{il} &= f_{kj}^{-1} f_{kl} && \left(\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix} \text{ sing. sq.}\right)\rangle.\end{aligned}$$

Proof: Reidemeister–Schreier rewriting process followed by Tietze transformations.

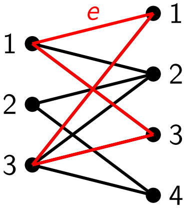
Graham-Houghton complex

Let S be an idempotent generated regular semigroup.

$GH(S)$: a 2-complex whose connected components are in a 1-1 correspondence with $\mathcal{D}$-classes of S.

D

$e_{(11)}$	e_{12}	e_{13}	
	e_{22}		e_{24}
e_{31}	e_{32}	e_{33}	e_{34}



Presentation – Approach #2

Theorem (Brittenham, Margolis, Meakin, 2009)

The fundamental group of $GH(S)$ at any point of its connected component C_e containing the edge $e \cong$ the maximal subgroup of $RIG(E(S))$ (and thus of $IG(E(S))$) containing e.

So,...

... let $\mathcal{T}$ be an arbitrary spanning tree of C_e. Then the maximal subgroup G of $e \in E$ in $\mathrm{IG}(E)$ (or $\mathrm{RIG}(E)$) is defined by the presentation:

$$\begin{aligned} \langle f_{ij} \mid f_{ij} &= 1 & &((i,j) \in \mathcal{T}), \\ f_{ij} f_{kj}^{-1} f_{kl} f_{il}^{-1} &= 1 & &((i,j,k,l) \text{ is a 2-cell})\rangle. \end{aligned}$$

Obviously, a clever choice of $\mathcal{T}$ may speed up the computation.

Remarks (1)

- Two types of relations:
  - Initial conditions: declaring some generators equal to 1 (or to each other in approach $\#1$);
  - Main relations: one per singular square.
- All relations of length ≤ 4.
- What can be defined by relations $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$?
- $\begin{bmatrix} 1 & b \\ a & c \end{bmatrix} \Rightarrow ab = c$.

Remarks (2)

- But: Every semigroup can be defined by relations of the form $ab = c$.
- Even better: Every finitely presented semigroup can be defined by finitely many relations of the form $ab = c$.
- Some more special squares ...
- $\begin{bmatrix} a & a \\ b & c \end{bmatrix} \Rightarrow b = c.$
- $\begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix} \Rightarrow a = 1.$

The freeness conjecture

Question

Which groups arise as maximal subgroups of $\mathrm{IG}(E)$ (and thus of $\mathrm{RIG}(E)$)?

- Work of Pastijn and Nambooripad ('70s and '80s) and McElwee (2002) led to the belief that these maximal subgroups must always be free groups (of a suitable rank).
- This conjecture was proved false by Brittenham, Margolis, and Meakin in 2009 who obtained the groups $\mathbb{Z} \oplus \mathbb{Z}$ (from a particular 73-element semigroup) and $\mathbb{F}^*$ for an arbitrary field $\mathbb{F}$.
- Finally, Gray and Ruškuc (2012) proved that every group arises as a maximal subgroup of some free idempotent generated semigroup (!!!).

Gray & Ruškuc (Israel J. Math., 2012)

Theorem

Every group is a maximal subgroup of some free idempotent generated semigroup (over a regular semigroup).

Theorem

Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup.

Remark

Maximal subgroups of free idempotent generated semigroups arising from finite semigroups have to be finitely presented by Reidemeister–Schreier.

Calculating the groups for natural examples of S

Some or all maximal subgroups in $\mathrm{IG}(E(S))$ have been calculated for the following S:

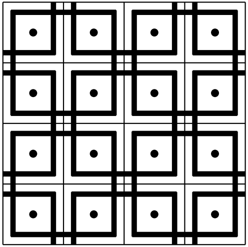
- Full transformation monoids: Gray, Ruškuc (symmetric groups, provided rank $\leq n-2$);
  (Proc. London Math. Soc., 2012)
- Partial transformation monoids: IgD (symmetric groups again);
  (Comm. Algebra, 2013)
- Full matrix monoid over a skew field: IgD, Gray (general linear groups, if rank $< n/3$, otherwise...);
  (Trans. Amer. Math. Soc., 2014)
- Endomorphism monoid of a free G-act: IgD, Gould, Yang (wreath products of G by symmetric groups).
  (J. Algebra, 2015)

Bands

Theorem (IgD, 2012)

For every left- or right seminormal band B, all maximal subgroups of $IG(B)$ are free. For every variety $\mathbf{V}$ *not contained in* $\mathbf{LSNB} \cup \mathbf{RSNB}$ *there exists $B \in \mathbf{V}$ such that $IG(B)$ contains a non-free maximal subgroup.*

An example of the GH-complex in a 20-element regular band (the top 2×2 $\mathcal{D}$-class not shown):



Bands

Question

Which groups arise as maximal subgroups of $\mathsf{IG}(B)$, B a (finite) band?

Answer (IgD, Ruškuc – IJAC, 2013): All of them! (Resp. all finitely presented ones.)

IgD & Ruškuc construction: set-up

Suppose we want to obtain

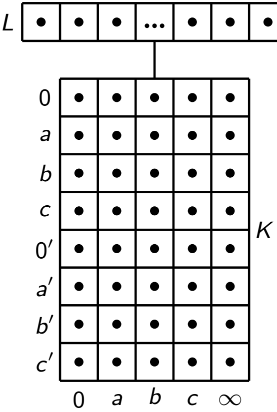
$$G = \langle a, b, c, \dots \mid ab = c, \dots \rangle$$

as a maximal subgroup of $\mathrm{IG}(B)$ for a band B.

- $I = \{0, a, b, c, \dots, 0', a', b', c', \dots\}$;
- $J = \{0, a, b, c, \dots, \infty\}$;
- $\mathcal{T} = \mathcal{T}_I^* \times \mathcal{T}_J$;
- the minimal ideal: $K = \{(\sigma, \tau) \,:\, \sigma, \tau \text{ constants}\}$;
- K is an $I \times J$ rectangular band.

IgD & Ruškuc construction: set-up

- $B = K \cup L$, where L is a left zero semigroup.
- We ensure this by virtue of every $(\sigma, \tau) \in L$ satisfying:
  - $\sigma^2 = \sigma$, $\tau^2 = \tau$;
  - $\ker(\sigma) = \{\{0, a, b, c\}, \{0', a', b', c'\}\}$;
  - thus σ is determined by its image $\{x, y\}$ transversing its kernel;
  - $\mathrm{im}(\tau) = \{0, a, b, c\}$;
  - thus τ is specified by $(\infty)\tau$.



IgD & Ruškuc construction: the action of L on K

Notation	**Indexing**	$\mathrm{im}(\boldsymbol{\sigma})$	$(\boldsymbol{\infty})\boldsymbol{\tau}$
(σ_0, τ_0)	–	$\{0, 0'\}$	0
(σ_a, τ_a)	$a \in A$	$\{0, a'\}$	a
$(\overline{\sigma}_a, \overline{\tau}_a)$	$a \in A$	$\{a, a'\}$	0
$(\sigma_{\mathbf{r}}, \tau_{\mathbf{r}})$	$\mathbf{r} = (ab, c) \in R$	$\{b, c'\}$	a

