Free idempotent generated semigroups and their maximal subgroups

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I'm free like a river
Flowin' freely to infinity
I'm free to be sure of what
I am and who I need not be
I'm much freer - like the meaning
Of the word 'free' that crazy man defines
Free - free like the vision that
The mind of only you are ever gonna see

Stevie Wonder: Free

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Hence:

What can we say about the structure of the free-est idempotent-generated (IG) semigroup with a fixed structure/configuration of idempotents ???

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Alternatively, biordered sets can be described as relational structures $(E(S), \leq^{(l)}, \leq^{(r)})$ with two quasi-orders and several simple rules/axioms (Easdown, Nambooripad, '80s).

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- To every regular semigroup S with idempotents E associate the free-est regular semigroup $\mathsf{RIG}(E)$ in whose idempotents form the same biordered set as in S.

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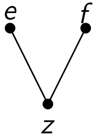
Let $S =$ 

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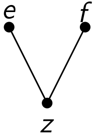
Let $S =$ 

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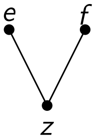
Let $S =$ 

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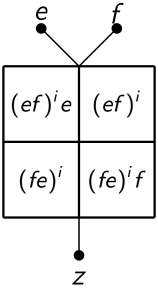

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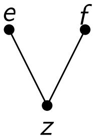
Let $S =$ 

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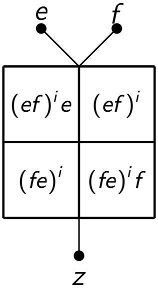

$$RIG(S) = \langle e, f, z \mid \mathrm{IG},\ ef = fe = z \rangle = S.$$

Example 2: 2×2 rectangular band

$S = \langle e_{ij} \mid e_{ij}e_{kl} = e_{il} \ (i, j, k, l \in \{1, 2\})\rangle$:

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$\mathrm{RIG}(S) = \mathrm{IG}(S)$.

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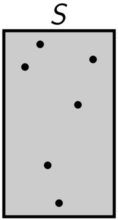
Question

Which groups arise as maximal subgroups of $\mathsf{IG}(E)$ (and thus of $\mathsf{RIG}(E)$)?

The big picture



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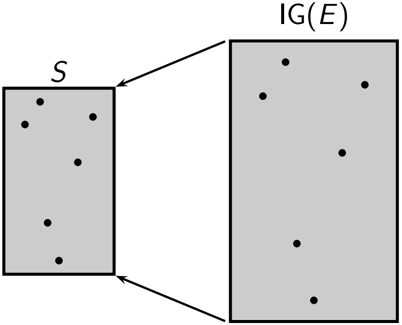

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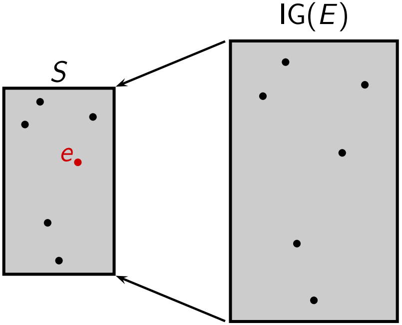

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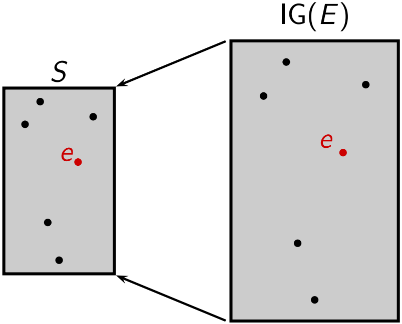

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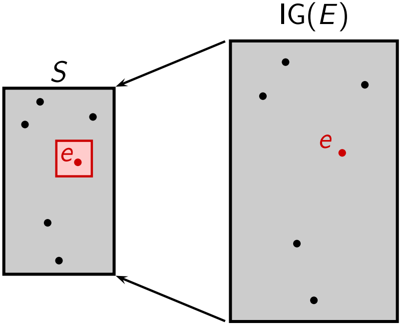

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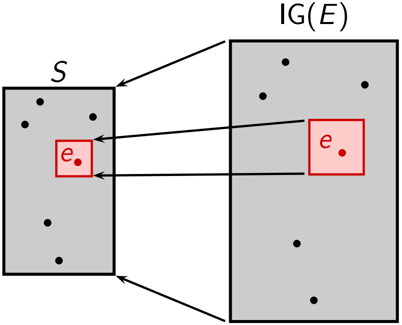

