One-relator groups, monoids, inverse monoids: An update on the word problem

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- (orientable) surface groups $\mathrm{Gp}\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \ldots [a_g, b_g] = 1 \rangle$

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Lallement (1977) and L. Zhang (1992) provided alternative proofs for the result about special monoids. (The proof of Zhang is particularly compact and elegant.)

The connection to the inverse realm

Adyan & Oganessyan (1987): The word problem for one-relator monoids can be reduced to the special case of

$$\mathrm{Mon}\langle X \mid asb = atc \rangle$$

where $a, b, c \in X$, $b \neq c$ and $s, t \in X^*$ (and their duals).

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This holds basically because $M = \mathrm{Mon}\langle X \mid asb = atc \rangle$ embeds into $I = \mathrm{Inv}\langle X \mid asbc^{-1}t^{-1}a^{-1} = 1 \rangle$.

The plot thickens

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This result follows from the existence of a particular one-relator group G and its finitely generated submonoid $N \leq G$ such that the membership problem for N in G is undecidable.

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Problem

Does every one-relator group $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem? If the answer is "no", characterise words w which do have this property.

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- ...defines certain Adyan-type or Baumslag-Solitar-type groups (Margolis, Meakin, Šunić, 2005);
- ...satisfies certain small-cancellation-type conditions (Juhász, 2012, 2014).

Tying everything together

Theorem (Ivanov, Margolis & Meakin, 2001)

If $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ *is* E*-unitary, then*

word problem for M = *prefix membership problem for* $G = \mathrm{Gp}\langle X \mid w = 1\rangle$.

E-unitary inverse semigroups = the "good guys" of inverse semigroup theory:

- For any $e \in E(S)$ and $x \in S$,
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If w is cyclically reduced, then $M = \mathrm{Inv}\langle X \mid w = 1\rangle$ is E-unitary.

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Here is a sample of such a type of result.

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(iii) the membership problem for $M \cap H$ in H is decidable;

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Let M be a submonoid of G such that:

(i) $L \subseteq M$;

(ii) both $M \cap H$ and $M \cap K$ are finitely generated, and

$$M = \mathrm{Mon}\langle (M \cap H) \cup (M \cap K) \rangle;$$

(iii) the membership problem for $M \cap H$ in H is decidable;

(iv) the membership problem for $M \cap K$ in K is decidable.

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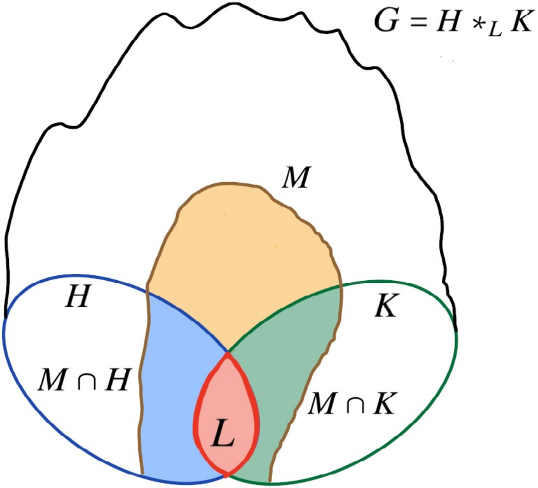
Then the membership problem for M in G is decidable.

Picture for Theorem A



Conservative factorisations (1)

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The factors w_i in a conservative factorisation are called pieces.

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  - Example: $G = \mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$ is a conservative factorisation

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Example

In $G = \mathrm{Gp}\langle a, b, x, y \mid axbaybaybaxbaybaxb = 1\rangle$, the Adyan overlap method produces a conservative factorisation

$$(axb)(ayb)(ayb)(axb)(ayb)(axb),$$

where the pieces axb and ayb have the unique marker letter property, so G has decidable prefix membership problem.

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$G = \mathrm{Gp}\langle a, b, c, d \mid (abcd)(acd)(ad)(abbcd)(acd) = 1\rangle$ is a conservative factorisation. Here, the pieces don't have the unique marker letter property.

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we obtain a conservative factorisation, where the pieces aba^{-1}, aca^{-1} and ad do have the unique marker letter property.

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(This answers a question harking back to the 1987 paper of Margolis, Meakin and Stephen.)

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$$(abab)(cdcd)(abab)(cdcd)(cdcd)(abab),$$

where the pieces *abab* and *cdcd* have no letters in common. So, G has decidable prefix membership problem.

Application 3: Exponent sum zero theorem

Let $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ where $t \in X$ and w (containing t) has t-exponent sum 0.

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Theorem
With the above notation, if G' is a free group and w is prefix t-positive, then G has decidable prefix membership problem.

Further applications

- Cyclically pinched groups: $\mathrm{Gp}\langle X, Y \mid uv^{-1} = 1 \rangle$ where $u \in \overline{X}^*$ and $v \in \overline{Y}^*$

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  - These include the orientable surface groups

$$\mathrm{Gp}\langle a_1, \ldots, a_n, b_1, \ldots, b_n \mid [a_1, b_1] \ldots [a_n, b_n] = 1\rangle$$

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- Some Adyan-type groups: $\mathrm{Gp}\langle X \mid uv^{-1} = 1 \rangle$, $u, v \in X^*$ are positive words such that the first letters of u, v are different and also the last letters of u, v are different (some new cases are covered)

A negative result and a problem

Theorem

There exists a finite alphabet X and a reduced word $w \in \overline{X}^*$ such that $G = \mathrm{Gp}\langle X \mid w = 1 \rangle$ has undecidable prefix membership problem.

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Characterise the words w such that $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem.

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Open Problem

Characterise the words w such that $G = \mathrm{Gp}\langle X \mid w = 1\rangle$ has decidable prefix membership problem. Are all cyclically reduced words among them?

Thank you!

Questions and comments to:

dockie@dmi.uns.ac.rs

Further information may be found at:

http://people.dmi.uns.ac.rs/∼dockie