Representing semigroups and groups by endomorphisms of Fraïssé limits

Part II. Groups: overt & covert

Igor Dolinka
dockie@dmi.uns.ac.rs

Department of Mathematics and Informatics, University of Novi Sad

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Sherlock Holmes: A Game of Shadows (2011)
It’s so overt, it’s covert – a more brutal version
Green’s relations

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\[ a \mathcal{L} b \iff S^1a = S^1b \iff (\exists u, v \in S^1) ua = b \text{ and } vb = a \]
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\[ \mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} \]
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\[ a R b \iff aS^1 = bS^1 \iff (\exists x, y \in S^1) ax = b & by = a \]

\[ a L b \iff S^1a = S^1b \iff (\exists u, v \in S^1) ua = b & vb = a \]

\[ \mathcal{D} = R \circ L = L \circ R \]

\[ \mathcal{H} = R \cap L \]
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\[
\begin{align*}
  a \mathcal{R} b & \iff aS^1 = bS^1 \iff (\exists x, y \in S^1) \ ax = b & \text{&} & by = a \\
  a \mathcal{L} b & \iff S^1a = S^1b \iff (\exists u, v \in S^1) \ ua = b & \text{&} & vb = a \\
  \mathcal{D} & = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R} \\
  \mathcal{H} & = \mathcal{R} \cap \mathcal{L} \\
  a \mathcal{J} b & \iff S^1aS^1 = S^1bS^1 \iff (\exists x, y, u, v \in S^1) \ uax = b & \text{&} & vby = a
\end{align*}
\]
The eggbox picture of a $\mathcal{D}$-class

![Diagram showing the eggbox picture of a $\mathcal{D}$-class with elements labeled $e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8$.](image-url)
The eggbox picture of a $\mathcal{D}$-class

Groups (overt): $\mathcal{H}$-classes shaded red (these are all isomorphic)
The eggbox picture of a $\mathcal{D}$-class

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maximal subgroups of a semigroup $= \mathcal{H}$-classes containing idempotents
Regularity

$a \in S$ is regular if

$$a = axa$$

for some $x \in S$. 
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for some \( x \in S \).

**Fact**

For any \( D \)-class \( D \), either all elements of \( D \) are regular or none of them.
Regularity

$a \in S$ is regular if

$$a = axa$$

for some $x \in S$.

**Fact**

For any $\mathcal{D}$-class $D$, either all elements of $D$ are regular or none of them.

Hence, $a$ is regular $\iff a \mathcal{D} e$ for and idempotent $e$. 
A regular $D$-class
A regular eggbox
A non-regular $\mathcal{D}$-class
A non-regular eggbox
Schützenberger groups – groups the never were

There is a ‘hidden’ / covert group capturing the structure of a (non-regular) \( \mathcal{D} \)-class \( D \), called the Schützenberger group of \( D \).
Schützenberger groups – groups the never were

There is a ‘hidden’ / covert group capturing the structure of a (non-regular) $D$-class $D$, called the Schützenberger group of $D$.

Namely, let $H$ be an $H$-class within a $D$-class $D$,
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There is a ‘hidden’ / covert group capturing the structure of a (non-regular) \( D \)-class \( D \), called the Schützenberger group of \( D \).

Namely, let \( H \) be an \( H \)-class within a \( D \)-class \( D \), and consider
\[
T_H = \{ t \in S^1 : Ht \subseteq H \}.
\]
There is a ‘hidden’ / covert group capturing the structure of a (non-regular) $\mathcal{D}$-class $D$, called the Schützenberger group of $D$.

Namely, let $H$ be an $\mathcal{H}$-class within a $\mathcal{D}$-class $D$, and consider $T_H = \{ t \in S^1 : Ht \subseteq H \}$.

