Representing semigroups and groups by endomorphisms of Fraïssé limits

Part II. Groups: overt & covert

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Sherlock Holmes: A Game of Shadows (2011)

It's so overt, it's covert – a more brutal version

Green's relations

The most fundamental tool in studying the structure of semigroups. (Named after J. Alexander "Sandy" Green (1926-2014).)

$$a \mathscr{R} b \iff aS^1 = bS^1 \iff (\exists x, y \in S^1)\, ax = b \,\&\, by = a$$

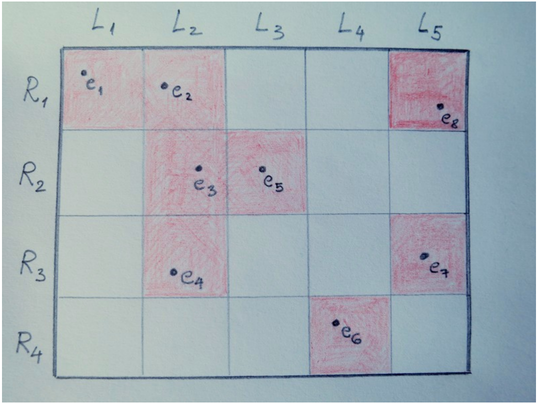
$$a \mathscr{L} b \iff S^1 a = S^1 b \iff (\exists u, v \in S^1)\, ua = b \,\&\, vb = a$$

$$\mathscr{D} = \mathscr{R} \circ \mathscr{L} = \mathscr{L} \circ \mathscr{R}$$

$$\mathscr{H} = \mathscr{R} \cap \mathscr{L}$$

$$a \mathscr{J} b \Leftrightarrow S^1 a S^1 = S^1 b S^1 \Leftrightarrow (\exists x, y, u, v \in S^1)\, uax = b \,\&\, vby = a$$

The eggbox picture of a $\mathscr{D}$-class



Groups (overt): $\mathscr{H}$-classes shaded red (these are all isomorphic)

maximal subgroups of a semigroup $=$ $\mathscr{H}$-classes containing idempotents

Regularity

$a \in S$ is regular if

$$a = axa$$

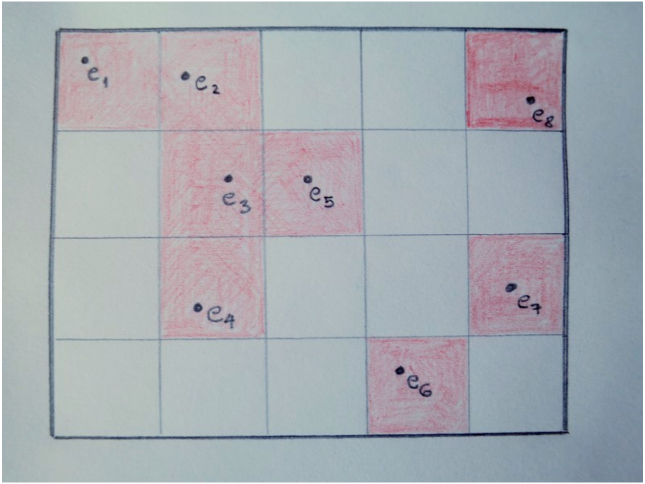
for some $x \in S$.

Fact

For any $\mathscr{D}$-class D, either all elements of D are regular or none of them.

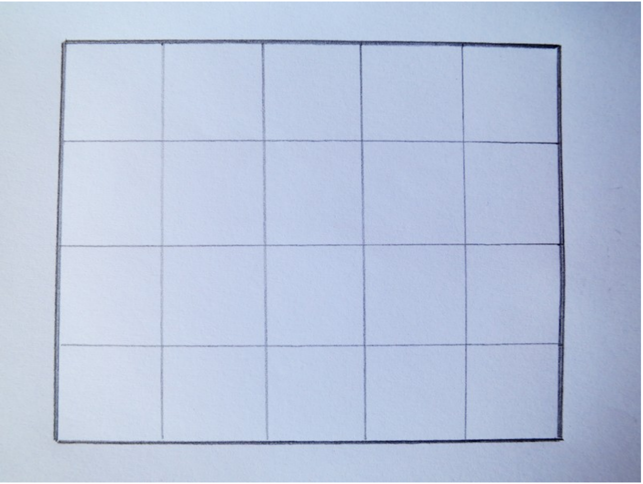
Hence, a is regular $\iff$ $a \mathrel{\mathscr{D}} e$ for and idempotent e.

