Free spectra of finite semigroups and the Seif Conjecture

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Semigroups and Applications
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I'm free like a river
Flowin' freely to infinity
I'm free to be sure of what
I am and who I need not be
I'm much freer - like the meaning
Of the word 'free' that crazy man defines
Free - free like the vision that
The mind of only you are ever gonna see

Stevie Wonder: Free

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$f_n(\mathbf{A}) =$ the number of operations $t : A^n \to A$ induced by terms (for semigroups: by words), the term operations of $\mathbf{A}$. Thus if $|A| = a$ then $f_n(\mathbf{A}) \leq a^{a^n}$.

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So, free spectra appear to be quite a useful tool in the general task of classifying finite algebras.

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Question

Is there a 'Neumann-Higman type' gap result for finite monoids?

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Seif proved ($\Rightarrow$) by showing that if S is a non-orthodox completely simple semigroup, then $f_n(S^1)$ is doubly exponential.

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Open Problem (Kitaev & Seif)

Determine the exact asymptotic behaviour of $\log f_n(B_2^1)$.

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- the chain is $\mathcal{C}$-coterminal, i.e. for each $M \in \mathcal{C}$ there exists a $k_M \geq 1$ such that $M \in \mathcal{V}_{k_M}$, and
- it is relatively 'easy' to obtain a log-polynomial bound (in terms of n) for $f_n(\mathcal{V}_k)$ for each $k \geq 1$ (which is usually achieved by gathering sufficient information about the equational problem (the word problem for free objects) of $\mathcal{V}_k$).

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A glimmer of hope:

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(Almeida & Escada, 2000/02).

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G_A might exercise a term-expressible left action ρ on Y_A^1: e.g. the left multiplication.

Inverse monoids: (not entirely impossible) dreams?

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Theorem (ID, 2011/12)

Let A be a Clifford inverse algebra such that the set of its subgroups generates a locally finite group variety $\mathcal{U}$, while ρ is a term-expressible action of G_A on Y_A^1. Then

$$\log f_n(Y_A^1 *_\rho G_A) \in \mathcal{O}(n(\log f_{n+1}(\mathcal{U}))^2).$$

In particular, if all subgroups of A are nilpotent of some bounded class, then any inverse monoid dividing $Y_A^1 *_\rho G_A$ has a log-polynomial free spectrum.

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This produces a host of examples of finite inverse monoids with a log-polynomial free spectrum.

Some corollaries

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Example

Let $S(G)$ be the inverse monoid obtained from a group G acting on the Brandt semigroup B_G by

$$g \cdot (x, y) = (xg^{-1}, y) \quad \text{and} \quad (x, y) \cdot g = (x, yg).$$

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$\log f_n(B_2^1) \in \mathcal{O}(n^3)$. *(Because B_2^1 embeds into $S(\mathbb{Z}_2)$.)*

The Higgins-Margolis (counter)example (1)

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As already mentioned, $\mathbf{EDA} \cap \overline{\mathbf{H}} \neq \mathbf{DA} * \mathbf{H}$ for any proper group pseudovariety $\mathbf{H}$ because in 1998 P.Higgins and S.Margolis constructed, for any finite group G, a finite aperiodic monoid S_G with commuting idempotents ($\Rightarrow \in \mathbf{EDA}$) such that if $G \notin \mathbf{H}$ then $S_G \notin \mathbf{DA} * \mathbf{H}$.

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This is a real, challenging test-example for the Seif conjecture!

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Open Problem

Determine the asymptotic behaviour of $\log f_n(S_G)$. In particular, what if $G = \mathbb{S}_3$?

TACK SÅ MYCKET!
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(THANK YOU!)

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