Basic results of semigroup theory (Green’s Lemma) show that each $\rho_t : H \rightarrow H$ ($t \in T_H$) defined by

$$h\rho_t = ht$$

is a permutation of $H$. 
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Hence, $S_H = \{ \rho_t : t \in T_H \}$ is a permutation group on $H$. 
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Hence, $S_H = \{ \rho_t : t \in T_H \}$ is a permutation group on $H$. This is the (right) Schützenberger group of $H$. 
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Fact
If both $H_1, H_2$ belong to $D$, then $S_{H_1} \cong S_{H_2}$.
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Fact
If $H$ is a group (so that $D$ is regular), then $S_H \cong H$. 
A classical example: $\mathcal{T}_X$

Fact
In $\mathcal{T}_X$ we have:

$$(1) \quad f \sim g \iff \ker(f) = \ker(g);$$

$$(2) \quad f \preceq g \iff \im(f) = \im(g);$$

$$(3) \quad f \succeq g \iff \rank(f) = \abs{\im(f)} = \abs{\im(g)} = \rank(g);$$

$$(4) \quad J = \succeq;$$

$$(5) \quad \text{if } e = e^2 \text{ and } \rank(e) = k, \text{ then } H_e \sim S_k;$$

$$(6) \quad \mathcal{T}_X \text{ is regular.}$$
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Fact
In $\mathcal{T}_X$ we have:

1. $f \mathrel{R} g \iff \ker(f) = \ker(g)$;

2. $f \mathrel{L} g \iff \text{im}(f) = \text{im}(g)$;

3. $f \mathrel{D} g \iff \text{rank}(f) = |\text{im}(f)| = |\text{im}(g)| = \text{rank}(g)$;

4. $J = \mathrel{D}$;

5. if $e = e_2$ and $\text{rank}(e) = k$, then $H_e \sim S_k$;

6. $\mathcal{T}_X$ is regular.
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1. $f R g \iff \ker(f) = \ker(g)$;
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3. $f D g \iff \text{rank}(f) = \text{rank}(g)$;
4. $J = D$;
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Igor Dolinka: Groups – overt & covert
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3. $f \mathcal{D} g \iff \operatorname{rank}(f) = |\operatorname{im}(f)| = |\operatorname{im}(g)| = \operatorname{rank}(g)$;
4. $\mathcal{J} = \mathcal{D}$;
5. If $e = e^2$ and $\operatorname{rank}(e) = k$, then $H_e \cong S_k$.
A classical example: $\mathcal{T}_X$

Fact

In $\mathcal{T}_X$ we have:

1. $f \equiv g \iff \ker(f) = \ker(g)$;
2. $f \ll g \iff \operatorname{im}(f) = \operatorname{im}(g)$;
3. $f \dashv g \iff \operatorname{rank}(f) = |\operatorname{im}(f)| = |\operatorname{im}(g)| = \operatorname{rank}(g)$;
4. $\mathcal{I} = \mathcal{D}$;
5. if $e = e^2$ and $\operatorname{rank}(e) = k$, then $H_e \cong S_k$;
6. $\mathcal{T}_X$ is regular.
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(i) $f \ R \ g \implies \ker(f) = \ker(g)$;

(ii) $f \ L \ g \implies \text{im}(f) = \text{im}(g)$. 

Remark

We must be careful with the notion of an ‘image’ of an endomorphism if our language contains relational symbols, because besides $\text{im}(f)$ we also have $\langle Af \rangle$, the induced substructure of $A$ on $Af$. 

Lemma

$f \ D \ g \implies \langle Af \rangle \cong = \langle Ag \rangle$. 

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Let $A$ be a first-order structure. Since $\text{End}(A) \leq \mathcal{T}_A$, if $f, g \in \text{End}(A)$ are $R$-$\mathcal{L}$-related in $\text{End}(A)$ they are certainly $R$-$\mathcal{L}$-related in $\mathcal{T}_A$. Hence,

(i) $f \mathrel{R} g \implies \ker(f) = \ker(g)$;

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**Remark**

We must be careful with the notion of an ‘image’ of an endomorphism if our language contains relational symbols, because besides $\text{im}(f)$ we also have $\langle Af \rangle$, the induced substructure of $A$ on $Af$.

**Lemma**

$f \mathrel{D} g \implies \langle Af \rangle \cong \langle Ag \rangle$. 
Proposition (Magill, Subbiah, 1974)

If \( f \in \text{End}(A) \) is regular, then \( \text{im}(f) = \langle Af \rangle \).
Regular elements in $\text{End}(A)$

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Lemma (Magill, Subbiah, 1974)

Let $f, g \in \text{End}(A)$ be regular. Then:

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(iii) $f D g \iff \text{im}(f) \sim = \text{im}(g)$;

(iv) if $e$ is idempotent, then $H e \sim = \text{Aut}(\text{im}(e)) \sim = \text{Aut}(\text{im}(f))$ for any $f \in D e$. 
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(iv) if $e$ is idempotent, then $H_e \cong \text{Aut}(\text{im}(e)) \cong \text{Aut}(\text{im}(f))$ for any $f \in D_e$. 
Proposition

Let $f \in \text{End}(A)$ and $H = H_f$. 

Schützenberger groups in $\text{End}(A)$
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Let $f \in \text{End}(A)$ and $H = H_f$.

(i) If $t \in T_H$, then $t|_{Af}$ is an automorphism of both $\langle Af \rangle$ and $\text{im}(f)$;
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(i) If $t \in T_H$, then $t|_{Af}$ is an automorphism of both $\langle Af \rangle$ and $\text{im}(f)$;

(ii) the mapping $\phi : \rho_t \mapsto t|_{Af}$ is an embedding of $S_H$ into $\text{Aut}(\langle Af \rangle) \cap \text{Aut}(\text{im}(f))$. 

So, what the heck are the images of (idempotent) endomorphisms of Fraïssé limits?

Call a Fraïssé class $\mathcal{C}$ neat if it consists of finite structures, and for each $n \geq 1$ the number of isomorphism types of $n$-generated structures in $\mathcal{C}$ is finite.
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Examples:
- relational structures
- Fraïssé classes of algebras contained in locally finite varieties
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Examples:

- relational structures
- Fraïssé classes of algebras contained in locally finite varieties

**Theorem (ID, 2012)**

Let $\mathcal{C}$ be a neat Fraïssé class enjoying the strict AP and the 1PHEP. Then there exists an (idempotent) endomorphism $f$ of $F$, the Fraïssé limit of $\mathcal{C}$, such that $A \cong \text{im}(f)$ if and only if $A$ is algebraically closed in $\overline{\mathcal{C}}$. 

An $L$-formula $\Phi(x)$ is primitive if it is of the form

$$\exists y \bigwedge_{i < k} \Psi_i(x, y)$$

where each $\Psi_i$ is a literal: an atomic formula or its negation.

Let $K$ be a class of $L$-structures. An $L$-structure $A$ is existentially (algebraically) closed (in $K$) if for any primitive (positive) formula $\Phi(x)$ and any tuple $a$ from $A$ we have $A \models \Phi(a)$ whenever there is an extension $A' \in K$ of $A$ such that $A' \models \Phi(a)$.
An $L$-formula $\Phi(x)$ is **primitive** if it is of the form

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Countable e.c. graphs: $R$ (Alice’s Restaurant property)
Graphs

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Countable a.c. graphs: any finite set of vertices has a common neighbour ($\Rightarrow$ infinitely many of them)
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In the rest of this talk we will be concerned with simple graphs and study $\text{End}(R)$. 
Graphs

Countable e.c. graphs: $R$ (Alice’s Restaurant property)

Countable a.c. graphs: any finite set of vertices has a common neighbour ($\Rightarrow$ infinitely many of them)

In the rest of this talk we will be concerned with simple graphs and study $\text{End}(R)$. However, all these results can be adapted for

- the random digraph,
- the random bipartite graph,
- the random (non-strict) poset,
- ...


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Graphs

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**Proposition**

A countable graph \((V, E)\) is a.c. if and only if there exists \( E' \subseteq E \) such that \((V, E') \cong R\) (that is, it is e.c.).
**Graphs**

Countable e.c. graphs: \( R \) (Alice’s Restaurant property)

Countable a.c. graphs: any finite set of vertices has a common neighbour (⇒ infinitely many of them)

In the rest of this talk we will be concerned with simple graphs and study \( \text{End}(R) \). However, all these results can be adapted for

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**Proposition**

A countable graph \((V, E)\) is a.c. if and only if there exists \( E' \subseteq E \) such that \((V, E') \cong R \) (that is, it is e.c.). Consequently, for any a.c. graph \( \Gamma \) there is a bijective homomorphism \( R \rightarrow \Gamma \).
Frucht’s Theorem (1939)

Any finite group is $\cong \text{Aut}(\Gamma)$ for a finite graph $\Gamma$. 
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de Groot / Sabidussi (1959/60) $\Rightarrow$ automorphism groups of countable graphs include all countable groups.
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de Groot / Sabidussi (1959/60) $\Rightarrow$ automorphism groups of countable graphs include all countable groups.

Name of the game: Strengthen this for countable a.c. graphs.
The team

Point Guard: Martyn Quick

Forward: “Baby” James Mitchell

Center: Jillian “Jay” McPhee

Shooting Guard: Robert “Bob” Gray

Power Forward: Dr. D
Happy 30th birthday, Jay !!! (July 28)
Automorphism groups of countable a.c. graphs

Theorem

Let $\Gamma$ be a countable graph. Then there exist $2^{\aleph_0}$ pairwise non-isomorphic countable a.c. graphs whose automorphism group is $\cong \text{Aut}(\Gamma)$. 
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Proof. For a (simple) graph $\Delta$, let $\Delta^\dagger$ denote its complement.

$\triangleright$ $\text{Aut}(\Delta^\dagger) = \text{Aut}(\Delta)$. 

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Proof. For a (simple) graph $\Delta$, let $\Delta^\dagger$ denote its complement.

- $\text{Aut}(\Delta^\dagger) = \text{Aut}(\Delta)$.
- $\Delta$ any graph, $\Lambda$ infinite locally finite graph $\Rightarrow (\Delta \uplus \Lambda)^\dagger$ is a.c.
Automorphism groups of countable a.c. graphs

Theorem
Let \( \Gamma \) be a countable graph. Then there exist \( 2^{\aleph_0} \) pairwise non-isomorphic countable a.c. graphs whose automorphism group is \( \cong \text{Aut}(\Gamma) \).

Proof. For a (simple) graph \( \Delta \), let \( \Delta^\dagger \) denote its complement.

\begin{itemize}
  \item Aut(\( \Delta^\dagger \)) = Aut(\( \Delta \)).
  \item \( \Delta \) any graph, \( \Lambda \) infinite locally finite graph \( \Rightarrow (\Delta \uplus \Lambda)^\dagger \) is a.c.
  \item The central idea – consider l.f. graphs \( L_S \) for \( S \subseteq \mathbb{N} \setminus \{0, 1\} \):
\end{itemize}
Proof (cont’d).

- Properties of $L_S$ ($S, T \subseteq \mathbb{N} \setminus \{0, 1\}$):
  - Each $L_S$ is rigid ($\text{Aut}(L_S) = 1$).
  - $L_S \cong L_T \iff S = T$.
Automorphism groups of countable a.c. graphs

Proof (cont’d).

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  - Each $L_S$ is rigid ($\text{Aut}(L_S) = 1$).
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- If $L_S$ is isomorphic to no connected component of $\Gamma$ (and this excludes only countably many choices of $S$), then

  \[
  \text{Aut}(\Gamma \cup L_S)^\dagger = \text{Aut}(\Gamma \cup L_S) \cong \text{Aut}(\Gamma) \times \text{Aut}(L_S) \cong \text{Aut}(\Gamma).
  \]
Proof (cont’d).

- Properties of $L_S$ ($S, T \subseteq \mathbb{N} \setminus \{0, 1\}$):
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- $S_1 \neq S_2$ yield non-isomorphic a.c. graphs.
Images of idempotent endomorphisms

Theorem (Bonato, Delić, 2000; ID, 2012)

Let $\Gamma$ be a countable graph. There exists an idempotent $f \in \text{End}(R)$ such that $\text{im}(f) \cong \Gamma$ if and only if $\Gamma$ is a.c.
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Theorem

If \( \Gamma \) is a countable a.c. graph, then there exists an (induced) subgraph \( \Gamma' \cong \Gamma \) of \( R \) such that there are \( 2^{\aleph_0} \) idempotent endomorphisms \( f \) of \( R \) such that \( \text{im}(f) = \Gamma' \).
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Proof.

At each stage of extending a homomorphism \( \phi : \Gamma \to R \) to an endomorphism \( \hat{\phi} \) of \( R = R \Gamma \), instead of mapping \( v_S \mapsto v_S \phi \), if \( \text{im}(\phi) \) is a.c. one can find a common neighbour \( w \) for \( S \phi \) within \( \text{im}(\phi) \).
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In this way, we achieve

\[
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In fact, at each stage there are infinitely many choices for \( w \), which results in \( \aleph_0^{\aleph_0} = 2^{\aleph_0} \) extensions.
The number of regular $\mathcal{D}$-classes with a given group $\mathcal{H}$-class

Theorem

(i) Let $\Gamma$ be a countable graph. Then there exist $2^{\aleph_0}$ distinct regular $\mathcal{D}$-classes of $\text{End}(R)$ whose group $\mathcal{H}$-classes are $\cong \text{Aut}(\Gamma)$.
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Corollary $\text{End}(R)$ has $2^{\aleph_0}$ regular $\mathcal{D}$-classes. (You know, the ones with eggs...)
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**Corollary**

$\text{End}(R)$ has $2^\aleph_0$ regular $\mathcal{D}$-classes.
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$\text{End}(R)$ has $2^{\aleph_0}$ regular $D$-classes. (You know, the ones with eggs...)
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Theorem

*Every regular \( \mathcal{D} \)-class of \( \text{End}(R) \) contains \( 2^{\aleph_0} \) many \( \mathcal{R} \)- and \( \mathcal{L} \)-classes.*
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**Proof.** Let $e$ be an idempotent endomorphism of $R$, and let $\Gamma = \text{im}(e)$ (a.c.).
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We already know that the identity mapping on $\Gamma$ can be extended to $f \in \text{End}(R)$ in $2^{\aleph_0}$ ways such that $\text{im}(f) = \text{im}(e)$. 
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However, all these idempotents are not $\mathcal{R}$-related.
The size of a regular eggbox

\textit{L}-classes: Key idea – construct the graph $\Gamma^\#$ from $\Gamma$ by replacing each edge by the following gadget:
**The size of a regular eggbox**

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![Diagram showing the transformation from $\Gamma$ to $\Gamma^\#$](image)
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$\Gamma$ a.c. $\implies \Gamma^\#$ a.c.
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\textbf{\L -classes:} Key idea – construct the graph $\Gamma^\#$ from $\Gamma$ by replacing each edge by the following gadget:

Construct $R$ around $\Gamma^\#$, so that $R = R_{\Gamma^\#}$.

$\Gamma$ a.c. $\rightarrow$ $\Gamma^\#$ a.c. Hence, the identity map on $\Gamma^\#$ can be extended to an endomorphism $g : R \rightarrow \Gamma^\#$. 
The size of a regular eggbox

For each binary sequence \( b = (b_i)_{i \in \mathbb{N}} \) define a map \( \psi_b \) on \( \Gamma^\# \) by

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v_{i,r} \psi_b = v_{i,b_i}
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for all \( i \in \mathbb{N} \) and \( r \in \{0, 1\} \).
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For each binary sequence $\mathbf{b} = (b_i)_{i \in \mathbb{N}}$ define a map $\psi_{\mathbf{b}}$ on $\Gamma^\#$ by

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$g\psi_b \in \text{End}(R)$ are idempotents, $\text{im}(g\psi_b) \cong \Gamma \Rightarrow$ all these idempotents are $\mathcal{D}$-related to $e$. 

The size of a regular eggbox
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\( g\psi_b \in \text{End}(R) \) are idempotents, \( \text{im}(g\psi_b) \cong \Gamma \Rightarrow \) all these idempotents are \( \mathcal{D} \)-related to \( e \).

Different images \( \Rightarrow \) they are not \( \mathcal{L} \)-related.
Non-regular eggboxes

Theorem
Let $\Gamma \not\cong R$ be a countable a.c. graph. Then there exists a non-regular endomorphism of $R$ such that $\text{im}(f) \cong \Gamma$ and $D_f$ contains $2^{\aleph_0}$ many $R$- and $L$-classes.
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The proof is a variation of the idea of $\Gamma^\#$ and binary sequences.
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There are $2^{\aleph_0}$ non-regular $D$-classes in $\text{End}(R)$. 
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Open Problem
Are there any non-regular eggboxes of some other size?
Schützenberger groups in $\text{End}(R)$

Let $\Gamma = (V_0, E_0)$ be a countable a.c. graph.
Schützenberger groups in $\text{End}(R)$

Let $\Gamma = (V_0, E_0)$ be a countable a.c. graph. Then, as we already know, there is a subset $F \subseteq E_0$ such that $(V_0, F) \cong R$. 

Proposition

Let $f$ be an injective endomorphism of $R = (V, E)$ as described above, with $V_f = V_0$. Then $S_H f \cong \text{Aut}(\langle V_0 \rangle) \cap \text{Aut}(\text{im}(f))$. 
Let $\Gamma = (V_0, E_0)$ be a countable a.c. graph. Then, as we already know, there is a subset $F \subseteq E_0$ such that $(V_0, F) \cong R$. Now build $R_{\Gamma} \cong R$ around $\Gamma$, and let $f : R_{\Gamma} \to (V_0, F)$ be an isomorphism.
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Schützenberger groups in $\text{End}(R)$

So, to show a universality result for Schützenberger groups in $\text{End}(R)$, one needs to extend the Frucht-de Groot-Sabidussi Theorem to countable a.c. graphs with 2-coloured edges (blue and red, say) where the ‘red graph’ is $\cong R$. 

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This is what we did via an involved construction that again uses the rigid graphs $L_S$ (for a particular countable family of sets $S$).
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This is what we did via an involved construction that again uses the rigid graphs $L_S$ (for a particular countable family of sets $S$).

**Theorem**

Let $\Gamma$ be any countable graph. There are $2^{\aleph_0}$ non-regular $D$-classes of $\text{End}(R)$ such that the Schützenberger groups of the $H$-classes within them are $\cong \text{Aut}(\Gamma)$. 

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A few words on posets

A poset \((P, \leq)\) is a.c. if for any finite \(A, B \subseteq P\) such that \(A \leq B\) there exists \(x \in P\) such that

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Now by the Birkhoff’s Representation Theorem any automorphism group of a countable/finite graph can be represented as the automorphism group of a countable/finite distributive lattice.

It follows all countable/finite groups arise as automorphism groups of countable/finite a.c. posets.
A few words on posets

However, for strict posets \((P, <)\) the notion of being a.c. changes: here we require that for all finite \(A < B\) we have \(x \in P\) such that

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Open Problem
What are the automorphism groups of countable a.c. strict posets? (I.e. what are the maximal subgroups of $\text{End}(P, <)$?)
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Related work: G. Behrendt (PEMS, 1992)
THANK YOU!

Questions and comments to:
dockie@dmi.uns.ac.rs

Further information may be found at:
http://people.dmi.uns.ac.rs/~dockie