Finite groups are big as semigroups

Igor Dolinka

dockie@dmi.uns.ac.rs

Department of Mathematics and Informatics, University of Novi Sad

The 13th NBSAN Meeting

York, November 21, 2012

Big algebraic structures

Big algebraic structures

Let $\mathcal{K}$ be a class of algebraic structures of a given similarity type (usually a variety or some other well-behaved class).

Big algebraic structures

Let $\mathcal{K}$ be a class of algebraic structures of a given similarity type (usually a variety or some other well-behaved class).

A finite algebra $B \in \mathcal{K}$ is $\mathcal{K}$-big

Big algebraic structures

Let $\mathcal{K}$ be a class of algebraic structures of a given similarity type (usually a variety or some other well-behaved class).

A finite algebra $B \in \mathcal{K}$ is $\mathcal{K}$-big if there exists a countably infinite algebra $A \in \mathcal{K}$

Big algebraic structures

Let $\mathcal{K}$ be a class of algebraic structures of a given similarity type (usually a variety or some other well-behaved class).

A finite algebra $B \in \mathcal{K}$ is $\mathcal{K}$-big if there exists a countably infinite algebra $A \in \mathcal{K}$ such that B is isomorphic to a maximal proper subalgebra of A.

Big algebraic structures

Let $\mathcal{K}$ be a class of algebraic structures of a given similarity type (usually a variety or some other well-behaved class).

A finite algebra $B \in \mathcal{K}$ is $\mathcal{K}$-big if there exists a countably infinite algebra $A \in \mathcal{K}$ such that B is isomorphic to a maximal proper subalgebra of A.

In other words, $A = \langle B, a \rangle$ for any $a \in A \setminus B$.

Motivation: Lattices

Motivation: Lattices

In 2001, Freese, Ježek and Nation published a paper where they fully described big lattices.

Motivation: Lattices

In 2001, Freese, Ježek and Nation published a paper where they fully described big lattices.

Theorem

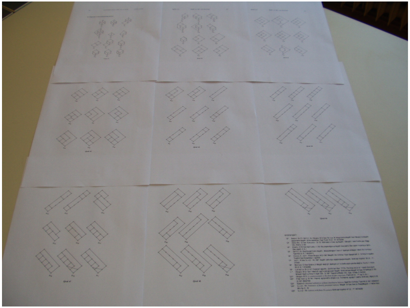
There exist a list of 145 lattices (in fact, only 81 of them, up to dual isomorphism) such that a finite lattice is big if and only if it contains one of the lattices from the list as a sublattice.

Motivation: Lattices

In 2001, Freese, Ježek and Nation published a paper where they fully described big lattices.

Theorem

There exist a list of 145 lattices (in fact, only 81 of them, up to dual isomorphism) such that a finite lattice is big if and only if it contains one of the lattices from the list as a sublattice.



Groups: serious issues

Groups: serious issues

Open Problem

Characterise big groups.

Groups: serious issues

Open Problem

Characterise big groups.

This is closely related to difficult Burnside-type problems.

Groups: serious issues

Open Problem

Characterise big groups.

This is closely related to difficult Burnside-type problems.

Ol'shanskii (1982): Constructed the first *Tarski monster group* — for each prime $p > 10^{75}$ there exists a 2-generated infinite group all of whose nontrivial proper subgroups have order p.

Groups: serious issues

Open Problem

Characterise big groups.

This is closely related to difficult Burnside-type problems.

Ol'shanskii (1982): Constructed the first *Tarski monster group* — for each prime $p > 10^{75}$ there exists a 2-generated infinite group all of whose nontrivial proper subgroups have order p.

$\Longrightarrow \mathbb{Z}_p$ is a big group for any prime $p > 10^{75}$.

Groups: serious issues

Open Problem

Characterise big groups.

This is closely related to difficult Burnside-type problems.

Ol'shanskii (1982): Constructed the first *Tarski monster group* — for each prime $p > 10^{75}$ there exists a 2-generated infinite group all of whose nontrivial proper subgroups have order p.
$\Longrightarrow \mathbb{Z}_p$ is a big group for any prime $p > 10^{75}$.

Adyan & Lysionok (1991): For any odd $n \geq 1003$ there exists a 2-generated infinite group G such that any proper subgroup of G is contained in a cyclic subgroup of order n.

Groups: serious issues

Open Problem

Characterise big groups.

This is closely related to difficult Burnside-type problems.

Ol'shanskii (1982): Constructed the first *Tarski monster group* — for each prime $p > 10^{75}$ there exists a 2-generated infinite group all of whose nontrivial proper subgroups have order p.

$\Longrightarrow \mathbb{Z}_p$ is a big group for any prime $p > 10^{75}$.

Adyan & Lysionok (1991): For any odd $n \geq 1003$ there exists a 2-generated infinite group G such that any proper subgroup of G is contained in a cyclic subgroup of order n.

$\Longrightarrow \mathbb{Z}_{2k+1}$ is a big group for any $k \geq 501$.

Relaxing the problem

Relaxing the problem

Problem (R. Gray, P. Marković, 2011)

Which finite groups are big with respect to the class of all semigroups?