A regular $\mathscr{D}$-class

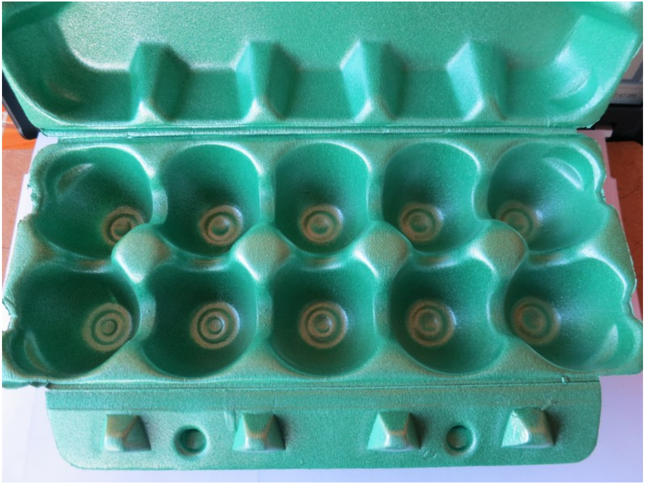


A regular eggbox

A non-regular $\mathscr{D}$-class



A non-regular eggbox



Schützenberger groups – groups the never were

There is a 'hidden' / covert group capturing the structure of a (non-regular) $\mathscr{D}$-class D, called the Schützenberger group of D.

Namely, let H be an $\mathscr{H}$-class within a $\mathscr{D}$-class D, and consider $T_H = \{t \in S^1 : \; Ht \subseteq H\}$.

Basic results of semigroup theory (Green's Lemma) show that each $\rho_t : H \to H$ $(t \in T_H)$ defined by

$$h\rho_t = ht$$

is a permutation of H.

Hence, $S_H = \{\rho_t : \; t \in T_H\}$ is a permutation group on H. This is the (right) Schützenberger group of H.

Schützenberger groups – groups the never were

Fact

If both H_1, H_2 belong to D, then $S_{H_1} \cong S_{H_2}$. Hence the Schützenberger group is really an invariant of a $\mathscr{D}$-class of a semigroup.

Fact

If H is a group (so that D is regular), then $S_H \cong H$.

A classical example: $\mathcal{T}_X$

Fact

In $\mathcal{T}_X$ we have:

(1) $f \mathscr{R} g \iff \ker(f) = \ker(g)$;

(2) $f \mathscr{L} g \iff \mathrm{im}(f) = \mathrm{im}(g)$;

(3) $f \mathscr{D} g \iff \mathrm{rank}(f) = |im(f)| = |\mathrm{im}(g)| = \mathrm{rank}(g)$;

(4) $\mathscr{J} = \mathscr{D}$;

(5) if $e = e^2$ and $\mathrm{rank}(e) = k$, then $H_e \cong \mathbb{S}_k$;

(6) $\mathcal{T}_X$ is regular.

End(A)

Let A be a first-order structure. Since $\mathrm{End}(A) \leq \mathcal{T}_A$, if $f, g \in End(A)$ are $\mathscr{R}$-/$\mathscr{L}$-related in $\mathrm{End}(A)$ they are certainly $\mathscr{R}$-/$\mathscr{L}$-related in $\mathcal{T}_A$. Hence,

(i) $f \ \mathscr{R} \ g \implies \ker(f) = \ker(g)$;

(ii) $f \ \mathscr{L} \ g \implies \mathrm{im}(f) = \mathrm{im}(g)$.

Remark

We must be careful with the notion of an 'image' of an endomorphism if our language contains relational symbols, because besides $\mathrm{im}(f)$ we also have $\langle Af \rangle$, the induced substructure of A on Af.

Lemma

$f \ \mathscr{D} \ g \implies \langle Af \rangle \cong \langle Ag \rangle$.

Regular elements in $\mathrm{End}(A)$

Proposition (Magill, Subbiah, 1974)

If $f \in End(A)$ is regular, then $\mathrm{im}(f) = \langle Af \rangle$.

Lemma (Magill, Subbiah, 1974)

Let $f, g \in End(A)$ be regular. Then:

(i) $f \mathscr{R} g \iff \ker(f) = \ker(g)$;

(ii) $f \mathscr{L} g \iff \mathrm{im}(f) = \mathrm{im}(g)$;

(iii) $f \mathscr{D} g \iff \mathrm{im}(f) \cong \mathrm{im}(g)$;

(iv) *if e is idempotent, then $H_e \cong \mathrm{Aut}(\mathrm{im}(e)) \cong \mathrm{Aut}(\mathrm{im}(f))$ for any $f \in D_e$.*

Schützenberger groups in End(A)

Proposition

Let $f \in \mathrm{End}(A)$ and $H = H_f$.