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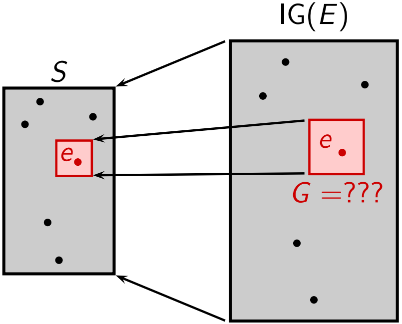
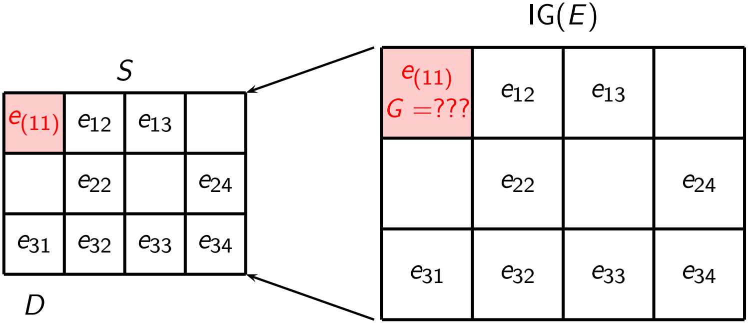


Let's zoom in



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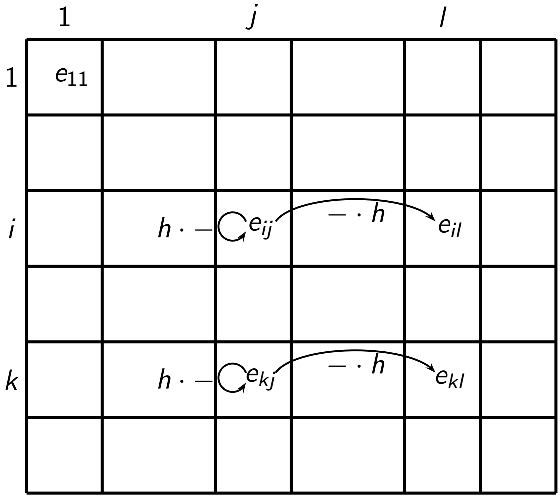
$$G = \langle f_{ij} \ (e_{ij} \in D \cap E) \mid ??? \rangle$$

Typical relations: $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$

- $h = h^2$



Singular square $\begin{bmatrix} e_{ij} & e_{il} \\ e_{kj} & e_{kl} \end{bmatrix} \Rightarrow$ relation $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$.

Presentation – Approach #1

Theorem (Nambooripad 1979; Gray, Ruškuc 2012)
The maximal subgroup G of $e \in E$ in $IG(E)$ or $RIG(E)$ is defined by the presentation:

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\begin{aligned}
\langle f_{ij} \mid f_{i,\pi(i)} &= 1 && (i \in I), \\
f_{ij} &= f_{il} && (\text{if } r_j e_{il} = r_l \text{ is a Schreier rep.}), \\
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Proof: Reidemeister–Schreier rewriting process followed by Tietze transformations.

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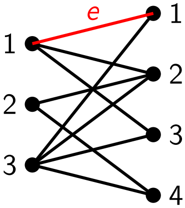

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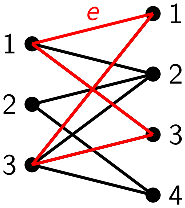

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Theorem (Brittenham, Margolis, Meakin, 2009)

The fundamental group of $GH(S)$ at any point of its connected component C_e containing the edge $e \cong$ the maximal subgroup of $RIG(E(S))$ (and thus of $IG(E(S))$) containing e.

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Obviously, a clever choice of $\mathcal{T}$ may speed up the computation.

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- What can be defined by relations $f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl}$?
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Question

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- Finally, Gray and Ruškuc (2012) proved that every group arises as a maximal subgroup of some free idempotent generated semigroup (!!!).

Gray & Ruškuc, 2012

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Every group is a maximal subgroup of some free idempotent generated semigroup (over a regular semigroup).

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Remaining Question

Is every finitely presented group a maximal subgroup in some free idempotent generated semigroup over a finite regular semigroup?

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- Endomorphism monoid of a free G-act: Gould, Yang.

Bands (aka idempotent semigroups)

Theorem (ID)

For every left- or right seminormal band B, all maximal subgroups of $IG(B)$ are free. For every variety $\mathbf{V}$ *not contained in* $\mathbf{LSNB} \cup \mathbf{RSNB}$ *there exists* $B \in \mathbf{V}$ *such that* $IG(B)$ *contains a non-free maximal subgroup.*

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Answer (ID, Ruškuc, 2013): All of them! (Resp. all finitely presented ones.)