Relaxing the problem

Problem (R. Gray, P. Marković, 2011)

Which finite groups are big with respect to the class of all semigroups?

Theorem (ID & N. Ruškuc)

A finite group G is big with respect to the class of all semigroups if and only if $|G| \geq 3$.

Relaxing the problem

Problem (R. Gray, P. Marković, 2011)

Which finite groups are big with respect to the class of all semigroups?

Theorem (ID & N. Ruškuc)

A finite group G is big with respect to the class of all semigroups if and only if $|G| \geq 3$.

Theorem (ID & NR)

Each finite semigroup S such that the kernel (the unique minimal ideal) of S contains a subgroup G such that $|G| \geq 3$ is a big semigroup.

Relaxing the problem

Problem (R. Gray, P. Marković, 2011)

Which finite groups are big with respect to the class of all semigroups?

Theorem (ID & N. Ruškuc)

A finite group G is big with respect to the class of all semigroups if and only if $|G| \geq 3$.

Theorem (ID & NR)

Each finite semigroup S such that the kernel (the unique minimal ideal) of S contains a subgroup G such that $|G| \geq 3$ is a big semigroup.

Also: we should take care of $\mathbb{Z}_2$ and the trivial group...

A simple, yet important fact

A simple, yet important fact

If S is a big (finite) semigroup such that it is $\cong$ a maximal proper subsemigroup of an infinite semigroup T, then T is called a witness for S.

A simple, yet important fact

If S is a big (finite) semigroup such that it is $\cong$ a maximal proper subsemigroup of an infinite semigroup T, then T is called a witness for S.

Lemma

If $T \supset S$ is a witness for a big semigroup S, then $T \setminus S$ is contained in a single $\mathscr{J}$-class of T.

A simple, yet important fact

If S is a big (finite) semigroup such that it is $\cong$ a maximal proper subsemigroup of an infinite semigroup T, then T is called a witness for S.

Lemma

If $T \supset S$ is a witness for a big semigroup S, then $T \setminus S$ is contained in a single $\mathscr{J}$-class of T. In particular, if S is a group, then T can have at most two $\mathscr{J}$-classes.

A simple, yet important fact

If S is a big (finite) semigroup such that it is $\cong$ a maximal proper subsemigroup of an infinite semigroup T, then T is called a witness for S.

Lemma

If $T \supset S$ is a witness for a big semigroup S, then $T \setminus S$ is contained in a single $\mathscr{J}$-class of T. In particular, if S is a group, then T can have at most two $\mathscr{J}$-classes.

Idea: Construct a witness Σ_S for S as an ideal extension of an infinite Rees matrix semigroup M by S^0,

A simple, yet important fact

If S is a big (finite) semigroup such that it is $\cong$ a maximal proper subsemigroup of an infinite semigroup T, then T is called a witness for S.

Lemma

If $T \supset S$ is a witness for a big semigroup S, then $T \setminus S$ is contained in a single $\mathscr{J}$-class of T. In particular, if S is a group, then T can have at most two $\mathscr{J}$-classes.

Idea: Construct a witness Σ_S for S as an ideal extension of an <u>infinite</u> Rees matrix semigroup M by S^0, so that $\Sigma = S \cup M$,

A simple, yet important fact

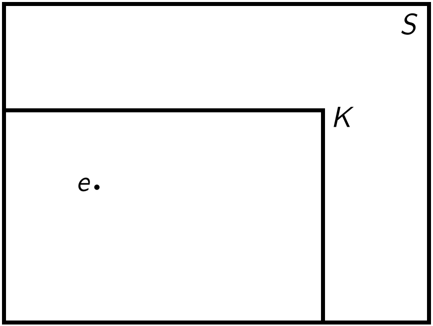
If S is a big (finite) semigroup such that it is $\cong$ a maximal proper subsemigroup of an infinite semigroup T, then T is called a witness for S.

Lemma

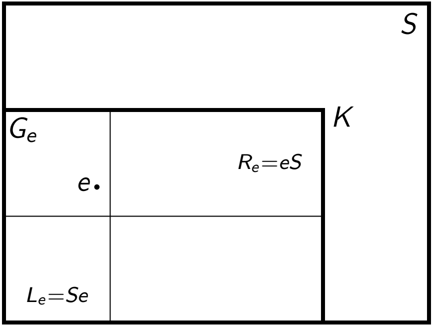
If $T \supset S$ is a witness for a big semigroup S, then $T \setminus S$ is contained in a single $\mathscr{J}$-class of T. In particular, if S is a group, then T can have at most two $\mathscr{J}$-classes.

Idea: Construct a witness Σ_S for S as an ideal extension of an <u>infinite</u> Rees matrix semigroup M by S^0, so that $\Sigma = S \cup M$, where S acts on M (from left and right) sufficiently 'transitively' to move around an arbitrary $a \in M$ along a generating set of M.