(i) *If $t \in T_H$, then $t|_{Af}$ is an automorphism of both $\langle Af \rangle$ and $\mathrm{im}(f)$;*

(ii) *the mapping $\phi : \rho_t \mapsto t|_{Af}$ is an embedding of S_H into $\mathrm{Aut}(\langle Af \rangle) \cap \mathrm{Aut}(\mathrm{im}(f))$.*

So, what the heck are the images of (idempotent) endomorphisms of Fraïssé limits?

Call a Fraïssé class **C** neat if it consists of finite structures, and for each $n \geq 1$ the number of isomorphism types of n-generated structures in **C** is finite.

Examples:

- relational structures
- Fraïssé classes of algebras contained in locally finite varieties

Theorem (ID, 2012)

Let $\mathbf{C}$ *be a neat Fraïssé class enjoying the strict AP and the 1PHEP. Then there exists and (idempotent) endomorphism* f *of* F*, the Fraïssé limit of* $\mathbf{C}$*, such that* $A \cong \mathrm{im}(f)$ *if and only if* A *is algebraically closed in* $\overline{\mathbf{C}}$*.*

Algebraically clo... wait, what?

An L-formula $\Phi(\mathbf{x})$ is primitive if it is of the form

$$(\exists \mathbf{y}) \bigwedge_{i<k} \Psi_i(\mathbf{x}, \mathbf{y})$$

where each Ψ_i is a literal: an atomic formula or its negation. No negation $\longrightarrow$ primitive positive formula.

Let $\mathbf{K}$ be a class of L-structures. An L-structure A is existentially (algebraically) closed (in $\mathbf{K}$) if for any primitive (positive) formula $\Phi(\mathbf{x})$ and any tuple $\mathbf{a}$ from A we have already $A \models \Phi(\mathbf{a})$ whenever there is an extension $A' \in \mathbf{K}$ of A such that $A' \models \Phi(\mathbf{a})$.

Graphs

Countable e.c. graphs: R (Alice's Restaurant property)
Countable a.c. graphs: any finite set of vertices has a common neighbour ($\Rightarrow$ infinitely many of them)

In the rest of this talk we will be concerned with simple graphs and study $\mathrm{End}(R)$. However, all these results can be adapted for

- the random digraph,
- the random bipartite graph,
- the random (non-strict) poset,
- ...

Proposition

A countable graph (V, E) is a.c. if and only if there exists $E' \subseteq E$ such that $(V, E') \cong R$ (that is, it is e.c.). Consequently, for any a.c. graph Γ there is a bijective homomorphism $R \to \Gamma$.

Frucht's Theorem (1939)

Any finite group is $\cong \mathrm{Aut}(\Gamma)$ for a finite graph Γ.

de Groot / Sabidussi (1959/60) $\Rightarrow$ automorphism groups of countable graphs include all countable groups.

Name of the game: Strengthen this for countable a.c. graphs.

The team

Point Guard:
Martyn Quick

Center:
Jillian "Jay" McPhee

Shooting Guard:
Robert "Bob" Gray

Forward:
"Baby" James Mitchell

Power Forward:
Dr. D

Happy 30th birthday, Jay !!! (July 28)

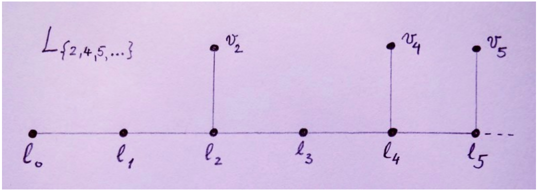
Automorphism groups of countable a.c. graphs

Theorem

Let Γ be a countable graph. Then there exist $2^{\aleph_0}$ pairwise non-isomorphic countable a.c. graphs whose automorphism group is $\cong \mathrm{Aut}(\Gamma)$.

Proof. For a (simple) graph Δ, let $\Delta^\dagger$ denote its complement.

- $\mathrm{Aut}(\Delta^\dagger) = \mathrm{Aut}(\Delta)$.
- Δ any graph, Λ infinite locally finite graph $\Rightarrow (\Delta \uplus \Lambda)^\dagger$ is a.c.
- The central idea – consider l.f. graphs L_S for $S \subseteq \mathbb{N} \setminus \{0, 1\}$:



Automorphism groups of countable a.c. graphs

Proof (cont'd).