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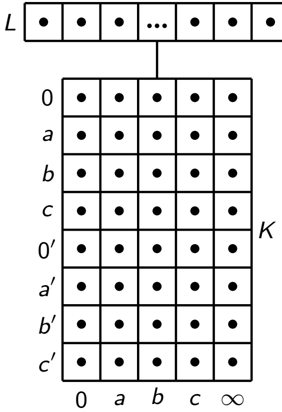
- $\mathcal{T} = \mathcal{T}_I^* \times \mathcal{T}_J$;
- the minimal ideal: $K = \{(\sigma, \tau) : \sigma, \tau \text{ constants}\}$;
- K is an $I \times J$ rectangular band.

New construction: set-up

- $B = K \cup L$, where L is a left zero semigroup.



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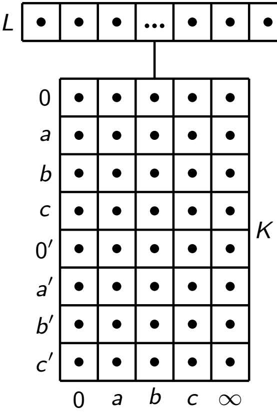

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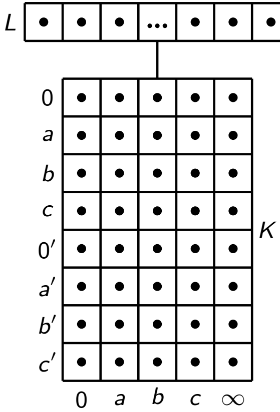

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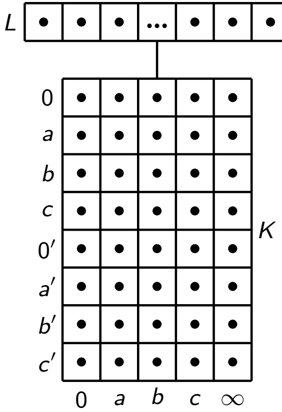

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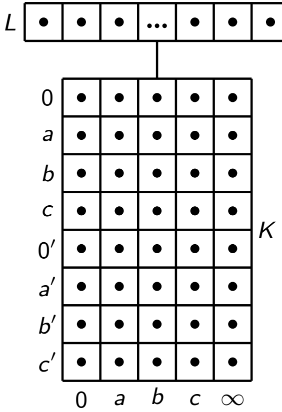

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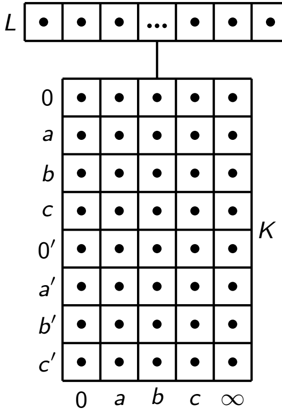

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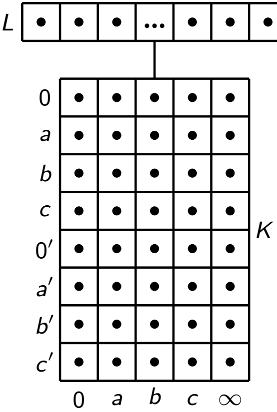

New construction: process

	0	a	b	c	∞
0	f_{00}	f_{0a}	f_{0b}	f_{0c}	$f_{0\infty}$
a	f_{a0}	f_{aa}	f_{ab}	f_{ac}	$f_{a\infty}$
b	f_{b0}	f_{ba}	f_{bb}	f_{bc}	$f_{b\infty}$
c	f_{c0}	f_{ca}	f_{cb}	f_{cc}	$f_{c\infty}$
$0'$	$f_{0'0}$	$f_{0'a}$	$f_{0'b}$	$f_{0'c}$	$f_{0'\infty}$
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Initial relations

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Subgroups of $\mathrm{IG}(B)$, B band

Theorem

For any group G there exists a band B such that $\mathrm{IG}(B)$ has a maximal subgroup isomorphic to G. Furthermore, if G is finitely presented, then B can be constructed to be finite.

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Remark

- B has two $\mathcal{D}$-classes, K and L.
- If $G = \langle A \mid R \rangle$ with $|A| = m$, $|R| = n$, then
  - K is a $(2m+2) \times (m+2)$ rectangular band;
  - L is a left zero semigroup of order $2m+n+1$.

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Open Problem

What about finite bands?

THANK YOU!

Questions and comments to:

dockie@dmi.uns.ac.rs

Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie