Free spectra of finite semigroups and the Seif Conjecture

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I'm free like a river
Flowin' freely to infinity
I'm free to be sure of what
I am and who I need not be
I'm much freer - like the meaning
Of the word 'free' that crazy man defines
Free - free like the vision that
The mind of only you are ever gonna see

Stevie Wonder: Free

What the heck are 'free spectra'?

Let $\mathbf{A}$ be an algebra (in the sense of universal algebra).

For each $n \geq 1$, the variety $\mathcal{V} = \mathrm{HSP}(\mathbf{A})$ generated by $\mathbf{A}$ contains a free object $\mathbf{F}_n(\mathcal{V})$ on an n-element set. The free spectrum of $\mathbf{A}$ is defined by

$$f_n(\mathbf{A}) = |\mathbf{F}_n(\mathcal{V})|.$$

Remark

If $\mathbf{A}$ generates a locally finite variety (main example: $\mathbf{A}$ is finite), then $(f_n(\mathbf{A}))_{n \geq 1}$ is a sequence of finite numbers.

Fact

$f_n(\mathbf{A}) =$ the number of operations $t : A^n \to A$ induced by terms (for semigroups: by words), the term operations of $\mathbf{A}$. Thus if $|A| = a$ then $f_n(\mathbf{A}) \leq a^{a^n}$.

Why free spectra?

Numerous examples from universal algebra show that there is a very intimate connection between the structural features of finite algebras and the asymptotic behaviour of their free spectra.

The Tame Congruence Theory (TCT) constructed in the 80s by D.Hobby and R.McKenzie (and developing ever since) gives a fair bit of an explanation why is this so.

So, free spectra appear to be quite a useful tool in the general task of classifying finite algebras.

Examples

- L – a nontrivial left zero band: $f_n(L) = n$
- RB_2 – a 2×2 rectangular band: $f_n(RB_2) = n^2$
- SL_2 – a two-element semilattice: $f_n(SL_2) = 2^n - 1$
- p – a prime: $f_n(\mathbb{Z}_p) = p^n$
- B_2 – a two-element Boolean algebra: $f_n(B_2) = 2^{2^n}$
- L_2 – a two element lattice: $f_n(L_2) = ?$ (doubly exponential)
- $f_n(\mathbb{S}_3) =$ doubly exponential

Theorem (P.Neumann & G.Higman, 60s)

Let G be a finite group. We have

$$\log f_n(G) \in \mathcal{O}(n^c)$$

if and only if G is nilpotent of class c. Otherwise, $f_n(G) =$ doubly exponential.

The significance of semigroups/monoids

In 1999, K.Kearnes discovered that the behaviour of $f_n(\mathbf{A})$ is a great deal governed by the free spectrum of an associated monoid, the twin monoid $\mathrm{Tw}(\mathbf{A})$.

Consider all $(n+1)$-ary term operations $f(x, \mathbf{y})$ of $\mathbf{A}$ with the property that $f(x, \mathbf{a})$ is the identity mapping on A for some $\mathbf{a} \in A^n$. Then all transformations of A of the form $f(x, \mathbf{b})$, $\mathbf{b} \in A^n$, are called the twins of the identity. $\mathrm{Tw}(\mathbf{A})$ is defined to be the submonoid of $\mathcal{T}_A$ generated by all twins of the identity.

Hence, the classification problem of finite algebras strongly depends on properties of free spectra of finite monoids!

Question

Is there a 'Neumann-Higman type' gap result for finite monoids?

The Seif Conjecture (2008)...

...attempts to supply an answer to that question.

- **DA** = the pseudovariety of all finite monoids in which every regular element is idempotent ($\Leftrightarrow$ each regular $\mathcal{J}$-class is a rectangular band)
- **EDA** = the pseudovariety of all finite monoids M such that $\langle E(M)\rangle \in \mathbf{DA}$ ($\Leftrightarrow$ each regular principal factor is orthodox)
- $\overline{\mathbf{G_{nil}}}$ = the pseudovariety of all finite monoids all of whose subgroups are nilpotent

Conjecture (S.W.Seif)

Let M be a finite monoid. Then $f_n(M)$ is not doubly exponential (or perhaps is even log-polynomial) if and only if $M \in \mathbf{EDA} \cap \overline{\mathbf{G_{nil}}}$.

Seif proved ($\Rightarrow$) by showing that if S is a non-orthodox completely simple semigroup, then $f_n(S^1)$ is doubly exponential.

Early partial results

- The conjecture is true if $M = S^1$, where S is completely simple (Seif).
  - The hardest part: to deal with the Brandt monoid B_2^1.
  - It turns out that asymptotically $\log f_n(B_2^1) \in [n^2, n^3]$.
- For any finite non-commutative band monoid M we have $\log f_n(M) \sim n^k \log n$ for some $k \geq 1$ (Seif & Wood).
- The conjecture is true if M is a completely regular monoid (ID, 2009).

Theorem

The free spectrum of any finite locally orthodox completely regular semigroup with nilpotent subgroups is log-polynomial.

Open Problem (Kitaev & Seif)

Determine the exact asymptotic behaviour of $\log f_n(B_2^1)$.

A general strategy

Suppose we have a class of finite monoids $\mathcal{C} \subseteq \mathbf{EDA} \cap \overline{\mathbf{G}_{\mathbf{nil}}}$ for which we want to prove the Seif Conjecture.

For example, this can be accomplished by constructing a chain $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_k \subseteq \ldots$ of locally finite (not necessarily finitely generated) monoid varieties (i.e. equational pseudovarieties) such that:

- the chain is $\mathcal{C}$-coterminal, i.e. for each $M \in \mathcal{C}$ there exists a $k_M \geq 1$ such that $M \in \mathcal{V}_{k_M}$, and
- it is relatively 'easy' to obtain a log-polynomial bound (in terms of n) for $f_n(\mathcal{V}_k)$ for each $k \geq 1$ (which is usually achieved by gathering sufficient information about the equational problem (the word problem for free objects) of $\mathcal{V}_k$).

Applications

(1) ID (2009): completely regular monoids; $\mathcal{C}$ = finite orthogroups with nilpotent subgroups

- Ingredients: The 'chain' is constructed by using the full force of Libor Polák's theory of CR semigroup varieties.

(2) Cs.Szabó et al. (2011): $\mathcal{R}$-trivial monoids

- Ingredients: Iterated semidirect products of semilattices ($\mathbf{Sl}^k$) + Stiffler's Theorem

(3) ID (2012): monoids from **DA**

- Ingredients: The bilateral semidirect product analogue of the previous approach ($\mathbf{Sl}^{(k)}$) + the result of Straubing-Thérien (although the generalisation doesn't go so smoothly as expected)

$$\log f_n(\mathbf{Sl}^k) \in \mathcal{O}(n^k) \qquad \log f_n(\mathbf{Sl}^{(k)}) \in \mathcal{O}(n^{2k-1})$$

Hopes and (broken) dreams

Because of (3), the problem would be solved if someone would came up with a miraculous formula for an arbitrary finite monoid M providing a polynomial upper bound for $\log f_n(M)$ in terms of

- $\log f_n(G)$, with G ranging through subgroups of M, and
- $\log f_n(\langle E(M) \rangle)$.

A glimmer of hope:

$$\mathbf{EDA} = \mathbf{DA} * \mathbf{G}$$

(Almeida & Escada, 2000/02).

But alas! Relativisation does not work: if $\mathbf{H}$ is any proper pseudovariety of groups, then

$$\mathbf{EDA} \cap \overline{\mathbf{H}} \supsetneq \mathbf{DA} * \mathbf{H}$$

(Higgins & Margolis, 2000). More on this in a minute.

Inverse algebras

Inverse algebra – if the natural order of an inverse monoid is a (meet-)semilattice order, one can equip the monoid by an additional operation $\wedge$, and this yields an 'inverse algebra'. Inverse algebras form a variety (J.Leech, 1995).

Clifford inverse algebra – inverse algebra in which the underlying monoid is Clifford.

If A is an inverse algebra, then we denote by

- Y_A – its underlying meet semilattice,
- G_A – its group of units.

G_A might exercise a term-expressible left action ρ on Y_A^1: e.g. the left multiplication.

Inverse monoids: (not entirely impossible) dreams?

Theorem (ID, 2011/12)

Let A be a Clifford inverse algebra such that the set of its subgroups generates a locally finite group variety $\mathcal{U}$, while ρ is a term-expressible action of G_A on Y_A^1. Then

$$\log f_n(Y_A^1 *_\rho G_A) \in \mathcal{O}(n(\log f_{n+1}(\mathcal{U}))^2).$$

In particular, if all subgroups of A are nilpotent of some bounded class, then any inverse monoid dividing $Y_A^1 *_\rho G_A$ has a log-polynomial free spectrum.

This produces a host of examples of finite inverse monoids with a log-polynomial free spectrum.

Some corollaries

Example

Let $S(G)$ be the inverse monoid obtained from a group G acting on the Brandt semigroup B_G by

$$g \cdot (x, y) = (xg^{-1}, y) \quad \text{and} \quad (x, y) \cdot g = (x, yg).$$

($S(G)$ was used by Reilly and Sapir, for example, for different purposes.)

Corollary

If G is a finite nilpotent group of class c, then $\log f_n(S(G)) \in \mathcal{O}(n^{2c+1})$.

Corollary

$\log f_n(B_2^1) \in \mathcal{O}(n^3)$. *(Because B_2^1 embeds into $S(\mathbb{Z}_2)$.)*

The Higgins-Margolis (counter)example (1)

As already mentioned, $\mathbf{EDA} \cap \overline{\mathbf{H}} \neq \mathbf{DA} * \mathbf{H}$ for any proper group pseudovariety $\mathbf{H}$ because in 1998 P.Higgins and S.Margolis constructed, for any finite group G, a finite aperiodic monoid S_G with commuting idempotents ($\Rightarrow\ \in \mathbf{EDA}$) such that if $G \notin \mathbf{H}$ then $S_G \notin \mathbf{DA} * \mathbf{H}$.

S_G is a submonoid of the symmetric inverse monoid I_{2k}, where $k = |G|$.

This is a real, challenging test-example for the Seif conjecture!

The Higgins-Margolis (counter)example (2)

Let G' be a disjoint copy of G. S_G consists of the following partial injections on the set $G \cup G'$:

- the identity mapping $\mathbf{1}_{G\cup G'}$,
- the bijections $b_g : G \to G'$, for each $g \in G$, defined by

$$b_g(x) = (xg)', \qquad x \in G,$$

- the Brandt semigroup consisting of the empty mapping and all mappings on $G \cup G'$ of rank 1.

Open Problem

Determine the asymptotic behaviour of $\log f_n(S_G)$. In particular, what if $G = \mathbb{S}_3$?

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(THANK YOU!)

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