- Properties of L_S ($S, T \subseteq \mathbb{N} \setminus \{0,1\}$):
  - Each L_S is rigid ($\mathrm{Aut}(L_S) = \mathbf{1}$).
  - $L_S \cong L_T \iff S = T$.
- If L_S is isomorphic to no connected component of Γ (and this excludes only countably many choices of S), then

$$\mathrm{Aut}(\Gamma \uplus L_S)^\dagger = \mathrm{Aut}(\Gamma \uplus L_S) \cong \mathrm{Aut}(\Gamma) \times \mathrm{Aut}(L_S) \cong \mathrm{Aut}(\Gamma).$$

- $S_1 \neq S_2$ yield non-isomorphic a.c. graphs.

Images of idempotent endomorphisms

Theorem (Bonato, Delić, 2000; ID, 2012)

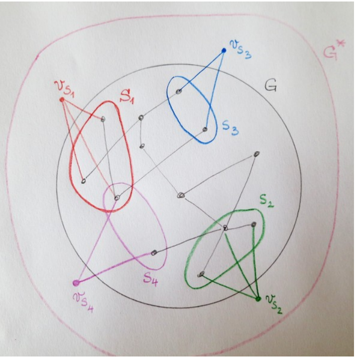
Let Γ be a countable graph. There exists an idempotent $f \in \mathrm{End}(R)$ such that $\mathrm{im}(f) \cong \Gamma$ if and only if Γ is a.c.

Theorem

If Γ is a countable a.c. graph, then there exists an (induced) subgraph $\Gamma' \cong \Gamma$ of R such that there are $2^{\aleph_0}$ idempotent endomorphisms f of R such that $\mathrm{im}(f) = \Gamma'$.

Images of idempotent endomorphisms

Proof.



At each stage of extending a homomorphism $\phi : \Gamma \to R_\Gamma$ to an endomorphism $\hat{\phi}$ of $R = R_\Gamma$, instead of mapping $v_S \mapsto v_{S\phi}$, if $\text{im}(\phi)$ is a.c. one can find a common neighbour w for $S\phi$ within $\text{im}(\phi)$.

In this way, we achieve

$$\text{im}(\hat{\phi}) = \text{im}(\phi).$$

In fact, at each stage there are infinitely many choices for w, which results in $\aleph_0^{\aleph_0} = 2^{\aleph_0}$ extensions.

The number of regular $\mathscr{D}$-classes with a given group $\mathscr{H}$-class

Theorem

(i) *Let* Γ *be a countable graph. Then there exist* $2^{\aleph_0}$ *distinct regular* $\mathscr{D}$*-classes of* $\mathrm{End}(R)$ *whose group* $\mathscr{H}$*-classes are* $\cong \mathrm{Aut}(\Gamma)$*.*

(ii) *Every regular* $\mathscr{D}$*-class contains* $2^{\aleph_0}$ *distinct group* $\mathscr{H}$*-classes.*

Corollary

$\mathrm{End}(R)$ *has* $2^{\aleph_0}$ *regular* $\mathscr{D}$*-classes. (You know, the ones with eggs...)*

The size of a regular eggbox

Theorem
Every regular $\mathscr{D}$-class of $\mathrm{End}(R)$ *contains* $2^{\aleph_0}$ *many* $\mathscr{R}$*- and* $\mathscr{L}$*-classes.*

Proof. Let e be an idempotent endomorphism of R, and let $\Gamma = \mathrm{im}(e)$ (a.c.).

$\mathscr{R}$-classes: Assume R is constructed as R_Γ.

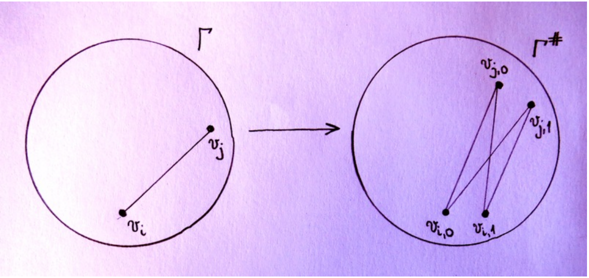
We already know that the identity mapping on Γ can be extended to $f \in \mathrm{End}(R)$ in $2^{\aleph_0}$ ways such that $\mathrm{im}(f) = \mathrm{im}(e)$.