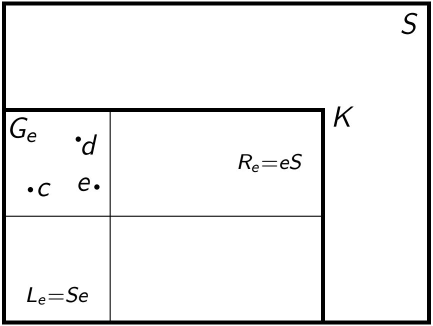
The kernel K of S



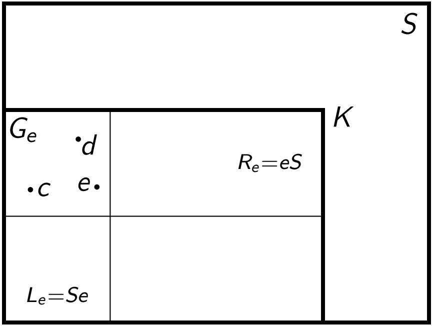
The kernel K of S



The kernel K of S

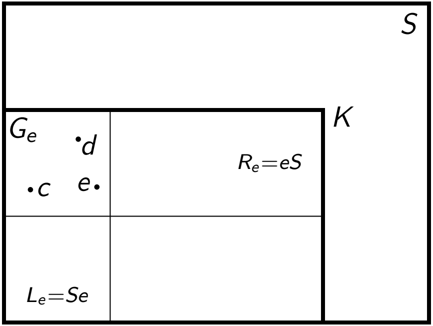


The kernel K of S



By assumption, we must have $|G_e| \geq 3$.

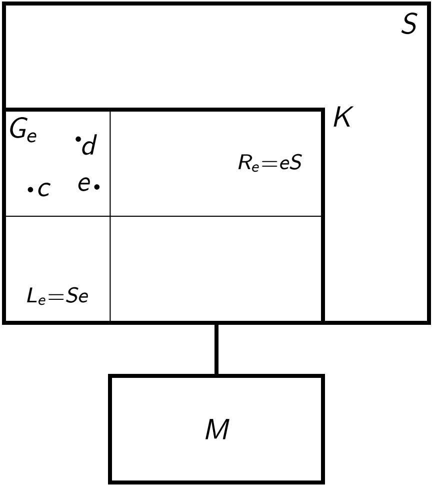
The kernel K of S



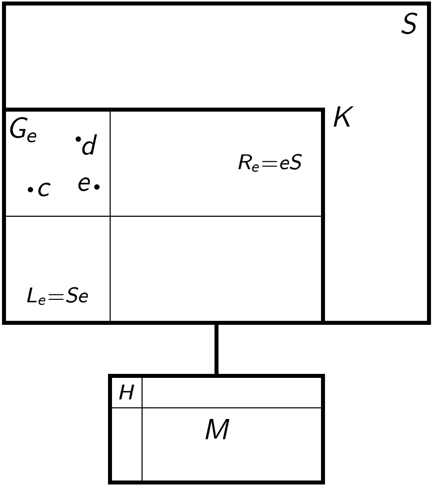
By assumption, we must have $|G_e| \geq 3$.

Thus we may fix two non-identity elements $c, d \in G_e = eSe$.

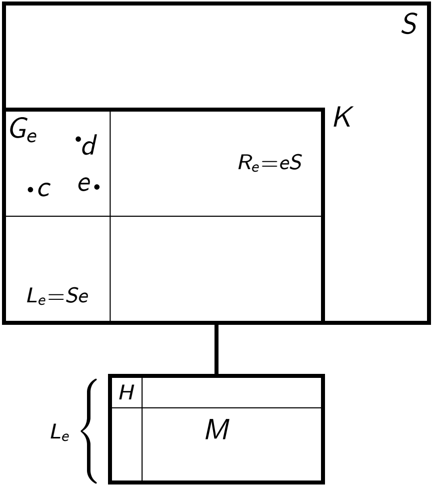
Σ_S



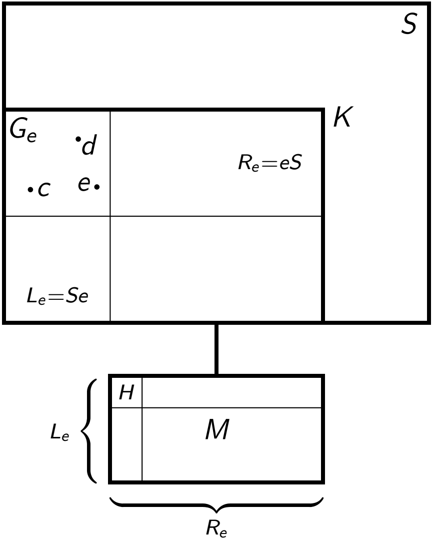
Σ_S



Σ_S



Σ_S



M: Choosing the structure group and the sandwich matrix

M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

Remark

By this, we have ensured that M is infinite, periodic, finitely generated, and has finitely many left and right ideals.

M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

Remark

By this, we have ensured that M is infinite, periodic, finitely generated, and has finitely many left and right ideals.

Consider a function $\lambda : L_e \cup R_e \to H$ with the following properties:

M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

Remark

By this, we have ensured that M is infinite, periodic, finitely generated, and has finitely many left and right ideals.

Consider a function $\lambda : L_e \cup R_e \to H$ with the following properties:

1. $\lambda(e) = 1_H$,

M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

Remark

By this, we have ensured that M is infinite, periodic, finitely generated, and has finitely many left and right ideals.

Consider a function $\lambda : L_e \cup R_e \to H$ with the following properties:

1. $\lambda(e) = 1_H$,
2. $\lambda(c) = \gamma_1$,

M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

Remark

By this, we have ensured that M is infinite, periodic, finitely generated, and has finitely many left and right ideals.

Consider a function $\lambda : L_e \cup R_e \to H$ with the following properties:

1. $\lambda(e) = 1_H$,
2. $\lambda(c) = \gamma_1$,
3. $\lambda(d) = \gamma_2$,

M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

Remark

By this, we have ensured that M is infinite, periodic, finitely generated, and has finitely many left and right ideals.

Consider a function $\lambda : L_e \cup R_e \to H$ with the following properties:

1. $\lambda(e) = 1_H$,
2. $\lambda(c) = \gamma_1$,
3. $\lambda(d) = \gamma_2$,
4. $\lambda(se) = \lambda(ese)$ for all $s \in S$.

M: Choosing the structure group and the sandwich matrix

Let H be a 2-generated infinite periodic group, $H = \langle \gamma_1, \gamma_2 \rangle$ (i.e. a counterexample to the Burnside conjecture).

Remark

By this, we have ensured that M is infinite, periodic, finitely generated, and has finitely many left and right ideals.

Consider a function $\lambda : L_e \cup R_e \to H$ with the following properties:

1. $\lambda(e) = 1_H$,
2. $\lambda(c) = \gamma_1$,
3. $\lambda(d) = \gamma_2$,
4. $\lambda(se) = \lambda(ese)$ for all $s \in S$. (That is, the value of λ on $L_e = Se$ is completely determined by its values on $G_e = eSe$.)

M: Choosing the sandwich matrix (2)

Recall that

$$M = \mathcal{M}(L_e, H, R_e, P)$$

where $P = [p_{a,b}]$ is a $R_e \times L_e$ matrix.

M: Choosing the sandwich matrix (2)

Recall that

$$M = \mathcal{M}(L_e, H, R_e, P)$$

where $P = [p_{a,b}]$ is a $R_e \times L_e$ matrix.

For $a \mathscr{R} e \mathscr{L} b$ we define

$$p_{a,b} = \lambda(a)^{-1}\lambda(ab)\lambda(b)^{-1}$$

M: Choosing the sandwich matrix (2)

Recall that

$$M = \mathcal{M}(L_e, H, R_e, P)$$

where $P = [p_{a,b}]$ is a $R_e \times L_e$ matrix.

For $a \mathscr{R} e \mathscr{L} b$ we define

$$p_{a,b} = \lambda(a)^{-1}\lambda(ab)\lambda(b)^{-1}$$

Remark

Since $be = b$ we have $p_{e,b} = 1_H$.

M: Choosing the sandwich matrix (2)

Recall that

$$M = \mathcal{M}(L_e, H, R_e, P)$$

where $P = [p_{a,b}]$ is a $R_e \times L_e$ matrix.

For $a \mathscr{R} e \mathscr{L} b$ we define

$$p_{a,b} = \lambda(a)^{-1}\lambda(ab)\lambda(b)^{-1}$$

Remark

Since $be = b$ we have $p_{e,b} = 1_H$.

The multiplication between S and M is defined by:

$$s \cdot (a, h, b) \quad = \quad (sa, \lambda(sa)\lambda(a)^{-1}h, b)$$

M: Choosing the sandwich matrix (2)

Recall that

$$M = \mathcal{M}(L_e, H, R_e, P)$$

where $P = [p_{a,b}]$ is a $R_e \times L_e$ matrix.

For $a \mathscr{R} e \mathscr{L} b$ we define

$$p_{a,b} = \lambda(a)^{-1}\lambda(ab)\lambda(b)^{-1}$$

Remark

Since $be = b$ we have $p_{e,b} = 1_H$.

The multiplication between S and M is defined by:

$$\begin{aligned} s \cdot (a, h, b) &= (sa, \lambda(sa)\lambda(a)^{-1}h, b) \\ (a, h, b) \cdot s &= (a, h\lambda(b)^{-1}\lambda(bs), bs) \end{aligned}$$

The definition is OK

Lemma

Σ_S is a semigroup.

The definition is OK

Lemma
Σ_S is a semigroup.

Remark
In the particular case when S is a group, the associativity of Σ_S boils down to an elementary fact in geometric group theory: there is a balanced labelling of the Cayley graph of S by elements of H such that two given non-loop edges are labelled by γ_1 and γ_2 respectively.

The definition is OK

Lemma

Σ_S is a semigroup.

Remark

In the particular case when S is a group, the associativity of Σ_S boils down to an elementary fact in geometric group theory: there is a balanced labelling of the Cayley graph of S by elements of H such that two given non-loop edges are labelled by γ_1 and γ_2 respectively. (A spanning tree argument...)

The definition is OK

Lemma

Σ_S is a semigroup.

Remark

In the particular case when S is a group, the associativity of Σ_S boils down to an elementary fact in geometric group theory: there is a balanced labelling of the Cayley graph of S by elements of H such that two given non-loop edges are labelled by γ_1 and γ_2 respectively. (A spanning tree argument...)

The definitions of λ, P and $\cdot$ between S and M are motivated by (and are one implementation of) this.

Proof of the Main Theorem (1)

Proof of the Main Theorem (1)

Let $h_0 \in H$ and $a \mathscr{R} e \mathscr{L} b$ be arbitrary.

Proof of the Main Theorem (1)

Let $h_0 \in H$ and $a \mathscr{R} e \mathscr{L} b$ be arbitrary.

Goal: Prove $T \equiv \langle S, (a, h_0, b) \rangle = \Sigma_S$.

Proof of the Main Theorem (1)

Let $h_0 \in H$ and $a \mathscr{R} e \mathscr{L} b$ be arbitrary.

Goal: Prove $T \equiv \langle S, (a, h_0, b) \rangle = \Sigma_S$.

There is no loss of generality in assuming that $a = b = e$, for otherwise $\exists s, t \in S$ such that $sa = bt = e$, and so

Proof of the Main Theorem (1)

Let $h_0 \in H$ and $a \mathscr{R} e \mathscr{L} b$ be arbitrary.

Goal: Prove $T \equiv \langle S, (a, h_0, b) \rangle = \Sigma_S$.

There is no loss of generality in assuming that $a = b = e$, for otherwise $\exists s, t \in S$ such that $sa = bt = e$, and so

$$s(a, h_0, b)t = (e, \lambda(sa)\lambda(a)^{-1}h_0\lambda(b)^{-1}\lambda(bt), e) \in T,$$

Proof of the Main Theorem (1)

Let $h_0 \in H$ and $a \mathscr{R} e \mathscr{L} b$ be arbitrary.

Goal: Prove $T \equiv \langle S, (a, h_0, b) \rangle = \Sigma_S$.

There is no loss of generality in assuming that $a = b = e$, for otherwise $\exists s, t \in S$ such that $sa = bt = e$, and so

$$s(a, h_0, b)t = (e, \lambda(sa)\lambda(a)^{-1}h_0\lambda(b)^{-1}\lambda(bt), e) \in T,$$

and we may continue working with

$$h_0' = \lambda(sa)\lambda(a)^{-1}h_0\lambda(b)^{-1}\lambda(bt)$$

instead of h_0.

Proof of the Main Theorem (2)

Revised goal: Prove $T \equiv \langle S, (e, h_0, e)\rangle = \Sigma_S$.

Proof of the Main Theorem (2)

Revised goal: Prove $T \equiv \langle S, (e, h_0, e) \rangle = \Sigma_S$.

Recall that we have picked $c, d \in G_e \setminus \{e\}$ carrying λ-labels γ_1 and γ_2, respectively.

Proof of the Main Theorem (2)

Revised goal: Prove $T \equiv \langle S, (e, h_0, e) \rangle = \Sigma_S$.

Recall that we have picked $c, d \in G_e \setminus \{e\}$ carrying λ-labels γ_1 and γ_2, respectively. Since H is periodic, $h_0^m = 1_H$ for some $m \in \mathbb{N}$.

Proof of the Main Theorem (2)

Revised goal: Prove $T \equiv \langle S, (e, h_0, e)\rangle = \Sigma_S$.

Recall that we have picked $c, d \in G_e \setminus \{e\}$ carrying λ-labels γ_1 and γ_2, respectively. Since H is periodic, $h_0^m = 1_H$ for some $m \in \mathbb{N}$. So, the following are elements of T:

Proof of the Main Theorem (2)

Revised goal: Prove $T \equiv \langle S, (e, h_0, e) \rangle = \Sigma_S$.

Recall that we have picked $c, d \in G_e \setminus \{e\}$ carrying λ-labels γ_1 and γ_2, respectively. Since H is periodic, $h_0^m = 1_H$ for some $m \in \mathbb{N}$. So, the following are elements of T:

$$\begin{aligned} (e, h_0, e)^m c (e, h_0, e)^m &= (e, 1_H, e)(c, \lambda(c), e) \\ &= (e, p_{e,c}\lambda(c), e) \\ &= (e, \gamma_1, e), \end{aligned}$$

Proof of the Main Theorem (2)

Revised goal: Prove $T \equiv \langle S, (e, h_0, e)\rangle = \Sigma_S$.

Recall that we have picked $c, d \in G_e \setminus \{e\}$ carrying λ-labels γ_1 and γ_2, respectively. Since H is periodic, $h_0^m = 1_H$ for some $m \in \mathbb{N}$. So, the following are elements of T:

$$\begin{aligned}
(e, h_0, e)^m c (e, h_0, e)^m &= (e, 1_H, e)(c, \lambda(c), e) \\
&= (e, p_{e,c}\lambda(c), e) \\
&= (e, \gamma_1, e),
\end{aligned}$$

$$\begin{aligned}
(e, h_0, e)^m d (e, h_0, e)^m &= (e, 1_H, e)(d, \lambda(d), e) \\
&= (e, p_{e,d}\lambda(d), e) \\
&= (e, \gamma_2, e).
\end{aligned}$$

Proof of the Main Theorem (3)

Therefore, $H_e = \{e\} \times H \times \{e\} \subseteq T$.

Proof of the Main Theorem (3)

Therefore, $H_e = \{e\} \times H \times \{e\} \subseteq T$.

However, then for any $a \mathscr{R} e \mathscr{L} b$ we have

$$aH_e b = \{a\} \times H \times \{b\} \subseteq T,$$

Proof of the Main Theorem (3)

Therefore, $H_e = \{e\} \times H \times \{e\} \subseteq T$.

However, then for any $a \mathscr{R} e \mathscr{L} b$ we have

$$aH_e b = \{a\} \times H \times \{b\} \subseteq T,$$

because

$$a(e, h, e)b = (a, \lambda(a)h\lambda(b), b),$$

and $x \mapsto \lambda(a)x\lambda(b)$ is a permutation of H.

Proof of the Main Theorem (3)

Therefore, $H_e = \{e\} \times H \times \{e\} \subseteq T$.

However, then for any $a \mathscr{R} e \mathscr{L} b$ we have

$$aH_e b = \{a\} \times H \times \{b\} \subseteq T,$$

because

$$a(e, h, e)b = (a, \lambda(a)h\lambda(b), b),$$

and $x \mapsto \lambda(a)x\lambda(b)$ is a permutation of H.

Hence, $L_e \times H \times R_e \subseteq T$, so $T = \Sigma_S$, Q.E.D.

The trivial (semi)group is not big

The trivial (semi)group is not big

Suppose, to the contrary, that S is a witness for $\{e\}$, $e \in E(S)$.

The trivial (semi)group is not big

Suppose, to the contrary, that S is a witness for $\{e\}$, $e \in E(S)$.

Both Se and eS are subsemigroups of S containing e, so $Se, eS \in \{\{e\}, S\}$.

The trivial (semi)group is not big

Suppose, to the contrary, that S is a witness for $\{e\}$, $e \in E(S)$.

Both Se and eS are subsemigroups of S containing e, so $Se, eS \in \{\{e\}, S\}$.

If $Se = eS = S$, then e is an identity element of S, and if $Se = eS = \{e\}$, then e is the zero of S.

The trivial (semi)group is not big

Suppose, to the contrary, that S is a witness for $\{e\}$, $e \in E(S)$.

Both Se and eS are subsemigroups of S containing e, so $Se, eS \in \{\{e\}, S\}$.

If $Se = eS = S$, then e is an identity element of S, and if $Se = eS = \{e\}$, then e is the zero of S.

In either case, for any $s \in S \setminus \{e\}$ we have $S = \langle e, s\rangle = \{e, s, s^2, \dots\}$, where s is not periodic (because S is infinite), so $\{e, s^2, s^4, \dots\}$ is a proper subsemigroup of S containing e.

The trivial (semi)group is not big

Suppose, to the contrary, that S is a witness for $\{e\}$, $e \in E(S)$.

Both Se and eS are subsemigroups of S containing e, so $Se, eS \in \{\{e\}, S\}$.

If $Se = eS = S$, then e is an identity element of S, and if $Se = eS = \{e\}$, then e is the zero of S.

In either case, for any $s \in S \setminus \{e\}$ we have $S = \langle e, s \rangle = \{e, s, s^2, \dots\}$, where s is not periodic (because S is infinite), so $\{e, s^2, s^4, \dots\}$ is a proper subsemigroup of S containing e.

If $Se = S$ and $eS = \{e\}$ ($Se = \{e\}$ and $eS = S$) then S is a left (resp. right) zero semigroup $\Longrightarrow$ every subset of S is a subsemigroup.

The trivial (semi)group is not big

Suppose, to the contrary, that S is a witness for $\{e\}$, $e \in E(S)$.

Both Se and eS are subsemigroups of S containing e, so $Se, eS \in \{\{e\}, S\}$.

If $Se = eS = S$, then e is an identity element of S, and if $Se = eS = \{e\}$, then e is the zero of S.

In either case, for any $s \in S \setminus \{e\}$ we have $S = \langle e, s \rangle = \{e, s, s^2, \dots\}$, where s is not periodic (because S is infinite), so $\{e, s^2, s^4, \dots\}$ is a proper subsemigroup of S containing e.

If $Se = S$ and $eS = \{e\}$ ($Se = \{e\}$ and $eS = S$) then S is a left (resp. right) zero semigroup $\Longrightarrow$ every subset of S is a subsemigroup. Contradiction!

Yet another useful...

Lemma

Let S be a big semigroup, and let T be any witness for S. Let J be the unique $\mathscr{J}$-class of T containing $T \setminus S$. Then J contains a J-primitive idempotent, that is, a minimal element in the restriction of the Rees order of idempotents of T to $J \cap E(T)$.

Yet another useful...

Lemma

Let S be a big semigroup, and let T be any witness for S. Let J be the unique $\mathscr{J}$-class of T containing $T \setminus S$. Then J contains a J-primitive idempotent, that is, a minimal element in the restriction of the Rees order of idempotents of T to $J \cap E(T)$.

Steps:

(i) There exist $a, b \in J$ such that $ab \in J$.

Yet another useful...