IgD & Ruškuc construction: the endgame

	0	a	b	c	∞
0	f_{00}	f_{0a}	f_{0b}	f_{0c}	$f_{0\infty}$
a	f_{a0}	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	f_{b0}	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	f_{c0}	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	$f_{0'0}$	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
a'	$f_{a'0}$	$f_{a'a}$	$f_{a'b}$	$f_{a'c}$	$f_{a'\infty}$
b'	$f_{b'0}$	$f_{b'a}$	$f_{b'b}$	$f_{b'c}$	$f_{b'\infty}$
c'	$f_{c'0}$	$f_{c'a}$	$f_{c'b}$	$f_{c'c}$	$f_{c'\infty}$

$$\xrightarrow[\,]{(\sigma_0,\tau_0)\;(\sigma_a,\tau_a)\;(\overline{\sigma}_a,\overline{\tau}_a)}$$

	0	a	b	c	∞
0	1	1	1	1	1
a	1	1	1	1	a
b	1	1	1	1	b
c	1	1	1	1	c
$0'$	1	a	b	c	1
a'	1	a	b	c	a
b'	1	a	b	c	b
c'	1	a	b	c	c

$$(\sigma_{\mathbf{r}},\tau_{\mathbf{r}}) \quad \mathbf{r} : ab = c$$

The word problem

A semigroup S with a finite generating set A has decidable word problem if there is an algorithm which for any two words $w_1, w_2 \in A^+$ decides whether or not they represent the same element of S.

Example

$S \cong \langle a, b \mid ab = ba \rangle$ has decidable word problem.

Some history:

- Markov (1947) and Post (1947): first examples of finitely presented semigroups with undecidable word problem;
- Turing (1950): finitely presented cancellative semigroup with undecidable word problem;
- Novikov (1955) and Boone (1958): finitely presented group with undecidable word problem.

The word problem for IG(E)

S – a semigroup, $E = E(S)$

Question

Does IG(E) have a decidable word problem if E is finite?

General facts:

- E finite $\Rightarrow$ every maximal subgroup of IG(E) is finitely presented,
- If IG(E) has a decidable word problem then every maximal subgroup of IG(E) must have a decidable word problem.

Hence, because of the Gray-Ruškuc result, the answer to the previous question is NO, because there is a finitely presented group with an undecidable WP. So, we obtain

Theorem

There exists a finite semigroup S such that $IG(E)$ has an undecidable word problem.

The word problem for IG(E) – reloaded

S – a semigroup, $E = E(S)$

Question (Updated)

Does IG(E) have a decidable word problem if E is finite and every maximal subgroup of IG(E) has a decidable word problem?

This question is the subject of the joint paper

IgD, R.D.Gray, N.Ruškuc: On regularity and the word problem for free idempotent generated semigroups, arXiv: 1412.5167, 33pp.

...just accepted few weeks ago in the Proc. London Math. Soc.

The good news

Question (Updated)

Does $\mathrm{IG}(E)$ have a decidable word problem if E is finite and every maximal subgroup of $\mathrm{IG}(E)$ has a decidable word problem?

Theorem

There exists an algorithm deciding whether $w \in E^+$ represents a regular element of $IG(E)$.

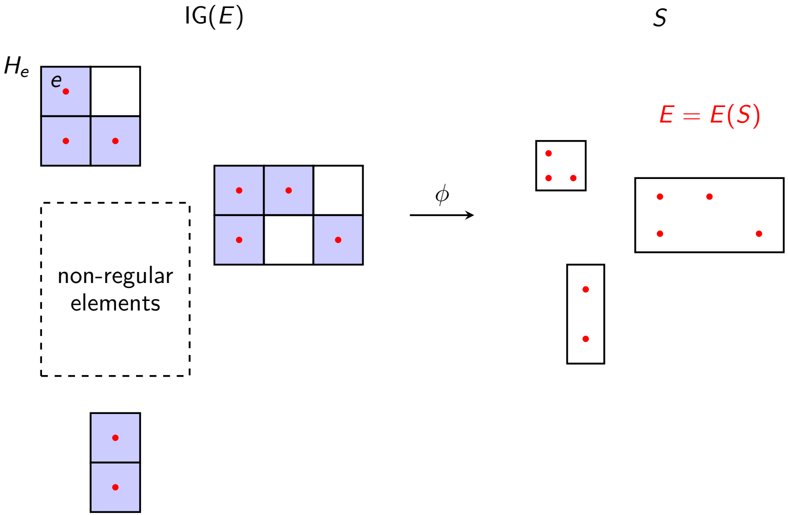
Theorem

If every maximal subgroup of $IG(E)$ has a solvable word problem, then there is an algorithm which, given $u, v \in E^+$ such that u represents a regular element, decides whether $u = v$ holds in $IG(E)$.

Corollary

The word problem for $RIG(E)$ is solvable iff the word problem for each of its maximal subgroups is solvable.

The general picture



The bad news

Question (Updated)

Does IG(E) have a decidable word problem if E is finite and every maximal subgroup of IG(E) has a decidable word problem?

IgD + RDG + NR: NO.

Theorem

There exists a finite band $B_{G,H}$ (constructed from a f.p. group G and its f.g. subgroup H) such that:

(i) *All maximal subgroups of $IG(B_{G,H})$ have decidable word problems.*

(ii) *The word problem for $IG(B_{G,H})$ is undecidable.*

For this result we make use of another decision problem...

The subgroup membership problem

Let G be a group with finite generating set A, and let H be a subgroup of G given by a finite set of words which generate H.

Then the membership problem for H in G is the problem of deciding, for an arbitrary word w over the generators A, whether or not w represents an element of the subgroup H.

Theorem (Mihailova, 1958)

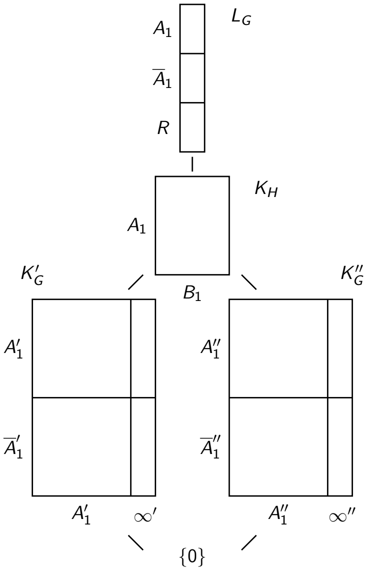
Their exists a finitely presented group G with a finitely generated subgroup H such that

- *G has a decidable word problem, but*
- *the membership problem for H in G is undecidable.*

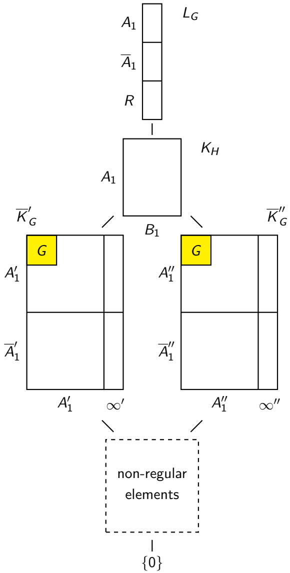
Example

$G = F \times F$ (F a f.g. free group), $H = \ker(\nu)$ ($\nu : F \to W$ a natural homomorphism onto a group with an undecidable WP).

The $B_{G,H}$ construction



Encoding the membership problem



Structure of IG$(B_{G,H})$

Each of $\overline{K}'_G$ and $\overline{K}''_G$ is a Rees matrix semigroup over G

$$\overline{K}'_G \cong I' \times G \times J', \quad \overline{K}''_G \cong I'' \times G \times J''.$$

For any word w over A the equality $(1', 1, 1')(1'', 1, 1'') = (1', w^{-1}, 1')(1'', w, 1'')$ holds in $\mathrm{IG}(B_{G,H}) \Leftrightarrow w \in H$.

Conclusion: If $\mathrm{IG}(B_{G,H})$ had a decidable word problem this would imply the membership problem for H in G is decidable, which is a contradiction. ↯

THANK YOU!

Questions and comments to:

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