All such f are idempotents, and $f \;\mathscr{D}\; e$, moreover, $f \;\mathscr{L}\; e$.

However, all these idempotents are not $\mathscr{R}$-related.

The size of a regular eggbox

$\mathscr{L}$-classes: Key idea – construct the graph $\Gamma^\sharp$ from Γ by replacing each edge by the following gadget:



Construct R around $\Gamma^\sharp$, so that $R = R_{\Gamma^\sharp}$.

Γ a.c. $\Longrightarrow$ $\Gamma^\sharp$ a.c. Hence, the identity map on $\Gamma^\sharp$ can be extended to an endomorphism $g : R \to \Gamma^\sharp$.

The size of a regular eggbox

For each binary sequence $\mathbf{b} = (b_i)_{i\in\mathbb{N}}$ define a map $\psi_{\mathbf{b}}$ on $\Gamma^\sharp$ by

$$v_{i,r}\psi_{\mathbf{b}} = v_{i,b_i}$$

for all $i \in \mathbb{N}$ and $r \in \{0,1\}$. Easy: $\psi_{\mathbf{b}} \in \mathrm{End}(\Gamma^\sharp)$ and $\mathrm{im}(\psi_{\mathbf{b}}) \cong \Gamma$ is induced by $\{v_{i,b_i} : \ i \in \mathbb{N}\}$.

$g\psi_{\mathbf{b}} \in \mathrm{End}(R)$ are idempotents, $\mathrm{im}(g\psi_{\mathbf{b}}) \cong \Gamma \Rightarrow$ all these idempotents are $\mathscr{D}$-related to e.

Different images $\Rightarrow$ they are not $\mathscr{L}$-related.

Non-regular eggboxes

Theorem

Let $\Gamma \not\cong R$ be a countable a.c. graph. Then there exists a non-regular endomorphism of R such that $\mathrm{im}(f) \cong \Gamma$ and D_f contains $2^{\aleph_0}$ many $\mathscr{R}$- and $\mathscr{L}$-classes.

The proof is a variation of the idea of $\Gamma^\sharp$ and binary sequences.

Theorem

There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes in $\mathrm{End}(R)$.

Open Problem

Are there any non-regular eggboxes of some other size?

Schützenberger groups in End(R)

Let $\Gamma = (V_0, E_0)$ be a countable a.c. graph. Then, as we already know, there is a subset $F \subseteq E_0$ such that $(V_0, F) \cong R$. Now build $R_\Gamma \cong R$ around Γ, and let $f : R_\Gamma \to (V_0, F)$ be an isomorphism. Then f is an injective endomorphism of R; if $F \neq E_0$ then f is non-regular.

Proposition

Let f be an injective endomorphism of $R = (V, E)$ as described above, with $Vf = V_0$. Then

$$S_{H_f} \cong \mathrm{Aut}(\langle V_0 \rangle) \cap \mathrm{Aut}(\mathrm{im}(f))$$

Schützenberger groups in End(R)

So, to show a universality result for Schützenberger groups in End(R), one needs to extend the Frucht-de Groot-Sabidussi Theorem to countable a.c. graphs with 2-coloured edges (blue and red, say) where the 'red graph' is $\cong R$.

This is what we did via an involved construction that again uses the rigid graphs L_S (for a particular countable family of sets S).

Theorem

Let Γ be any countable graph. There are $2^{\aleph_0}$ non-regular $\mathscr{D}$-classes of End(R) *such that the Schützenberger groups of the $\mathscr{H}$-classes within them are $\cong$* Aut(Γ).

See arXiv:1408.4107 for details.

A few words on posets

A poset $(P, \leq)$ is a.c. if for any finite $A, B \subseteq P$ such that $A \leq B$ there exists $x \in P$ such that

$$A \leq x \leq B.$$

Hence, any lattice is a.c. when considered as a poset (but not as an algebra!).

Now by the Birkhoff's Representation Theorem any automorphism group of a countable/finite graph can be represented as the automorphism group of a countable/finite distributive lattice.

It follows all countable/finite groups arise as automorphism groups of countable/finite a.c. posets.

A few words on posets

However, for strict posets $(P, <)$ the notion of being a.c. changes: here we require that for all finite $A < B$ we have $x \in P$ such that

$$A < x < B.$$

Open Problem

What are the automorphism groups of countable a.c. strict posets? (I.e. what are the maximal subgroups of $\mathrm{End}(\mathbb{P}, <)$?)

Related work: G. Behrendt (PEMS, 1992)

THANK YOU!

Questions and comments to:

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