Lemma

Let S be a big semigroup, and let T be any witness for S. Let J be the unique $\mathscr{J}$-class of T containing $T \setminus S$. Then J contains a J-primitive idempotent, that is, a minimal element in the restriction of the Rees order of idempotents of T to $J \cap E(T)$.

Steps:

(i) There exist $a, b \in J$ such that $ab \in J$.

(ii) There exists $t \in J$ such that $t^n \in J$ for all $n \in \mathbb{N}$.

Yet another useful...

Lemma

Let S be a big semigroup, and let T be any witness for S. Let J be the unique $\mathscr{J}$-class of T containing $T \setminus S$. Then J contains a J-primitive idempotent, that is, a minimal element in the restriction of the Rees order of idempotents of T to $J \cap E(T)$.

Steps:

(i) There exist $a, b \in J$ such that $ab \in J$.

(ii) There exists $t \in J$ such that $t^n \in J$ for all $n \in \mathbb{N}$.

(iii) J contains an idempotent.

Yet another useful...

Lemma

Let S be a big semigroup, and let T be any witness for S. Let J be the unique $\mathscr{J}$-class of T containing $T \setminus S$. Then J contains a J-primitive idempotent, that is, a minimal element in the restriction of the Rees order of idempotents of T to $J \cap E(T)$.

Steps:

(i) There exist $a, b \in J$ such that $ab \in J$.

(ii) There exists $t \in J$ such that $t^n \in J$ for all $n \in \mathbb{N}$.

(iii) J contains an idempotent.

(iv) J contains a J-primitive idempotent.

$\mathbb{Z}_2$ is not a big semigroup (1)

$\mathbb{Z}_2$ is not a big semigroup (1)

Assume to the contrary, that T is a witness for $\mathbb{Z}_2 = \{e, a\}$.

$\mathbb{Z}_2$ is not a big semigroup (1)

Assume to the contrary, that T is a witness for $\mathbb{Z}_2 = \{e, a\}$.

Since $T \setminus \mathbb{Z}_2$ is contained in a single $\mathscr{J}$-class J of T, there are two possibilities:

$\mathbb{Z}_2$ is not a big semigroup (1)

Assume to the contrary, that T is a witness for $\mathbb{Z}_2 = \{e, a\}$.

Since $T \setminus \mathbb{Z}_2$ is contained in a single $\mathscr{J}$-class J of T, there are two possibilities:

1. $T = J$ is simple,

$\mathbb{Z}_2$ is not a big semigroup (1)

Assume to the contrary, that T is a witness for $\mathbb{Z}_2 = \{e, a\}$.

Since $T \setminus \mathbb{Z}_2$ is contained in a single $\mathscr{J}$-class J of T, there are two possibilities:

1. $T = J$ is simple, or
2. T has precisely two $\mathscr{J}$-classes: $\mathbb{Z}_2$ and J.

$\mathbb{Z}_2$ is not a big semigroup (1)

Assume to the contrary, that T is a witness for $\mathbb{Z}_2 = \{e, a\}$.

Since $T \setminus \mathbb{Z}_2$ is contained in a single $\mathscr{J}$-class J of T, there are two possibilities:

1. $T = J$ is simple, or
2. T has precisely two $\mathscr{J}$-classes: $\mathbb{Z}_2$ and J.

In either case, J is the kernel of T and, since it contains a J-primitive idempotent that must also be T-primitive, it follows that J is completely simple.

$\mathbb{Z}_2$ is not a big semigroup (2)

Case 1: $T \cong \mathcal{M}(I, G, \Lambda, P)$, and G has a subgroup of order 2.

$\mathbb{Z}_2$ is not a big semigroup (2)

Case 1: $T \cong \mathcal{M}(I, G, \Lambda, P)$, and G has a subgroup of order 2.

Now $\mathbb{Z}_2$ is not big as a group (F+J+N — easy), so if G is infinite, there is a proper subgroup G_1 of G properly containing $\mathbb{Z}_2$, destroying T as a witness.

$\mathbb{Z}_2$ is not a big semigroup (2)

Case 1: $T \cong \mathcal{M}(I, G, \Lambda, P)$, and G has a subgroup of order 2.

Now $\mathbb{Z}_2$ is not big as a group (F+J+N — easy), so if G is infinite, there is a proper subgroup G_1 of G properly containing $\mathbb{Z}_2$, destroying T as a witness.

Thus G must be finite, so at least one of the index sets I, Λ are infinite.

$\mathbb{Z}_2$ is not a big semigroup (2)

Case 1: $T \cong \mathcal{M}(I, G, \Lambda, P)$, and G has a subgroup of order 2.

Now $\mathbb{Z}_2$ is not big as a group (F+J+N — easy), so if G is infinite, there is a proper subgroup G_1 of G properly containing $\mathbb{Z}_2$, destroying T as a witness.

Thus G must be finite, so at least one of the index sets I, Λ are infinite.

At the same time, notice that we must have

$$T = \langle G_{i\mu}, (j, h, \nu) \rangle$$

for some $i \in I$, $\mu \in \Lambda$, and any $(j, h, \nu) \in T \setminus G_{i\mu}$.

$\mathbb{Z}_2$ is not a big semigroup (2)

Case 1: $T \cong \mathcal{M}(I, G, \Lambda, P)$, and G has a subgroup of order 2.

Now $\mathbb{Z}_2$ is not big as a group (F+J+N — easy), so if G is infinite, there is a proper subgroup G_1 of G properly containing $\mathbb{Z}_2$, destroying T as a witness.

Thus G must be finite, so at least one of the index sets I, Λ are infinite.

At the same time, notice that we must have

$$T = \langle G_{i\mu}, (j, h, \nu) \rangle$$

for some $i \in I$, $\mu \in \Lambda$, and any $(j, h, \nu) \in T \setminus G_{i\mu}$.

However, $\langle G_{i\mu}, (j, h, \nu) \rangle \subseteq G_{i\mu} \cup G_{j\mu} \cup G_{i\nu} \cup G_{j\nu} \subsetneq T$.

$\mathbb{Z}_2$ is not a big semigroup (2)

Case 1: $T \cong \mathcal{M}(I, G, \Lambda, P)$, and G has a subgroup of order 2.

Now $\mathbb{Z}_2$ is not big as a group (F+J+N — easy), so if G is infinite, there is a proper subgroup G_1 of G properly containing $\mathbb{Z}_2$, destroying T as a witness.

Thus G must be finite, so at least one of the index sets I, Λ are infinite.

At the same time, notice that we must have

$$T = \langle G_{i\mu}, (j, h, \nu) \rangle$$

for some $i \in I$, $\mu \in \Lambda$, and any $(j, h, \nu) \in T \setminus G_{i\mu}$.

However, $\langle G_{i\mu}, (j, h, \nu) \rangle \subseteq G_{i\mu} \cup G_{j\mu} \cup G_{i\nu} \cup G_{j\nu} \subsetneq T$.
Contradiction!

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$.

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$. Furthermore, e can be assumed to be the identity of T, for otherwise $\{e, a\} \subsetneq eTe \subsetneq T$.

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$. Furthermore, e can be assumed to be the identity of T, for otherwise $\{e, a\} \subsetneq eTe \subsetneq T$.

Hence, each element of T is an alternating product of a and f.

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$. Furthermore, e can be assumed to be the identity of T, for otherwise $\{e, a\} \subsetneq eTe \subsetneq T$.

Hence, each element of T is an alternating product of a and f.

We have $faf \mathrel{\mathscr{J}} f$

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$. Furthermore, e can be assumed to be the identity of T, for otherwise $\{e, a\} \subsetneq eTe \subsetneq T$.

Hence, each element of T is an alternating product of a and f.

We have $faf \mathrel{\mathscr{J}} f \Longrightarrow f = t_1(faf)t_2 = ft_1faft_2f$ for some $t_1, t_2 \in J^1$.

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$. Furthermore, e can be assumed to be the identity of T, for otherwise $\{e, a\} \subsetneq eTe \subsetneq T$.

Hence, each element of T is an alternating product of a and f.

We have $faf \mathrel{\mathscr{J}} f \implies f = t_1(faf)t_2 = ft_1faft_2f$ for some $t_1, t_2 \in J^1$.

Therefore, for some $k \geq 1$ we have

$$(faf)^k = faf \cdots faf = f$$

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$. Furthermore, e can be assumed to be the identity of T, for otherwise $\{e, a\} \subsetneq eTe \subsetneq T$.

Hence, each element of T is an alternating product of a and f.

We have $faf \mathscr{J} f \implies f = t_1(faf)t_2 = ft_1faft_2f$ for some $t_1, t_2 \in J^1$.

Therefore, for some $k \geq 1$ we have

$$(faf)^k = faf \cdots faf = f$$

$\implies |J| \leq 4k + 1$ (i.e. J is finite).

$\mathbb{Z}_2$ is not a big semigroup (3)

Case 2: T is an ideal extension of a completely simple semigroup J by $\mathbb{Z}_2^0$.

So, T has an idempotent $f \neq e$, whence $T = \langle a, f \rangle$. Furthermore, e can be assumed to be the identity of T, for otherwise $\{e, a\} \subsetneq eTe \subsetneq T$.

Hence, each element of T is an alternating product of a and f.

We have $faf \mathrel{\mathscr{J}} f \implies f = t_1(faf)t_2 = ft_1faft_2f$ for some $t_1, t_2 \in J^1$.

Therefore, for some $k \geq 1$ we have

$$(faf)^k = faf \cdots faf = f$$

$\implies |J| \leq 4k + 1$ (i.e. J is finite). Contradiction!

OK, girls & boys, the last slide of this talk is SOOOOO predictable...

OK, girls & boys, the last slide of this talk is SOOOOO predictable...

Open Problem

Characterise big semigroups.

OK, girls & boys, the last slide of this talk is SOOOOO predictable...

Open Problem

Characterise big semigroups.

Igor, now remember to make a sketch on the black-/white-board...
(For what is a lecture without a nice drawing...?)

Also, don't forget some handwaving to finish it off nicely. ♡

THANK YOU!

Questions and comments to:

dockie@dmi.uns.ac.rs

Further information may be found at:

http://sites.dmi.rs/personal/dolinkai